SERIES CHEAT SHEET

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Tests

Theorem (n^{th} Term Test for Divergence). *If the sequence $\{a_n\}$ does not converge to zero, then the series $\sum a_n$ diverges.*

***WARNING*: If $\lim_{n\to\infty} a_n = 0$ then the test is inconclusive!**

Theorem (The Integral Test). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Assume that there exists an integer $N \geq 0$ and a function f such that for all $x \geq N$ and for all $n \geq N$*

1. $a_n > 0$,
2. $f(n) = a_n$, *and*
3. f *is*
   - *positive,*
   - *continuous, and*
   - *decreasing*

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges.

Theorem (The Comparison Tests). *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and assume there exists some number N such that*

$$0 < a_n \leq b_n$$

is satisfied whenever $n \geq N$.

(i) *If $\sum a_n$ diverges, then $\sum b_n$ also diverges.*
(ii) *If $\sum b_n$ converges, then $\sum a_n$ also converges.*

Theorem (The Limit Comparison Test). *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and assume there exists some number N such that*

$$0 < a_n, b_n$$

is satisfied whenever $n \geq N$. If there exists some number $c > 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

then either

- $\sum a_n$ *and* $\sum b_n$ *both converge, or*
- $\sum a_n$ *and* $\sum b_n$ *both diverge.*

Theorem (Alternating Series Test). *Let $\{b_n\}$ be a sequence. If there exists some N such that for all $n \leq N$*

(1) $0 < b_n$
(2) $b_{n+1} \leq b_n$
(3) $\lim_{n\to\infty} b_n = 0$

then the Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

converges.

Theorem (Ratio Test). *Let $\{a_n\}$ be a sequence and let*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(ii) If $1 < L \leq \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
(iii) If $L = 1$, then the test is inconclusive.

Theorem (Root Test). *Let $\{a_n\}$ be a sequence and let*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(i) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(ii) If $1 < L \leq \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
(iii) If $L = 1$, then the test is inconclusive.

Known Convergent Series

Geometric Series, $|r| < 1$: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \frac{a}{1-r}$

p-series, $1 < p$: $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

Known Divergent Series

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$

Geometric Series, $1 \leq |r|$: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$

p-series, $p \leq 1$: $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$