A KEY POINT: Work (Energy) is done if a displacement occurs at the same location and in the same direction as a force. It does not matter whether or not the force caused the displacement, only that the force and displacement coexist in the same direction and location.

LESSON OBJECTIVES
1. Explain The Principle of Virtual Work.
2. Calculate truss deflections using the principle of virtual work.
3. Solve 1-DOI trusses by employing Virtual Work to calculate the needed deflections.

READING
- Attached examples (online version of this handout), from Hibbeler (we will use the same symbols)
- Attached (online version of this handout), examples from Leet, Uang, and Gilbert
  - Symbols different from Hibbeler, but concepts are identical

TERMS:
- N = Real bar forces in a truss
- n = Virtual bar forces in a truss
- n(NL/AE) = Virtual work in a truss bar

GENERAL PROCEDURE FOR TRUSS DEFLECTIONS BY VIRTUAL WORK:
1. Do a Real Truss Analysis for the real loads to get N’s (real bar forces).
2. Put a virtual force at the location and direction of interest. e.g., if you want to know the vertical deflection at point G, you need to imagine a vertical virtual force at G. Usually, you place a unit virtual force.
3. Do a Virtual Truss Analysis to get n’s. (virtual bar forces)
4. The Sum of nNL/AE = virtual force*(Δ). Solve for Δ. If your virtual force was a unit force, than the sum of internal virtual work is simply Δ. The sign and direction of the deflection is consistent with the direction of your assigned virtual force.

ANNOUNCEMENTS:
EXAM 6: THIS FRIDAY?
HOMEWORK ASSIGNMENT (Due Wednesday. Standard Assignment. Presentation Counts):

1. Determine the horizontal movement of Point A in the truss, due to the applied load.
   Given: all bars have $E=10000 \text{ ksi}$. Diagonal bars have $A=0.2 \text{ in}^2$. All other bars have $A=1.0 \text{ in}^2$

2. Determine the horizontal movement of Point A in the truss, due to the applied load. Hint: you already have the truss analyses completed, from the previous problem.
   Given: all bars have $E=10000 \text{ ksi}$. Diagonal bars have $A=0.2 \text{ in}^2$. All other bars have $A=1.0 \text{ in}^2$

3. Determine the reactions and bar forces for the indeterminate truss. THIS IS NOT REALLY A NEW ANALYSIS.
   You have determined how far Point A would move to the right due to the applied forces (problem 1). You have determined how far Point A would move to the left due to the unit force (problem 2). Use this information to deduce the reaction at A. Similarly, you can use the previous two analyses to deduce the other reactions and bar forces without having to perform a new truss analysis (its all Superposition. If you think hard about Superposition of the previous results, you don’t have to perform a new truss analysis).
   Given: all bars have $E=10000 \text{ ksi}$. Diagonal bars have $A=0.2 \text{ in}^2$. All other bars have $A=1.0 \text{ in}^2$

CONGRATULATIONS. YOU HAVE DONE AN INDETERMINATE TRUSS ANALYSIS. THAT’S HOW ITS DONE. 3 STEPS:

1. REMOVE A REDUNDANT REACTION AND DETERMINE THE MOVEMENT OF THAT POINT.
2. APPLY A UNIT LOAD TO THAT POINT AND DETERMINE THAT MOVEMENT.
3. USE SUPERPOSITION (ESSENTIALLY, THE RATIO OF THE PREVIOUS TWO DEFORMATIONS) TO DEDUCE THE REDUNDANT FORCE. NOW, ITS JUST STATICS.
8.9 Method of Virtual Work: Trusses

We can use the method of virtual work to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.

**External Loading.** For the purpose of explanation let us consider the vertical displacement $\Delta$ of joint $B$ of the truss in Fig. 8–31. Here a typical element of the truss would be one of its members having a length $L$. If the applied loadings $P_1$ and $P_2$ cause a linear elastic material response, then this element deforms an amount $\Delta L = NL/AE$, where $N$ is the normal or axial force in the member, caused by the loads. Applying Eq. 8–20, the virtual-work equation for the truss is therefore

$$1 \cdot \Delta = \sum \frac{nNL}{AE} \quad (8–22)$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of $\Delta$

$n =$ internal virtual normal force in a truss member caused by the external virtual unit load

