Timing Risk: A User Guide

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Abstract

Timing risk refers to a situation in which the *timing* of an economically important event is unknown (risky) from the perspective of an economic decision maker. While this special class of dynamic stochastic control problems has many applications in economics, the methods used to solve them are less well-known and not easily accessible within a unified, comprehensive discussion. We provide a textbook example of timing risk and show how to solve the model under various assumptions about the nature of the timing risk. Our goal is to provide a comprehensive survey of various models and methods as a user guide for researchers.

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1 Introduction

Economic models often involve optimization over an uncertain event. In some settings, the uncertainty is over the magnitude of the event only–for example, what will the income tax rate be in the future, or how large of a bequest might an individual receive. However, in other settings, the key economic uncertainty is regarding the *timing* of the event itself. For example, *when* might tax policy change or *when* might an individual receive a bequest.[1] This second type of uncertainty, regarding the risky timing of an economically important event, is the focus of this paper. We provide a textbook summary of timing risk models and methods that can be applied to various economic questions.

Economists in several fields use models with timing risk to study problems specific to their particular research area, such as investment decisions of firms, or saving/consumption decisions over the life cycle. Despite the importance of timing risk in many applications, typical textbook treatments of dynamic optimization and optimal control do not cover these methods. This is partly due to a recent surge in interest in timing risk applications. The various methods for solving problems with timing risk are scattered across a number of recent papers in the literature—and each paper provides solution techniques that apply to a specific timing risk application that has important nuances that distinguish it from other applications. This paper provides a comprehensive summary of the various methods, with the hope that such a summary would be useful to researchers as an accessible and unified user guide.

[1] The interaction of timing and magnitude uncertainty is also the subject of economic investigation and can be incorporated into the timing uncertainty framework presented in this paper.

The timing risk models that we discuss are best suited to study situations where the risky event only happens once, or not at all.[2] For example, a spouse making a household life-cycle saving consumption decision faces a risk of their spouse dying. The timing of their spouse passing away is uncertain, and will only happen once. Similarly, in the context of disability insurance, an individual faces a risk of becoming (permanently) disabled that either happens once (the individual becomes disabled) or not at all. Timing risk models are not suited to study uncertain events that happen repeatedly, like the risk of becoming unemployed or idiosyncratic wage shocks.

We consider two different dimensions of timing risk: the maximum date at which the uncertainty is resolved (bounded vs unbounded), and the type of shock that occurs (stock or flow). We use the phrase "bounded timing risk" to refer to problems where the timing risk is resolved within the planning horizon of the economic decision maker, and "unbounded timing risk" to refer to problems where the timing risk is (potentially) resolved outside the planning horizon of the decision maker (or the uncertainty is never resolved–i.e., the uncertain event may never occur). An example of bounded timing risk would be the receipt of an accidental bequest from a parent. From the perspective of the child, the bequest will arrive sometime before the end the child's planning horizon (the maximum age of the child). An example of unbounded timing risk would be the risk of Social Security benefits being reduced: the policy reform may occur after the end of an individual's planning horizon (life cycle) or government may *never* enact the reform. Thus, from the perspective of an individual solving a consumption/saving problem, the timing uncertainty is unbounded. Notice that in each case, the risk of the event need not be constant over time and the methods summarized in

[2] This is also true of regime switching models *without* timing risk.

this paper allow for non-stationary timing risk.

Within each type of timing risk (bounded vs. unbounded) we are careful to distinguish between two types of risky events: level shocks which are one-time shocks to the level of a state variable, and flow shocks which are shocks that change the state equation and thus the evolution of the state variable going forward after the date of the shock. An example of a level shock would be a stock market crash, such as the Great Depression, which reduces the value of an individual's assets, or the receipt of a lump-sum bequest, which increases an individual's assets. An example of a flow shock would be a change in payroll taxes which increases or decreases take-home pay for an individual every period after the policy change is enacted. We derive the necessary conditions to solve timing risk problems for all four combinations of bounded vs unbounded risk with stock or flow shocks.

