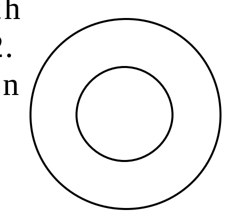
LVAIC Mathematics Contest - Oct. 25, 1997

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. Show that the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ always has an integer solution, where the numbers 3, 4, -5, 2 and -4 are randomly assigned as the unspecified coefficients.

2. Show that the number $19^m 97^n + 2$ can never be a prime number for any choice of positive integers m and n.

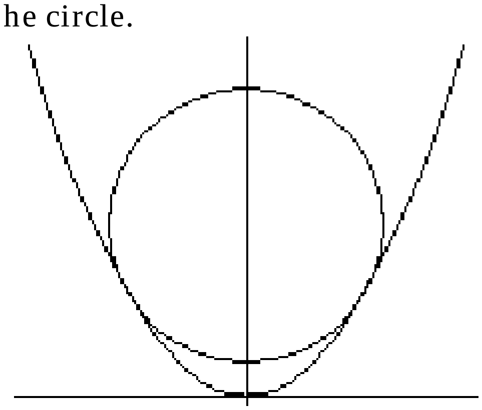
3. Two concentric circles are drawn, forming an annulus, with the inner circle of radius 1 and the outer circle of radius 2. (The annulus includes the two circles and the region between them.)



a) One point is picked at random on the outer circle and one point is picked at random on the inner circle.
What is the probability that the line segment between them lies entirely within the annulus?
b) This time two points are picked at random on the outer circle. What is the probability that the line segment between them lies entirely within the annulus?

4. Let $f(x) = \dfrac{x^{1997}}{x-1}$. Find $f^{(1997)}(0)$, the 1997th derivative of $f(x)$ evaluated at $x = 0$.

5. The figure shows the parabola $y = x^2$ with a circle of radius 1 inscribed in it. Find the center of the circle.

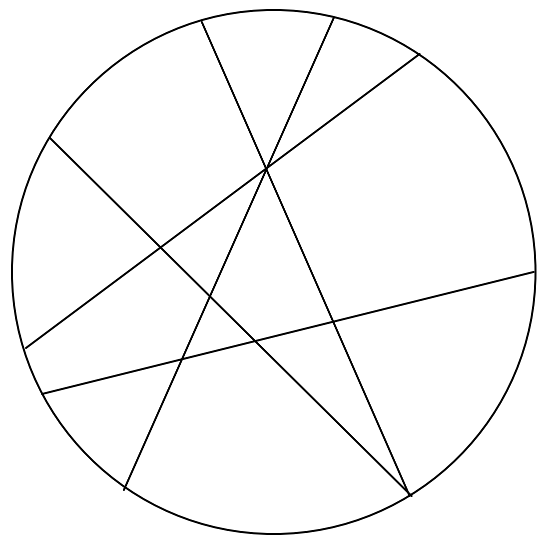


6. 1997 ping pong balls, numbered 1, 2, …, 1997, are in a big box. You reach in and pull out 2 balls, and record the smaller of the two numbers drawn, then put these two balls back in the box. Suppose you systematically pull out

every possible pair of balls, record the smaller number, then replace the balls. What will the average of all the numbers you wrote down be? Express your answer as simply as possible.

7. A two-person cooperative game is played as follows: The players are given the number 1000 initially. Player 1 is allowed to add or subtract any multiple of 7 from 1000, then passes the result to player 2. Player 2 is then allowed to add or subtract any multiple of 10 from the number player 1 passed, then passes the result back to player 1. The game continues in this way, with the players passing numbers back and forth, each player adding or subtracting multiples of 7 (in the case of player 1) or 10 (in the case of player 2), until they get the number 1. Show that they can eventually achieve their goal. What is the minimum number of steps needed to reach 1?

8. The continent of LVAIConia is circular, with all country boundaries made by lines that meet the circle in two points. See the picture for one possible map having 13 countries. We wish to color the countries on the map so that no two countries which share a border receive the same color. (Two countries that share only one point can receive the same color.) Prove that the minimum number of colors which will always work for any such map (where the country boundaries are lines that meet the circle twice) is 2.
Your proof needs to work for any such map, not just the one drawn.)



9. Prove that, for any x ? 0, $\dfrac{\sin x}{x} = \cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)\cdots$

10. Suppose $0 < a \le b$ are real numbers. Find $\lim_{n\to\infty}\left(a^n + b^n\right)^{1/n}$