$\Delta =$ external joint displacement caused by the real loads on the truss

$N =$ internal normal force in a truss member caused by the real loads

$L =$ length of a member

$A =$ cross-sectional area of a member

$E =$ modulus of elasticity of a member

The formulation of this equation follows naturally from the development in Sec. 8.8. Here the external virtual unit load creates internal virtual forces $n$ in each of the truss members. The real loads then cause the truss joint to be displaced $\Delta$ in the same direction as the virtual unit load, and each member is displaced $NL/AE$ in the same direction as its respective $n$ force. Consequently, the external virtual work $1 \cdot \Delta$ equals the internal virtual work or the internal (virtual) strain energy stored in all the truss members, that is, $\sum nNL/AE$. 
Temperature. In some cases, truss members may change their length due to temperature. If $\alpha$ is the coefficient of thermal expansion for a member and $\Delta T$ is the change in its temperature, the change in length of a member is $\Delta L = \alpha \Delta T L$. Hence, we can determine the displacement of a selected truss joint due to this temperature change from Eq. 8–20, written as

$$1 \cdot \Delta = \sum n \alpha \Delta T L$$  \hspace{1cm} (8–23)

where

- $1$ = external virtual unit load acting on the truss joint in the stated direction of $\Delta$
- $n$ = internal virtual normal force in a truss member caused by the external virtual unit load
- $\Delta$ = external joint displacement caused by the temperature change
- $\alpha$ = coefficient of thermal expansion of member
- $\Delta T$ = change in temperature of member
- $L$ = length of member

Fabrication Errors and Camber. Occasionally, errors in fabricating the lengths of the members of a truss may occur. Also, in some cases truss members must be made slightly longer or shorter in order to give the truss a camber. Camber is often built into a bridge truss so that the bottom cord will curve upward by an amount equivalent to the downward deflection of the cord when subjected to the bridge’s full dead weight. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from direct application of Eq. 8–20, written as

$$1 \cdot \Delta = \sum n \Delta L$$  \hspace{1cm} (8–24)

where

- $1$ = external virtual unit load acting on the truss joint in the stated direction of $\Delta$
- $n$ = internal virtual normal force in a truss member caused by the external virtual unit load
- $\Delta$ = external joint displacement caused by the fabrication errors
- $\Delta L$ = difference in length of the member from its intended size as caused by a fabrication error

A combination of the right sides of Eqs. 8–22 through 8–24 will be necessary if both external loads act on the truss and some of the members undergo a thermal change or have been fabricated with the wrong dimensions.
Procedure for Analysis

The following procedure may be used to determine a specific displacement of any joint on a truss using the method of virtual work.

Virtual Forces \( n \)

- Place the unit load on the truss at the joint where the desired displacement is to be determined. The load should be in the same direction as the specified displacement, e.g., horizontal or vertical.
- With the unit load so placed, and all the real loads removed from the truss, use the method of joints or the method of sections and calculate the internal \( n \) force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

Real Forces \( N \)

- Use the method of sections or the method of joints to determine the \( N \) force in each member. These forces are caused only by the real loads acting on the truss. Again, assume tensile forces are positive and compressive forces are negative.

Virtual-Work Equation

- Apply the equation of virtual work, to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding \( n \) and \( N \) forces when substituting these terms into the equation.
- If the resultant sum \( \Sigma n N L / AE \) is positive, the displacement \( \Delta \) is in the same direction as the unit load. If a negative value results, \( \Delta \) is opposite to the unit load.
- When applying \( 1 \cdot \Delta = \Sigma n \alpha \Delta T L \), realize that if any of the members undergoes an increase in temperature, \( \Delta T \) will be positive, whereas a decrease in temperature results in a negative value for \( \Delta T \).
- For \( 1 \cdot \Delta = \Sigma n \Delta L \), when a fabrication error increases the length of a member, \( \Delta L \) is positive, whereas a decrease in length is negative.
- When applying any formula, attention should be paid to the units of each numerical quantity. In particular, the virtual unit load can be assigned any arbitrary unit (lb, kip, N, etc.), since the \( n \) forces will have these same units, and as a result the units for both the virtual unit load and the \( n \) forces will cancel from both sides of the equation.
Determine the vertical displacement of joint C of the steel truss shown in Fig. 8–33a. The cross-sectional area of each member is $A = 0.5$ in$^2$ and $E = 29(10^3)$ ksi.