The timing risk problems considered in this paper can be solved analytically up to an unknown constant. After working through the mathematical solution techniques for timing risk problems in sections 2 and 3, we outline a computational process to numerically solve each type of problem in section 4. In section 5 we discuss welfare and we discuss time consistency in section 6. Regime-switching optimal control problems are solved recursively, by breaking the timing-risk problem into two discrete sections: the optimal control problem the decision maker solves after the shock occurs, and the optimal control problem the decision maker solves before the shock occurs (which embeds the solution from the post-shock problem). We show that the solution to timing-risk problem is time-consistent. A decision maker follows the solution to the pre-shock subproblem in a time consistent way until the shock occurs, and then switches to the solution of the post-shock subproblem as soon as the shock occurs.

Finally we consider the information set of the decision maker. Our baseline analysis throughout the paper assumes that the decision maker has full information about the timing uncertainty they face. They know the full distribution of possible shock dates, but they do not have any additional advanced information about when the shock will occur. We consider two alternative information assumptions in section 7. First, we consider the case of early information, where the decision maker learns when the shock will occur prior to the shock occurring. For example, changes to monetary policy might be announced in advance (forward guidance), or fiscal policy changes might be implemented with a lag that gives decision makers some time to adjust before the policy takes effect. Second, we consider the opposite case where the decision maker has less than full information about the timing risk they face. We consider the assumption of ambiguity where the decision maker knows they face a risk, but they do not have any information about the distribution of that risk. For example, an individual in an aging country may face a risk that their public pension benefit is reduced, but they might not have any (reliable) information about the possible dates of the policy change, since it is difficult to forecast the legislative process.

The timing risk models considered in this paper are an application of two-stage regime switching optimal control theory where the switch date is stochastic rather than being known ex-ante or a choice variable of the decision maker. Examples of this type of model can be found in studies on resource extraction (Dasgupta and Heal (1974)), operations research (Kamien and Schwartz (1971)), and environmental catastrophe (Clarke and Reed (1994)). There are several timing uncertainty examples in life-cycle consumption and saving, including the death of a spouse (Cottle Hunt and Caliendo (2021a)), receiving a bequest (Cottle Hunt and Caliendo (2021c)), retiring stochastically from the labor force (Caliendo et al. (2022)), or rare event risks like the Great Depression (Cottle Hunt and Caliendo (2021b)). Timing uncertainty models have also been used to study policy uncertainty with applications in fiscal policy (Bi et al. (2013), Davig et al. (2010) and Davig and Foerster (2014)), Social Security reform (Caliendo et al. (2019), Kitao (2018), Cottle Hunt (2020), and Nelson (2017)), Medicare reform (Michelangeli and Santoro (2013)), and tax policy as it impacts firms' investment decisions (Stokey (2016)).[3] It is our hope that the tools presented in this paper will be useful to researchers as they seek to answer additional questions with timing uncertainty in the future.

2 Review: Standard Problem without Timing Risk

Time is continuous and is indexed by t. From the perspective of the decision maker (DM), the planning interval starts at $t = 0$ and ends at $t = T$. At each moment in time the DM chooses a control $c(t)$ that generates instantaneous payoff $u(t, c(t))$. Note that our writing of the payoff function is general enough to accommodate time discounting and survival risk. The DM is endowed with a state variable $k(0)$ and is constrained by a terminal value $k(T) = 0$ (without loss of generality, assume $k(0) = k(T) = 0$, although any other assumptions will

[3] Regime switching models with structural uncertainty (i.e., uncertainty regarding the characteristics of the new regime, but not the timing of the switch) have been used in the resource extraction literature (Hoel (1978)) and later in the technology adoption literature (Hugonnier and Pommeret (2006), Pommeret and Schubert (2009), Abel and Eberly (2012)). These models, as well as timing uncertainty models, are extensions of two-stage regime switching models such as those found in Kamien and Schwartz (1971), Kemp and Long (1977), Tomiyama (1985), and Makris (2001). Additional applications of regime switching models include irreversible pollution accumulation (Tahvonen and Withagen (1996)), the optimal extraction of natural resources (Hoel (1978), Amit (1986), Boucekkine et al. (2013b), and Boucekkine et al. (2013c)), economic growth (Dogan et al. (2011)), capital controls (Boucekkine et al. (2013a)), and technology adoption (Boucekkine et al. (2004), Saglam (2011), and Abel and Eberly (2012)).

work), and by a state equation

$$\dot{k}(t) = g(t, c(t), k(t)).$$

In a standard Pontryagin problem, the function g would either be invariant to t or would vary with t in a manner that is either known ex ante or chosen by the DM. In this case, subject to these constraints, the DM would maximize

$$U = \int_0^T u(t, c(t))dt.$$

3 Timing Risk

We now consider problems where there is a shock to the state system occurring at an unknown time t. The shock date is a continuous random variable with PDF $\phi(t)$ and CDF $\Phi(t)$.

