Finite Horizon Life-cycle Learning

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Abstract

I develop a new theory of bounded rationality, which I call Finite Horizon Life-cycle Learning. This approach embeds finite horizon learning in a medium-scale overlapping generations model with government debt. Agents in the model have a finite forecasting or planning horizon that is (potentially) less than the length of their life cycle. Agents use adaptive expectations to forecast endogenous variables over their finite planning horizon, and make decisions conditional on their forecasts. I find that the determinate, high capital steady state of the model is Expectationally-stable for all planning horizons, while the low capital steady state is not Expectationally-stable. The transition dynamics of the model differ depending on the planning horizon. I compare the finite horizon life-cycle learning model to the benchmark rational expectation model, and provide examples of transitions based on changes to social security policy or changes due to a recession under both modeling assumptions.

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1 Introduction

Standard life-cycle models model households as forward-looking optimizers who rely on either perfect foresight (in the absence of shocks) or rational expectations (in an expected utility framework) to solve their inter-temporal problem. The rational expectations assumption also underlies most macroeconomic models. Although common, the assumption of rational expectations is restrictive and requires agents in models to possess detailed knowledge about the structure of the model economy. Rational expectations also requires households to look very far into the future to plan their saving and consumption choices over the *entire* life cycle. This paper relaxes the assumption of rational expectations by assuming that (i) households make forecasts using adaptive learning, and (ii) households only make forecasts over a finite planning horizon. This paper combines the main elements of the adaptive learning literature (for example, Sargent (1993), and Evans and Honkapohja (2001)), and the short planning horizon literature (for example, Caliendo and Aadland (2007)). The majority of the adaptive learning literature has focused on macroeconomic questions, such as the effectiveness of monetary policy. Less attention has focused on life-cycle questions regarding aging, retiring, and social security. This paper develops a framework to explore the response of households to changing demographics, social security policy changes, and recessions.

This paper answers the question: how do households respond to announced or surprise policy changes over their life cycle if they have a finite planning horizon? A growing literature assumes that households do not look forward over their entire life cycle when making saving and consumption choices. Park and Feigenbaum (2018) show that a short planning horizon model is better able to fit U.S. consumption data than a traditional life-cycle model. A short planning horizon delivers a hump-shaped consumption profile as well as a drop in consumption at retirement (see also Caliendo and Aadland (2007)). Findley and Caliendo (2014) show that a short planning horizon drives the consumption choices of households in a model with both hyperbolic discounting and short planning horizons. Woodford (2018) argues that a traditional assumption of infinite horizon optimization under rational expectations implicitly assumes "unrealistic cognitive abilities of economic decision makers" and proposes a boundedly rational model in which households only optimize over a finite horizon.

This paper develops a new theory of bounded rationality, which I call Finite Horizon Life-cycle Learning, that is conceptually in-line with observed behavior, and explores the implications of an aging society and social security policy changes in that framework. Survey data suggest that a majority of U.S. households age fifty and over feel a planning horizon of the "next few years" or less is the "time period that is most important" for planning consumption and spending.[1] The main contribution of this

[1]This question is from the Health and Retirement Study; Fifty-seven percent of respondents report

paper is to develop a model of household decision making that is consistent with using a short planning horizon and also using adaptive expectations. This paper embeds the main ideas of Finite Horizon Learning, developed by Branch, Evans, and McGough (2013), into a life-cycle model with finitely lived agents. The model developed in this paper differs from existing short-horizon papers by using adaptive learning rather than an alternative behavioral primitive. Adaptive learning is the main alternative to rational expectations in macroeconomic models, and thus the main assumption of this paper.[2]

The first result of the paper is to show that the underlying rational expectations equilibrium is stable under finite horizon learning. The steady state equilibrium in an overlapping generations model with capital and debt is not unique. In a two-period model, it can be shown analytically that there are at most two steady states and at fewest zero steady states.[3] In multi-period models, the same underlying relationship exists, but can only be demonstrated numerically. I examine the equilibria of the multi-period overlapping generations model with capital and debt and show that the high capital steady-state is stable under finite horizon life-cycle learning learning (or Exceptionally-stable) for all finite learning horizons. I also show that the low-capital steady state is not stable under learning for any finite horizon.[4]

The second contribution of this paper is to illustrate the trade-offs a household would face in they extended their planning horizon. Specifically, a longer planning horizon would increase the forecast errors the household makes (because they are forecasting further into the future), but it also allows the household to incorporate announced policy changes into their decisions sooner. In contrast, reducing the planning horizon reduces forecast errors (and thus brings household consumption closer to the optimal rational expectations baseline), but also limits the household's ability to respond to announced policy that is beyond the end of the planning horizon. This trade-off is explored in the context of a simple announced policy, a more realistic social security reform in response to an aging population, and finally in relation to surprise recessions.