**Solution**

**Virtual Forces $n$.** Only a vertical 1-k load is placed at joint C, and the force in each member is calculated using the method of joints. The results are shown in Fig. 8–33b. Positive numbers indicate tensile forces and negative numbers indicate compressive forces.

**Real Forces $N$.** The real forces in the members are calculated using the method of joints. The results are shown in Fig. 8–33c.

**Virtual-Work Equation.** Arranging the data in tabular form, we have

<table>
<thead>
<tr>
<th>Member</th>
<th>$n$ (k)</th>
<th>$N$ (k)</th>
<th>$L$ (ft)</th>
<th>$nNL$ (k$^2$·ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>0.333</td>
<td>4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>$BC$</td>
<td>0.667</td>
<td>4</td>
<td>10</td>
<td>26.67</td>
</tr>
<tr>
<td>$CD$</td>
<td>0.667</td>
<td>4</td>
<td>10</td>
<td>26.67</td>
</tr>
<tr>
<td>$DE$</td>
<td>-0.943</td>
<td>-5.66</td>
<td>14.14</td>
<td>75.42</td>
</tr>
<tr>
<td>$FE$</td>
<td>-0.333</td>
<td>-4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>$EB$</td>
<td>-0.471</td>
<td>0</td>
<td>14.14</td>
<td>0</td>
</tr>
<tr>
<td>$BF$</td>
<td>0.333</td>
<td>4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>$AF$</td>
<td>-0.471</td>
<td>-5.66</td>
<td>14.14</td>
<td>37.71</td>
</tr>
<tr>
<td>$CE$</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

$\Sigma 246.47$

Thus

$$1 \text{ k} \cdot \Delta_C = \sum \frac{nNL}{AE} = \frac{246.47}{29(10^3) \text{ k/ft}}$$

Converting the units of member length to inches and substituting the numerical values for $A$ and $E$, we have

$$1 \text{ k} \cdot \Delta_C = \frac{(246.47 \text{ k$^2$·ft})(12 \text{ in./ft})}{(0.5 \text{ in}^2)(29(10^3) \text{ k/ft})}$$

$\Delta_C = 0.204$ in.  

*Ans.*
Example 8–15

Determine the vertical displacement of joint C of the steel truss shown in Fig. 8–34a. Due to radiant heating from the wall, member AD is subjected to an increase in temperature of \(\Delta T = +120^\circ F\). Take \(\alpha = 0.6(10^{-5})/\circ F\) and \(E = 29(10^3)\) ksi. The cross-sectional area of each member is indicated in the figure.

![Diagram of truss with labeled forces and dimensions]

**Fig. 8–34**

**SOLUTION**

**Virtual Forces \(n\).** A vertical 1-k load is applied to the truss at joint C, and the forces in the members are computed, Fig. 8–34b.

**Real Forces \(N\).** Since the \(n\) forces in members AB and BC are zero, the \(N\) forces in these members do not have to be computed. Why? For completion though, the entire real-force analysis is shown in Fig. 8–34c.

**Virtual-Work Equation.** Both loads and temperature affect the deformation; therefore, Eqs. 8–22 and 8–23 are combined. Working in units of kips and inches, we have

\[
1 \cdot \Delta_c = \sum nNL \frac{AE}{\Sigma \alpha \Delta T L} = \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]} + \frac{(-0.75)(120)(10)(12)}{1.5[29(10^3)]} + \frac{(1)(0.6(10^{-5}))(120)(8)(12)}{3[29(10^3)]}
\]

\[
\Delta_c = 0.658 \text{ in.} \quad \text{Ans.}
\]
Under the action of the 30-kip load, joint $B$ of the truss in Figure 10.7a displaces to $B'$ (the deflected shape is shown by the dashed lines). Using virtual work, compute the components of displacement of joint $B$. For all bars, $A = 2\text{ in}^2$ and $E = 30,000\text{ kips/in}^2$.

**Solution**

To compute the horizontal displacement $\delta_x$ of joint $B$, we apply a dummy load of 1 kip horizontally at $B$. Figure 10.7b shows the reactions and bar forces $F_Q$ produced by the dummy load. With the dummy load in place, we apply the real load of 30 kips to joint $B$ (indicated by the dashed arrow). The 30-kip load produces bar forces $F_P$, which deform the truss. Although both the dummy and the real loading now act dependently on the structure, for clarity we show the forces and deformations produced by the real load, $P = 30\text{ kips}$, separately on the sketch in Figure 10.7a.