The nature of timing risk can differ across applications and we seek to carefully distinguish between different settings that imply subtle changes to the solution techniques. For example, while the timing of the shock is unknown in advance, in some applications the shock is sure to occur *before* date T. We will call this case *bounded timing risk*. Let the latest possible shock date be $t' < T$. Hence, $\Phi(t') = 1$.

In other applications, the shock may occur at the end (or beyond the end) of the planning period. We will call this case *unbounded timing risk*. Assume the shock can occur anytime on the interval $[0, \infty]$ and $\int_0^\infty \phi(t) = 1$. This includes the case in which the risk is resolved by the terminal planning date T (i.e., $\phi(t) = 0$ for all t beyond T) and for cases in which the shock need not strike before T (i.e., $\phi(t) > 0$ for some t beyond T).

Moreover, within each of these two cases, the shock to the state system could take the form of a permanent *flow shock* to the functional form of the state equation, or it could take the form of a one-time *level shock* to the quantity of the state variable. In the case of the flow shock, the function form of the state equation switches from $g(.)$ to $g_2(.)$ at the time of the shock. In the case of a level shock, the quantity of the state variable jumps (up or down) discretely by an amount Δ, while the functional form of the state equation g stays the same before and after the shock. Importantly, in either case, the timing of the shock is unknown ex ante and that is the key feature of the problems at hand.

Timing risk is a stationary process in some applications; that is, the risk of the shock is constant (exponentially distributed). But this would not be the case in general, and a number of examples involve non-stationary timing risk. Therefore, all of the methods summarized in this paper are general enough to handle stationary and non-stationary timing risk.

Beyond the assumptions that are made about the nature of the timing risk itself, the researcher must make assumptions about (1) how much information the DM has about the timing risk, and (2) how the DM behaves in the face of this information. Unless we say otherwise, we assume the DM has full information about the distribution of the timing risk, and that they behave optimally in the face of this risk.

We seek to provide a simple guide for solving timing risk problems recursively. We break the problem into a deterministic problem from the perspective of the time of the shock and then work backwards to the dynamic stochastic (time 0) problem.

3.1 Problem A: Bounded Timing Risk

Step 1. Post-shock subproblem:

Given shock date $t \leq t' < T$ and state variable $k(t)$, the optimal control path $c(z)$ for $z \in [t, T]$, after the shock has occurred, is the solution to

$$\max_{c(z)_{z \in [t,T]}} : U_2 = \int_t^T u(z, c(z))dz.$$

For the case of a permanent *flow shock* to the state equation, the constraints for the post-shock problem are

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given, } k(T) = 0, \, t \text{ given.}$$

Alternatively, for the case of the one-time *level shock* to the state variable, the constraints are

$$\dot{K}(z) = g(z, c(z), K(z))$$

$$K(t) = k(t) + \Delta$$

$$K(T) = 0$$

$$k(t) \text{ given, } t \text{ given.}$$

Here $K(t)$ is the state variable post-shock and is notationally different from the state variable pre-shock to accommodate the level shock itself.

In either case, we denote the solution to this deterministic sub-problem as $c_2^*(z, t, k(t))_{z\in[t,T]}$. And solution continuation utility is a function of the timing of the shock t and the state variable at the time of the shock $k(t)$

$$U_2^*(t, k(t)) = \int_t^T u(z, c_2^*(z, t, k(t)))dz.$$

This will be nested into the next step of the procedure.

