LVAIC Mathematics Contest - Oct. 22, 1994

Do as many of these problems as you can. Your solutions must be complete and your work justified to receive full credit.

1. Consider the sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \ldots$ (The sequence of numbers is formed by taking 2^n copies of the fraction $\frac{1}{2^n}$.) Find the sum of the first 1994 terms of this sequence.

2. Let S = {1, 2, …, 1994}. What is the largest number of integers you can select from S so that no three of the numbers selected will add up to a multiple of 3?

3. Let S be a set of m points on the positive x-axis and let T be a set of n points on the positive y-axis in the xy plane, where m and n are positive integers. Suppose all possible line segments with one endpoint in S and the other in T are drawn.
a) How many line segments will be drawn?
b) Assuming no three segments meet in one intersection point, how many crossings will there be among all the segments? (Don't count the endpoints as crossings.)

4. Let ABC be a triangle having |AB| = 1 and |AC| = 2 with the property that among all such triangles, the distance from the point A to the line determined by points B & C is as large as possible. Find |BC|.

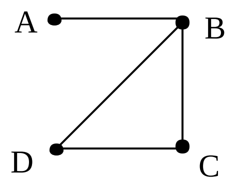
5. Let A be the set of integer lattice points in the plane, i.e., A = {(m,n): m and n are integers}. Let G be the graph of a parabola $y = ax^2 + bx + c$ (for some real numbers a, b and c). For each of the following, give an example of a parabola satisfying the condition:

i. G contains no points of A
ii. G contains exactly one point of A
iii. G contains exactly two points of A
iv. G contains an infinite number of points of A.

Show that these are the only possibilities.

6. Cities A, B, C, and D are connected by bridges as in the diagram. Suppose that each bridge has a probability of 0.1 of falling down during a storm and these failures are independent

of one another. Find the probability that the cities are still connected to each other (so that it is possible to get from any city to any other) after the storm.



7. Let A be a subset of 10 consecutive positive integers with the property that any two integers in A are relatively prime (that is, they have no common factors). What is the largest numbers of integers A could have? Give an example of 10 consecutive integers and that subset A.

8. A sequence is defined as follows: $a_1 = N$ (a positive integer) and a_n = the product of the digits of a_{n-1}. (Assume all numbers are written in the usual base 10 way.) Assume that the product of a single digit is just that digit. Show that, for any initial value N, $\lim_{n \to \infty} a_n$ exists.

9. Five points are located on the x-axis at x = -5, x = 0, x = 1, x = 2 and x = 17. For each of the following, find a point P on the x-axis which minimizes:

a) the sum of the squares of the distances from P to the given points.

b) the sum of the distances from P to the given points.

c) the maximum of the distances from P to the given points.