With the bar forces established, we use Equation 10.24 to compute $\delta_x$:

$$\sum Q\delta_p = \sum F_Q \frac{F_P L}{AE}$$

$$\sum Q\delta_p = \sum F_Q \frac{F_P L}{AE}$$

$$\frac{1\text{ kip}}{(\delta_x)} = \frac{5}{3} \frac{50(20 \times 12)}{2(30,000)} + \left(\frac{-4}{3}\right) \left(-\frac{40}{2}(16 \times 12)\right)$$

$$\delta_x = 0.5\text{ in} \rightarrow \text{ Ans.}$$

To compute the vertical displacement $\delta_y$ of joint $B$, we apply a dummy load of 1 kip vertically at joint $B$ (see Figure 10.7c) and then apply the real load. Since the value of $F_Q$ in bar $AB$ is zero (see Figure 10.7c), no energy is stored in that bar and we only have to evaluate the strain energy stored in bar $BC$. Using Equation 10.24, we compute

$$\sum Q\delta_p = \sum F_Q \frac{F_P L}{AE}$$

$$\frac{1\text{ kip}}{(\delta_y)} = \frac{(-1)(-40)(16 \times 12)}{2(30,000)} = 0.128\text{ in} \rightarrow \text{ Ans.}$$

**Figure 10.7:** (a) Real loads ($P$-system producing bar forces $F_P$). (b) Dummy load ($Q$-system producing $F_Q$ forces) used to compute the horizontal displacement of $B$. The dashed arrow indicates the actual load that creates the forces $F_P$ shown in (a). (c) Dummy load ($Q$-system) used to compute the vertical displacement of $B$. 

The following examples are from a different text (Lee et al. Uang). Hence, they use a different nomenclature ($F_P$ and $F_Q$, rather than $N$ and $n$). The concepts are, of course, the same.
Compute the horizontal displacement $\delta_x$ of joint $B$ of the truss shown in Figure 10.8a. Given: $E = 30,000$ kips/in$^2$, area of bars $AD$ and $BC = 5$ in$^2$; area of all other bars = 4 in$^2$.

**Solution**

The $F_P$ bar forces produced by the $P$ system are shown in Figure 10.8a, and the $F_Q$ bar forces and reactions produced by a dummy load of 1 kip directed horizontally at joint $B$ are shown in Figure 10.8b. Table 10.1 lists the terms required to evaluate the strain energy $U_Q$ given by the right side of Equation 10.24. Since $E$ is constant, it is factored out of the summation and not included in the table.

Substituting $\Sigma F_QF_P L/A = 1025$ into Equation 10.24 and multiplying the right side by 12 to convert feet to inches give

$$\Sigma Q\delta_P = \Sigma F_Q \frac{F_P L}{AE} = \frac{1}{E} \Sigma F_Q \frac{F_P L}{A}$$

(10.24)

$$1 \text{ kip} (\delta_x) = \frac{1}{30,000} (1025)(12)$$

$$\delta_x = 0.41 \text{ in} \rightarrow \text{ Ans.}$$

**TABLE 10.1**

<table>
<thead>
<tr>
<th>Member</th>
<th>$F_Q$ kips</th>
<th>$F_P$ kips</th>
<th>$L$ ft</th>
<th>$A$ in$^2$</th>
<th>$F_QF_P L/A$ kips$^2$-ft/in$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>+1</td>
<td>+80</td>
<td>20</td>
<td>4</td>
<td>+400</td>
</tr>
<tr>
<td>$BC$</td>
<td>0</td>
<td>+100</td>
<td>25</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$CD$</td>
<td>0</td>
<td>-80</td>
<td>20</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$AD$</td>
<td>-\frac{1}{4}</td>
<td>-100</td>
<td>25</td>
<td>5</td>
<td>+625</td>
</tr>
<tr>
<td>$BD$</td>
<td>0</td>
<td>-60</td>
<td>15</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

$\Sigma F_QF_P L/A = 1025$

**Figure 10.8:** (a) $P$ system actually loads; (b) $Q$ system.