Step 2. Pre-shock t=0 subproblem:

Facing random variable t, at time 0 the DM faces a dynamic stochastic control problem and maximizes expected utility

$$\max_{c(t)_{t\in[0,t']}} : \int_0^{t'} \left([1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t))\right) dt$$

subject to

$$\dot{k}(t) = g(t, c(t), k(t))$$

$$k(0) = 0,\ k(t') \text{ free.}$$

Note that this is a *free-endpoint* optimal control problem. The uncertainty is sure to resolve itself before the end of the planning horizon, so while the ultimate terminal value of the state variable is fixed, the state variable at the last possible shock date $t' < T$ is free to be chosen optimally and hence the necessary conditions will involve a Transversality Condition. Form the Hamiltonian

$$\mathcal{H} = [1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)) + \lambda(t)g(t, c(t), k(t)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0,$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial k(t)},$$

$$\lambda(t') = 0.$$

The solution to this subproblem is $(c_1^*(t), k_1^*(t))_{t\in[0,t']}$. The individual follows this contingent timepath as long as the shock has not yet hit. Once the shock strikes at time t, the individual immediately jumps onto the path $c_2^*(z, t, k_1^*(t))_{z\in[t,T]}$ for the remainder of the life cycle. Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a pre-shock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable at each moment.

3.2 An Important Clarification

Whether we are dealing with stock or level shock, a technical difficulty arises when solving the pre-shock subproblem in the setting with bounded timing risk. Note that the Maximum Condition is

$$\frac{\partial \mathcal{H}}{\partial c(t)} = [1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t)) + \lambda(t)\frac{\partial}{\partial c(t)}g(t, c(t), k(t)) = 0.$$

Evaluate this condition at $t = t'$

$$[1 - \Phi(t')] \frac{\partial}{\partial c(t)} u(t', c(t')) + \lambda(t') \frac{\partial}{\partial c(t)} g(t', c(t'), k(t')) = 0.$$

Note that $\Phi(t') = 1$ by definition, so the condition simplifies to

$$\lambda(t') \frac{\partial}{\partial c(t)} g(t', c(t'), k(t')) = 0.$$

Now, note that imposing the Transversality Condition $\lambda(t') = 0$ causes the above condition to be satisfied automatically for any choice of the terminal pair $(c(t'), k(t'))$. Hence, the Transversality Condition does not appear to provide the needed endpoint condition and instead the Maximum Condition and Multiplier Equation generate a family of solution paths rather than a unique solution. However, there is a simple answer to this apparent complication: use the *limiting case* of the first-order conditions as follows.

Let us rewrite the Maximum Condition as

$$\frac{\frac{\partial}{\partial c(t)} u(t, c(t))}{\frac{\partial}{\partial c(t)} g(t, c(t), k(t))} = -\frac{\lambda(t)}{1 - \Phi(t)}.$$

Noting the indeterminate form

$$\frac{\lambda(t')}{1 - \Phi(t')} = \frac{0}{0},$$

we apply L'Hôpital's Rule and then impose the Transversality Condition

$$\begin{aligned}
\frac{\frac{\partial}{\partial c(t)}u(t',c(t'))}{\frac{\partial}{\partial c(t)}g(t',c(t'),k(t'))} &= -\lim_{t\to t'}\frac{\lambda(t)}{1-\Phi(t)} \\
&= \lim_{t\to t'}\frac{\dot{\lambda}(t)}{\dot{\Phi}(t)} \\
&= \lim_{t\to t'}\frac{\dot{\lambda}(t)}{\phi(t)} \\
&= \lim_{t\to t'}\frac{-\frac{\partial\mathcal{H}}{\partial k(t)}}{\phi(t)} \\
&= \lim_{t\to t'}\frac{-\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t,k(t))-\lambda(t)\frac{\partial}{\partial k(t)}g(t,c(t),k(t))}{\phi(t)} \\
&= -\frac{\partial}{\partial k(t)}U_2^*(t',k(t')).
\end{aligned}$$

Hence, written in final form we have

$$\frac{\frac{\partial}{\partial c(t)}u(t',c(t'))}{\frac{\partial}{\partial c(t)}g(t',c(t'),k(t'))} = -\frac{\partial}{\partial k(t)}U_2^*(t',k(t')).$$

This condition provides the needed endpoint restriction to derive a unique solution from the first-order equations. That is, in addition to the Maximum Condition and Multiplier Equation, the solution path (c,k) must have terminal values $(c(t'),k(t'))$ that satisfy this condition. In some of our research we have referred to this condition as the "Stochastic Continuity Condition" or the "Limiting Case" of the Transversality Condition.