This paper also explores the oscillatory dynamics introduced by embedding adaptive learning into a life-cycle model. The oscillatory dynamics propagate shocks

that "the next few months", "next year", or "next few years" is the "time period that is most important" for "planning your family's saving and spending." The remaining 43% of respondents choose "the next five to ten years" or "more than ten years." Data available at the HRS website (http://hrsonline.isr.umich.edu/data/index.html).

[2]One appeal of adaptive learning models is that adaptive learning matches survey experimental data on expectations better than rational expectations. See Branch (2004), Adam (2007), and Pfajfar and Santoro (2010).

[3]See Azariadis (1993) chapters 8 and 20 for a discussion of planar overlapping generations models.

[4]See Evans and Honkapohja (2001) and (2009 for a discussion of stability under learning or Expectational-stability.

through the economy creating more realistic business cycle fluctuations.[5] These oscillatory dynamics depend on both the planning horizon length, as well as the gain parameter used in the adaptive learning algorithm. I also compare the dynamics of the learning model to the standard rational expectations model.

1.1 Finite-Horizons and Adaptive Learning

In contrast to the model developed in this paper, rational expectations macroeconomic models often use an infinitely-lived representative agent to describe the household sector of the economy. This assumption is also common in adaptive learning models.[6] Adaptive learning can be embedded into the household decision process in infinite horizon models via "Euler-equation learning" (see as one example, Evans and Ramey (2006)), or "Infinite Horizon Learning" (see as one example, Preston (2005)). Under the assumption of Euler-equation learning, the household is assumed to make a decision based on its (one-step ahead) Euler-equation where the expectation of future unknown variables (for example consumption) is formed adaptively. Under the assumption of infinite horizon learning, the household is assumed to forecast unknown variables for all periods into the future using an adaptive rule and then make decisions conditional on those forecasts.

Branch, Evans, and McGough (2013) combine the two main approaches of Euler-equation learning and infinite horizon learning by developing a learning model called "Finite Horizon Learning." Under the assumption of finite horizon learning, the infinitely-lived representative agent makes decisions based on adaptive forecasts over a finite horizon. The authors develop two varieties of finite horizon learning: N-step Euler-equation learning and N-step optimal learning which correspond to Euler-equation learning and infinite horizon learning, respectively.[7]

This paper embeds the main ideas of N-step optimal learning in a multi-period overlapping generations model. The key differences between an infinite-horizon representative agent economy and an overlapping generations economy is that the former is populated by a single, representative agent that lives forever, while the later is populated by agents of different ages who enter the model as young workers, grow old, potentially retire, and eventually pass away. An overlapping generations framework is key to model questions related to aging, demographic changes, life-cycle saving,

[5]Eusepi and Preston (2011) and Branch and McGough (2011) demonstrate models with adaptive learning better fit business cycle data than rational expectations models. See also Bullard and Duffy (2001) and Adam et al. (2016) for asset price volatility in learning models.

[6]A notable exception is the use of adaptive learning in overlapping generations models with money; for example, Woodford (1990) or Bullard and Russell (1999).

[7]N-step Euler-equation learning is the same as one-step ahead Euler-equation learning when the planning forecast is only period; N-step optimal learning converges to infinite horizon leaning as the forecast horizon approaches infinity. The reserve is not true.

pensions, and social security. A forecasting horizon greater than one period is necessary to study households responses to announced policy in a learning model. By developing a framework to look at announced policy in a overlapping generations economy with learning, this paper contributes to a growing literature that examines the response to anticipated policy in models with adaptive expectations. See Evans et al. (2009), Mitra et al. (2013), Gasteiger and Zhang (2014), and Caprioli (2015) for examples of anticipated fiscal policy in adaptive learning models.

When the planning horizon is equal to the length of the life-cycle, the model in this paper is similar to a learning model developed by Bullard and Russell (1999). In their overlapping generations model, agents forecast inflation looking forward over their life cycle, and embed their forecasts into their first order conditions and life-time budget constraint. This paper differs from Bullard and Russell (1999) in two main ways. First, agents have a finite planning horizon that can be less than the length of their life-cycle. Second, the underling model in this paper includes government debt rather than money.

