LVAIC Mathematics Contest – November 13, 2016

Do as many of these problems as you can.
No calculators of any sort, notes, or reference materials are allowed.
Your solutions must be complete and your work justified to receive full credit.
Each solution must be written on a separate piece of paper with your team's code on it.

1. A numerical code of five distinct integers has been sent. It is your job to decipher the code that was sent. Correctly deduce the five number code given the following information. For every row there is a series of circles. A filled in circle represents that a number in that row is correct and in the correct place. A hollow circle represents a digit that is in the code but in the wrong place. Correctly decipher this code and explain the reasoning behind your answer.

5	2	3	1	0	●
8	9	7	6	5	○ ○ ●
4	0	1	9	8	● ●
5	3	7	2	6	○ ○ ○

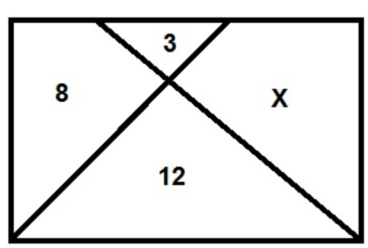
(Note that the first line means that exactly one of the numbers 52310 is correct and in the correct place in the code.)

2. Eighteen people are in a room and they are assigned the integers 1 through 18. They decide to form 9 teams of 2 individuals to compete in a mathematics competition that is similar but not the same as the LVAIC competition. Someone notices that if you add the two numbers of the teammates for each team, the result is always a perfect square. (This would only be noticed at a mathematics competition.) Who is person #1 teamed up with?

3. A positive integer *n* is randomly chosen between 1 and 20 (inclusive). What is the probability that n! will be a multiple of 2016?

4. Find all solutions (x,y) to the equation $|||x-y|-x|-y| = 0$.

5. Two crossing line segments divide a rectangle into four regions as pictured below. The areas of three of the regions are given in the diagram. Determine X, the area of the fourth region.



6. You have a computer program that performs the following operation. If the number input into the program is divisible by 5, divide the input number by 5. Otherwise, add 2 to the input number. The program then uses the output of this step as the next input before repeating the procedure. You randomly pick an integer between 1 and 2016 (all numbers equally likely) as the first input and then decide to go on a three week vacation to the Bahamas. You stop the program when you return, and you have no idea how many times the program ran, but you do know that it ran many, many times. What numbers are possible for the output when you finally stopped the program and what is the probability of obtaining these numbers?

7. You have an 8-liter bowl which is full of fresh water. You only have two empty bowls: a 5-liter bowl and a 3-liter bowl. You want to divide the water into two equal portions (4 liters + 4 liters). Describe a strategy that allows you to achieve your goal.

8. Call a quadruple of positive integers (a,b,c,d) good if $a < b < c < d$ and $\gcd(a,b) = 1$, $\gcd(b,c) = 2$, $\gcd(c,d) = 3$, and $\gcd(a,c) = \gcd(a,d) = \gcd(b,d) = 1$ where $\gcd(a,b)$ is the greatest common divisor of the integers a and b.

(a) Show that for any positive integer N there is a good quadruple with $a > N$.

(b) Find an explicit good quadruple with $a > 2016$.

9. Suppose there is a head of mathematics department (let's call him Gary) who doesn't like to talk to students. The floor on which his department is located has *n* offices, aligned on one side of a corridor. Each day, Gary decides to randomly move his office either one office to the left or one office to the right, if possible. If he gets to the end of the hall, he cannot choose the office at the other end of the hall (wrapping around), but must choose the office immediately next to him as his next office. He also cannot stay in the same office for two consecutive days. A student (let's call him Ben) wants to speak to Gary, but can only check one office per day.

a) Show that Ben has a strategy that allows him to meet Gary in a finite number of days.

b) According to your strategy, what is the maximum number of days required for Ben to find Gary? Of all strategies proposed for this competition, whichever team or teams that have the smallest maximum will receive full credit.

10. An isosceles right triangle has vertices (0,0), (0,1) and (1,0) in the xy plane. It "rolls" along the x-axis, pivoting at (1,0) until it lands with the hypotenuse lying along the x-axis. Then it continues to "roll," until the triangle is standing up again, as in the picture. As the triangle moves, it traces out the dashed curve in the picture. Find the total area between that curve and the x-axis.