3.3 Problem B: Unbounded Timing Risk

Step 1. Post-shock subproblem:

If the shock strikes at $t<T$ with state variable $k(t)$, the optimal control path $c(z)$ for

$z \in [t, T]$, after the shock has occurred, is the solution to

$$\max_{c(z)_{z\in[t,T]}} : U_2 = \int_t^T u(z, c(z))dz.$$

For the case of a permanent *flow-shock* to the state equation, the constraints for the post-shock problem are

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given, } k(T) = 0, \ t \text{ given.}$$

Alternatively, for the case of the one-time *level shock* to the state variable, the constraints are

$$\dot{K}(z) = g(z, c(z), K(z))$$

$$K(t) = k(t) + \Delta$$

$$K(T) = 0$$

$$k(t) \text{ given, } t \text{ given.}$$

Here $K(t)$ is the state variable post-shock and is notationally different from the state variable pre-shock to accommodate the level shock itself.

In either case, we denote the solution to this sub-problem as $c_2^*(z, t, k(t))_{z\in[t,T]}$. And solution continuation utility is a function of the timing of the shock t and the state variable at the time of the shock $k(t)$

$$U_2^*(t, k(t)) = \int_t^T u(z, c_2^*(z, t, k(t)))dz.$$

This will be nested into the next step of the procedure.

Step 2. Pre-shock t=0 subproblem:

Facing random variable t, at time 0 the DM maximizes expected utility

$$\max_{c(t)_{t\in[0,T]}} : \int_0^T \left([1-\Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t))\right) dt$$

subject to

$$\dot{k}(t) = g(t, c(t), k(t))$$

$$k(0) = 0,\ k(T) = 0.$$

Note that this is a *fixed-endpoint* optimal control problem. Because the distribution of the timing risk extends at least to and potentially beyond the end of the planning horizon, the individual must make an optimal pre-shock plan that extends all the way to the end of the planning horizon. Hence, the constraint on the terminal value of the state variable provides the needed endpoint condition (rather than a Transversality Condition). Form the Hamiltonian

$$\mathcal{H} = [1-\Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)) + \lambda(t)g(t, c(t), k(t)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0,$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial k(t)}.$$

The solution to this subproblem is $(c_1^*(t), k_1^*(t))_{t \in [0,T]}$. The individual follows this contingent timepath as long as the shock has not yet hit. Once the shock strikes, the individual immediately jumps onto the path $c_2^*(z, t, k_1^*(t))_{z \in [t,T]}$ for the remainder of the life cycle. Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a pre-shock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable at each moment.

4 Computational Method for the Pre-Shock Subproblem

For Problems A and B, and for both cases (flow shock and level shock), the first-order conditions end up being complicated enough (even for the simplest possible applications of timing risk) that we cannot solve for fully closed-form solutions to the pre-shock subproblem $(c_1^*(t), k_1^*(t))$. However, for typical problems in economics with $\partial g/\partial c = -1$ (as in the case of life-cycle consumption saving problems where an extra unit of consumption reduces the amount saved one for one), we can in fact obtain a closed-form Euler equation $\dot{c}(t)$, and from there a simple computational method can be used to anchor to Euler equation to an initial condition.

To obtain the Euler equation, recall the Hamiltonian

$$\mathcal{H} = [1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)) + \lambda(t)g(t, c(t), k(t)),$$

with Maximum Condition and Multiplier Equation

$$\frac{\partial \mathcal{H}}{\partial c(t)} = [1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t)) - \lambda(t) = 0$$

$$\dot{\lambda}(t) = -\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t, k(t)) - \lambda(t)\frac{\partial}{\partial k(t)}g(t, c(t), k(t)).$$

Differentiating the Maximum Condition with respect to t

$$\frac{d}{dt}\left([1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t))\right) - \dot{\lambda}(t) = 0,$$

and combining with the Multiplier Equation gives

$$\frac{d}{dt}\left([1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t))\right) + \phi(t)\frac{\partial}{\partial k(t)}U_2^*(t, k(t)) + \lambda(t)\frac{\partial}{\partial k(t)}g(t, c(t), k(t)) = 0.$$

Finally, inserting $\lambda(t) = [1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t))$ from the Maximum Condition gives the final form of the Euler Equation

$$\frac{d}{dt}\left([1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t))\right) = -\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t, k(t)) - [1 - \Phi(t)]\frac{\partial}{\partial c(t)}u(t, c(t))\frac{\partial}{\partial k(t)}g(t, c(t), k(t)).$$

This Euler Equation governs the dynamics of the optimal $c(t)$ path. All that remains is to identify the unknown initial value $c(0)$, which can be done with a simple computation technique:

Step 1. Guess $c(0)$.

