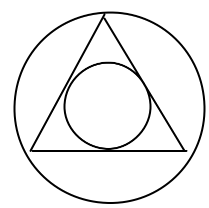
LVAIC Mathematics Contest - Oct. 19, 1996

Do as many of these problems as you can. Your solutions must be complete and your work justified to receive full credit.

1. A 500 pound mixture is 99% water and 1% salt by weight. How much water must be evaporated to make the mixture 98% water?

2. A bracelet is made by putting 4 spherical beads on a circular wire. How many different bracelets can be made using one blue, one red, one white, and one yellow bead? (If one bracelet can be obtained from another by flipping the entire bracelet over, then these two bracelets are condsidered identical.)

3. The pattern below with the equilateral triangle, the inscribed circle and the circumscribed circle has been found in wheat fields in England. Some people claim this pattern was made by extraterrestrial visitors (or producers of *The X-Files*).
Let T denote the triangle, S_1 the smaller circle and S_2 the larger circle.
Which is greater,



$$\frac{\text{Area}(S_1)}{\text{Area}(T)} \quad \text{or} \quad \frac{\text{Area}(T)}{\text{Area}(S_2)}$$

4. Find a polynomial p(x) of degree 2 such that p(2)=5 and p(3)=2.

5. The numbers from 1 to 1824 (= 19*96) are randomly put in an array having 19 (horizontal) rows and 96 (vertical) columns. A *column sum* is just the sum of the 19 numbers in a column. Similarly, a *row sum* is just the sum of the 96 numbers in a row.
Determine whether or not each of the following statements is true. If the statement is true, give a proof. If it's false, give a counterexample.

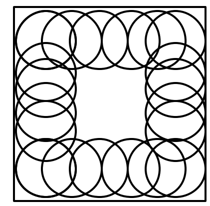
   a) The smallest column sum is always smaller than the smallest row sum.

   b) The largest column sum is always smaller than the largest row sum.

6. A (common?) mistake by students learning algebra is $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$. Find all real numbers a and b which satisfy this equation or show that there are none. (Assume a ? 0 and b ? 0.)

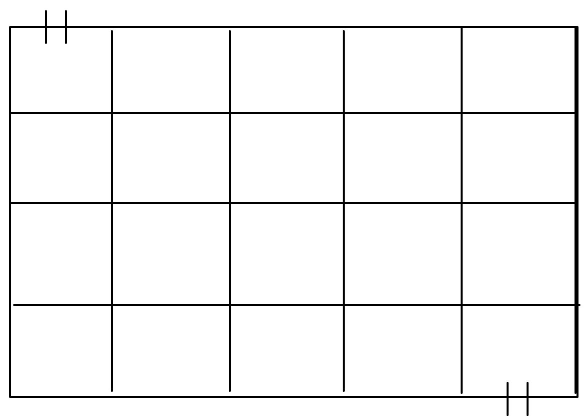
7. You are given a function $f(x)$ which satisfies the equation $f(x+1) = f(x)$ for all real numbers x. You are also given $\int_0^1 f(x)e^{-x}dx = 1$. Find $\int_0^\infty f(x)e^{-x}dx$.

8. A circle of radius r rolls along the inside of a square whose side length is 1 in such a way that the circle is always touching a side of the square. (See the picture.) What should the radius r be so that the area covered by the circle as it rolls all the way around the inside of the square is as large as possible?



9. A large rectangular building contains m*n small square offices laid out in a grid. (See the picture for the case m = 5, n = 4 case.) There are doors between any two offices sharing a wall as well as the two doors indicated in the picture in the lower right and the upper left. You are a supervisor who must visit each office. To save time, you wish to enter the building through the lower right door, visit each office exactly once, then leave the building through the upper left door. Is this always possible?

For full credit on this problem, explain carefully for which values of m and n this will be possible and show how to do it. Also describe the values of m and n for which it is impossible, and explain why you can't do it.



10. Consider the following procedure: Given any positive integer n, if n is odd, add 9 to n; otherwise, divide n by 2. This gives you a new positive integer, to which you can reapply this process. If you continue to iterate this process, you should see one or more long-term patterns emerge. Your mission is to determine what these long term patterns are and, for each initial positive integer n, a way to determine which long-term pattern will occur. Justify all of your answers completely.