LVAIC Mathematics Contest – November 8, 2015

Do as many of these problems as you can.
No calculators of any sort, notes, or reference materials are allowed.
Your solutions must be complete and your work justified to receive full credit.
Each solution must be written on a separate piece of paper with your team's code on it.

1. Even though this is the first problem, you're already thinking about taking a vacation and getting the most out of your relaxing time. With your brain still on mathematics, you create the following equation:

$$\begin{array}{r} \text{MOST} \\ +\ \underline{\text{MOST}} \\ \text{TOKYO} \end{array}$$

where each of the letter represents a distinct digit and neither M nor T are equal to 0. Before you leave for Japan, find all possible digits for the letters.

2. You decide to go to your local farmers' market for some fresh fruit. The farmers' market has three barrels of fruit: one is labeled "Apples," one is labeled "Oranges," and one is labeled "Apples and Oranges." Unfortunately, this particular farmers' market is notorious for never labeling their barrels correctly. You decide to blindly reach into only one barrel and randomly pick out a fruit. After observing the fruit, you know what the proper labels for the three barrels should be. Explain the strategy you used to determine which barrel has which fruit.

3. A *k*-sequence is an increasing sequence of *k* positive integers such that the positive difference between every pair is also the greatest common divisor (gcd) of that pair. An example of a 3-sequence is {9, 10, 12} because 10-9=1 (which is the gcd of 9 and 10), 12-9=3 (which is the gcd of 9 and 12), and 12-10=2 (which is the gcd of 10 and 12).

a) Find a 4-sequence.
b) Find a 5-sequence.
c) Prove that *k*-sequences exist for all positive integers $k > 1$.

4. While working on this test, you get bored and start doodling in 3-dimensions. You decide to start at the point (0,0,0). You then draw a line to the point (1,0,0), adding 1 to the x-coordinate. Then, you draw a line to the point (1,2,0), adding 2 to the y-coordinate. Next, you draw a line to the point (1,2,3), adding 3 to the z-coordinate. After this, you add 4 to the x-coordinate, moving to the point (5,2,3). You continue the pattern. Thus, your path goes as follows:

$$(0,0,0) \rightarrow (1,0,0) \rightarrow (1,2,0) \rightarrow (1,2,3) \rightarrow (5,2,3) \rightarrow (5,7,3) \rightarrow (5,7,9) \rightarrow (12,7,9) \rightarrow \ldots$$

Where will you be after taking exactly 2015 steps?

5. Let a_n be the number of ways to write the integer *n* using only powers of 3, where each number may be used up to 3 times. For example, $a_6 = 2$ since the only ways to write the number 6 using this technique are $6 = 3 + 3$ and $6 = 3 + 1 + 1 + 1$.

a) Find a_{2015}.
b) Find any integer *n* such that $a_n = 100$.

6. Welcome to Elvayak, a strange land with even stranger inhabitants. Each inhabitant has two characteristics, veracity and clarity. Clarity types are sharp and confused. A sharp inhabitant believes all true statements and does not believe all false statements. A confused inhabitant believes all false statements and does not believe all true statements. Veracity types are trustworthy and devious. A trustworthy inhabitant always makes true statements about her beliefs, and a devious inhabitant always lies about her beliefs. For example, an inhabitant who was both confused and devious could make the statement 2 + 2 = 4. The statement is true, but since she is confused, she believes the statement is false. Since she is devious, she will lie about her beliefs, and make the statement. You meet three inhabitants, Alice, Betty, and Charlie, who make the following statements.

Alice: "I am trustworthy."

Betty: "I am sharp."

Charlie: "I am confused."

Alice: "Betty is confused."

Betty: "Charlie is confused."

Charlie: "Alice is sharp."

Determine both the veracity type and clarity type of each of the three inhabitants.

7. Show that for each positive integer n, there is a positive integer m whose digits consist only of 0s and 1s such that m is divisible by n.

8. There are $6n$ rocks on a table, each with a positive integer between 1 and $6n$ written on it. All the integers are different. Two players alternate moves. A move consists of taking one of the rocks from the table and putting it in that player's rock pile. At the end of the game, the second player wins if the sum of the numbers in his or her pile is divisible by 3. Otherwise, the first player wins. Which player has a winning strategy and what is that strategy?

9. Find all real number triples (x, y, z) such that when any one of these numbers is added to the product of the other two, the result is 2.

10. Looking around, you note the following picture on the wall, featuring rotating squares inscribed in each other, continuing on infinitely. Not being a fan of modern art, you start randomly throwing darts at the picture. If you continue to throw darts, on average within how many squares could you expect that each dart would be contained?