Step 2. Using a system of equations (Euler Equation, the law of motion $\dot{k}(t)$, and the initial condition $k(0)$) to simulate the $c(t)$ and $k(t)$ paths.

Step 3.

- For the case of *bounded timing risk* (Problem A), use the system to simulate $c(t)$ and $k(t)$ up to the timing boundary $t = t'$ and check to see if the "Stochastic Continuity Condition" holds.

- For the case of *unbounded timing risk* (Problem B), use the system to simulate $c(t)$ and $k(t)$ up to the end of the planning horizon $t = T$ and check to see if the terminal constraint on the state variable holds.

Step 4. If the check in Step 3 is met, then stop. If not, go back to Step 1 and repeat.

5 Welfare

In some applications, the researcher may wish to measure the welfare effect of a government policy. In other applications, the researcher may wish to measure the welfare effect of the timing risk itself. In either case, ex ante expected utility will need to be calculated.

For the case of bounded timing risk (Problem A),

$$\mathbb{E}(U) = \int_0^{t'} \phi(t) \left(\int_0^t u(v, c_1^*(v))dv + \int_t^T u(v, c_2^*(v, t, k_1^*(t)))dv \right) dt.$$

Ex ante, the DM does not know when the shock will hit, so expected utility is a probability weighted average of the utility associated with a continuum of shock dates.

For the case of unbounded timing risk (Problem B),

$$\begin{aligned} \mathbb{E}(U) \quad &= \quad \int_0^T \phi(t) \left(\int_0^t u(v, c_1^*(v))dv + \int_t^T u(v, c_2^*(v, t, k_1^*(t)))dv \right) dt \\ &\quad + \left(\int_T^\infty \phi(t)dt \right) \times \left(\int_0^T u(t, c_1^*(t))dt \right). \end{aligned}$$

Here notice that we must account for the fact that the shock may never strike, in which case the individual will be on the pre-shock path c_1^* for the entirety of the planning interval.

6 A Remark on Time Consistency

It is worth noting that the timing risk component of the problem is inherently time consistent by the laws of probability. For example, suppose we are standing at time t_0 and the shock has not yet hit. Further suppose the individual has been following the optimal pre-shock path $(c_1^*(t), k_1^*(t))_{t \in [0, t_0]}$. While the results below hold for both Problems A and B, to ease notation we focus on Problem A.

From the perspective of time t_0, the timing PDF and CDF, conditional on the shock not yet occurring by date t_0 are

$$\phi_0(t)_{t\in[t_0,t']} = \frac{\phi(t)}{\int_{t_0}^{t'} \phi(t)dt} = \frac{\phi(t)}{1-\Phi(t_0)}$$

$$\Phi_0(t)_{t\in[t_0,t']} = \int_{t_0}^{t} \frac{\phi(v)}{1-\Phi(t_0)} dv = \frac{\Phi(t)-\Phi(t_0)}{1-\Phi(t_0)}.$$

Furthermore,

$$1-\Phi_0(t) = \frac{1-\Phi(t)}{1-\Phi(t_0)}.$$

Given these conditional distributions, if the shock has not yet hit, then the DM standing at time t_0 would seek to solve

$$\max_{c(t)_{t\in[t_0,t']}} : \int_{t_0}^{t'} \left(\frac{1-\Phi(t)}{1-\Phi(t_0)} u(t,c(t)) + \frac{\phi(t)}{1-\Phi(t_0)} U_2^*(t,k(t)) \right) dt.$$

Note the proportionality between this objective functional and the time 0 functional

$$\max_{c(t)_{t\in[0,t']}} : \int_{0}^{t'} \left([1-\Phi(t)]u(t,c(t)) + \phi(t)U_2^*(t,k(t)) \right) dt,$$

which in turn implies that the solution to the time t_0 problem will coincide with the solution to the time 0 problem (i.e., the solution is time inconsistent).

7 Alternative Assumptions about Information

In our baseline model, the DM has full information about the timing of the risk and does not know when the shock will strike until it happens. Here we explore other assumptions about the role of information. Specifically, we consider two alternative assumptions about the information available to the DM: early information and no information.

7.1 Early Information

In our baseline scenario, the DM has no early warning about the timing of the shock. In other words, while the DM has full information about the distribution of the timing risk, the exact timing of the shock cannot be foreseen in advance. Instead, the DM employs an optimal hedging strategy as the solution to the dynamic stochastic problem as described above.

In this section we consider another possible assumption. Imagine a scenario where a policy change from one state to another is eminent but the date is uncertain. It is common knowledge that an announcement will be made at time t^*, and the announcement will unveil the future date $t \geq t^*$ at which time the new policy will take effect. That is, the timing of the shock t is revealed early at $t^* \leq t$. Notice the DM still faces ex ante timing risk, but in this case the timing risk is resolved at the information revelation date t^*. For instance, in 6 weeks the Fed will announce the length of time for which it's current policy position will endure before switching to a new policy (from say expansion to tightening, etc).

From the perspective of time 0, the DM knows that at time $t^* > 0$ they will learn the timing of the shock. Let's say the shock is distributed over the support $[t^*, T]$, which corresponds to our unbounded timing risk examples above, although other assumptions are possible too. Let's also model a flow shock to lighten notation. As with the other cases, we solve this problem recursively, though this is a simpler problem because all of the timing risk gets resolved at a known, single point.

Step 1. Information Revelation Date:

At the information revelation date t^* the DM learns the timing of the shock $t \geq t^*$, and given state variable $k(t^*)$, the optimal control path $c(z)$ for $z \in [t^*, T]$, after the information has been revealed, is the solution to

$$\max_{c(z)_{z\in[t^*,T]}} : U_2 = \int_{t^*}^{T} u(z, c(z))dz,$$

subject to

$$\dot{k}(z) = g(z, c(z), k(z)), \text{ for } z \in [t^*, t],$$

$$\dot{k}(z) = g_2(z, c(z), k(z)), \text{ for } z \in [t, T],$$

$$k(t^*) \text{ given}, \ k(T) = 0, \ t^* \text{ given}, \ t \text{ given}.$$

This is a standard two-stage control problem without timing risk and conventional methods can be applied to find the solution $c_2^*(z, t, t^*, k(t^*))_{z\in[t^*,T]}$. The DM follows this timepath for all points in time z from the information revelation date t^* through the end of the planning horizon T, and note that this path depends on the announced shock date t as well as the level of the state variable at the announcement date $k(t^*)$.

Solution utility as of date t^* is a function of the shock date t and the level of state variable at the announcement date $k(t^*)$

$$U_2^*(t, t^*, k(t^*)) = \int_{t^*}^{T} u(z, c_2^*(z, t, t^*, k(t^*)))dz.$$

Step 2. Pre-shock t=0 subproblem:

Facing random variable t, at time 0 the DM maximizes expected utility

$$\max_{c(v)_{v\in[0,t^*]}} : \int_0^{t^*} u(v, c(v))dv + \int_{t^*}^{T} \phi(t) U_2^*(t, t^*, k(t^*))dt,$$

subject to

$$\dot{k}(v) = g(v, c(v), k(v)), \text{ for } v \in [0, t^*],$$

$$k(0) = 0, \ k(t^*) \text{ free.}$$

Note that this is a *free-endpoint* optimal control problem because the DM is free to choose the level of the state variable at the information revelation date. In selecting this amount, the DM is conscious of all of the potential shock dates and accordingly forms a probability weighted continuation value of the state variable.

To solve this problem, form the Hamiltonian

$$\mathcal{H} = u(v, c(v)) + \lambda(v) g(v, c(v), k(v)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(v)} = 0,$$

$$\dot{\lambda}(v) = -\frac{\partial \mathcal{H}}{\partial k(v)},$$

and the Transversality Condition

$$\lambda(t^*) = \frac{\partial}{\partial k(t^*)} \int_{t^*}^{T} \phi(t) U_2^*(t, t^*, k(t^*)) dt.$$

The solution to this subproblem is $(c_1^*(v), k_1^*(v))_{v \in [0,t^*]}$. The DM follows this contingent timepath up to the information revelation date t^*, at which time they learn the shock date $t \geq t^*$ and immediately jumps onto the path c_2^* for the remainder of the life cycle (the DM does not wait for the shock to actually materialize to jump onto the c_2^* path). Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a pre-shock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable.

7.2 No Information (Ambiguity)

At the other extreme, another possibility is that the DM has *no information* about the distribution of timing risk. Suppose the DM knows there will be a regime switch at some point, but the probabilities are unknown. For instance, suppose fiscal policy reform is necessary to balance the government's budget, but it is not possible to know the probability that reform happens at a given date.

To lighten notation, let's focus on the case of the flow shock (switch in the law of motion of the state variable from g to g_2). Let's further suppose for the moment that the switch to g_2 is a bad thing from the DM's perspective, and therefore the later the shock occurs the better. Then in a Maximin fashion, the DM standing at t would rationally plan for the shock to happen *immediately* (which is the worst case scenario) and optimize accordingly.

Standing at time $t < T$ with state variable $k(t)$, the DM optimizes under a worst case scenario of the shock happening at that moment and therefore g_2 rather than g governing the law of motion for the state variable for the remainder of the planning interval. Therefore the optimal control path $c(z)$ for $z \in [t, T]$ is the solution to

$$\max_{c(z)_{z\in[t,T]}} : U_2 = \int_t^T u(z, c(z))dz,$$

subject to

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given, } k(T) = 0, \ t \text{ given.}$$

Denote the solution to this sub-problem as $c_2^*(z, t, k(t))_{z\in[t,T]}$.

But unlike the baseline problems studied above, this problem is *time inconsistent*. If the shock does not hit immediately as planned at date t, then the DM will need to reoptimize at the next moment in time because the state variable will have evolved according to g rather than g_2 as planned. The DM will follow the control plan only at the moment the plan is made t and will need to reoptimize in the next moment. Hence, the DM's actual control choice (for as long as the shock has not yet hit) at time t is $c_2^*(t, t, k(t))$ where $k(t)$ follows the pre-shock law of motion and boundary constraints

$$\dot{k}(t) = g(t, c_2^*(t, t, k(t)), k(t))$$

$$k(0) = 0,\, k(T) = 0.$$

Note that the DM's actual control path is the *envelope* of initial values of the many planned control paths. Each planned path, including the initial value of that planned path, is a utility-maximizing path based on the worst-case scenario of an immediate regime shift from g to g_2. Yet the state variable continues to evolve according to the pre-shock regime g as long as the shock has not yet hit. Eventually, at some point the shock does hit, and the decision maker is finally correct at that moment and stays on the post-shock path from there forward rather than reoptimizing. In sum, Maximin behavior under timing ambiguity involves solving a sequence of time-inconsistent, deterministic control problems, as opposed to our baseline case with risk only (and no ambiguity about that risk) where the DM solves an ex-ante control problem that is time consistent.[4]

8 Conclusion

This paper is a user guide to modeling timing uncertainty—the date at which an economically significant event takes place is unknown from the perspective of the decision maker. We outline the steps to solve a model with timing risk recursively. The first step is to solve the post-shock, deterministic subproblem. The second step is to embed the solution to the post-shock subproblem into the dynamic stochastic (time 0) problem. We provide details

[4]Other examples are possible too. Suppose the switch from g to g_2 is a good thing in the eyes of the DM. If the DM has no information about the distribution of the timing risk, then a Maximin strategy would be to assume the switch to g_2 never comes. Now the optimization problem is time consistent during the pre-shock period because the DM's predictions are correct so far. Then, when the shock eventually hits, the DM will need to reoptimize to factor in the good news. Hence, an initial, deterministic control problem based on the worst case is followed until good news hits, and then a new, deterministic control problem governs choices from that point forward.

on how to solve models where the shock occurs within the planning horizon of the decision maker (bounded risk) and models where the shock potentially occurs outside the planning horizon of the decision maker (unbounded risk). We distinguish between flow shocks and stock shocks which either change the state equation or the level of the state variable. We also provide guidance on how to solve the model computationally, how to conduct welfare analysis, and how changing the information assumptions alters the problem. Our hope is that this user guide will be helpful for researchers who want to write down an economic model with timing risk. The tools summarized in this paper can be applied to many economic disciplines including economic growth, natural resource extraction, firm investment decisions and valuations, life-cycle consumption and saving, fiscal policy analysis, monetary policy analysis, and political economics.

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