Similarly, when the planning horizon is equal to the length of the life cycle, the learning model in this paper is related to “optimal learning” or “infinite horizon learning” developed by Marcet and Sargent (1989), and emphasized by Preston (2005 and 2006). Under infinite horizon learning, the representative agents forecasts endogenous variables into the infinite future and embeds those forecasts into the life-time budget constraint to make a saving consumption decision. The main difference between infinite horizon learning and this model (when the planning horizon is equal to the length of the life-cycle), is that agents in this model are finitely lived, and plan age-specific saving and consumption for each of the remaining periods of their life. The behavior of a given cohort depends on their age-specific Euler-equations and their (homogeneous) expectations about future endogenous variables. In contrast, in an infinite horizon learning model, the behavior of the representative agent can be summarized by a single Euler-equation iterated forward and embedded in the life-time budget constraint.

2 Model

2.1 Households

Each household is comprised of a single individual, who makes a savings consumption decision each period of life in order to maximize their utility. Each household lives for J periods, and retires exogenously at date $T \leq J$. They receive wage income w_t for labor inelastically supplied in period t. The gross real return on savings in period t is given by R_{t+1}. Consumption is indicated by c, savings (or asset allocations) is given by a, the social security payroll tax is given by τ, and social security benefits are indicated by z. Superscripts on variable indicate life-cycle stage (i.e., age), and subscripts indicate time period.[8]

The household maximization problem is given by (1) and (2), with $E_t^*(x)$ indicating the time t expectation of x. The star indicates that the expectations need not be rational.

$$\max_{a^j_{t+j-1}} E_t^* \sum_{j=1}^{J} \beta^{j-1} u(c^j_{t+j-1}) \tag{1}$$

$$c^j_{t+j-1} + a^j_{t+j-1} \leq R_{t+j-1} a^{j-1}_{t+j-2} + y^j_{t+j-1} \qquad \text{for } j = 1, \cdots, J \tag{2}$$

Here y^j indicates gross labor income during working life and social security income after retirement:

$$y^j_{t+j-1} = (1 - \tau_{t+j-1}) w_{t+j-1} \quad \text{for } j < T$$
$$y^j_{t+j-1} = z^j_{t+j-1} \quad \text{for } j \geq T.$$

Households optimally choose to exhaust all of their resources in the final period of life and thus $a^J_{t+J-1} = 0$.

2.2 Demographics

The economy begins in period zero with J cohorts. The initial young enter the economy with zero assets and all other cohorts enter with assets corresponding to their age a^j_{-1} for $j = 2, \cdots, J$. Young agents in successive cohorts enter the model with zero assets. The number of young born in period t is given by N_t. Each cohort is larger than the previous such that:

$$N_t = (1 + n_t) N_{t-1}.$$

The population at time t is the sum of all living cohorts. The working population is the sum of all living cohorts who have not reached the exogenous retirement age.

2.3 Production

Output is produced according to a Cobb-Douglass function that turns capital and labor into the consumption good Y_t:

$$Y_t = F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}.$$

[8]As an example, a^2_{t+1} is age 2 savings (saving in the second stage of life), in time period $t+1$.

The parameter A represents total factor productivity and α is the capital share of income. Capital and labor are paid their marginal products according to profit maximization of a competitive firm. The gross real interest rate R_t is given by:

$$R_t = F_K(K_t, L_t) + 1 - \delta \tag{3}$$

with δ representing the rate of depreciation. The wage rate w_t is given by

$$w_t = F_H(K_t, L_t). \tag{4}$$

Technology, or total factor productivity A is assumed to be constant unless there is a recession, in which case it falls and then returns to its original value.

2.4 Government

The government taxes wages, pay benefits to retirees, and potentially issues debt in a modified pay-as-go social security system. The payroll tax rate is given by τ_t. The tax is composed to two parts, a baseline tax rate τ_t^0, and an incremental tax τ_t^1 that responds to the level of government debt:

$$\tau_t = \tau_t^0 + \tau_t^1(B_t/L_t).$$

If $\tau^1 > 0$, taxes increases as government debt increases. This slows the accumulation of debt and increases the basin of attraction of the stable steady state. This type of tax is called a Leeper tax (see Davig et al. (2011) as one example). In the policy experiments that follow, the majority of the tax burden will come from the baseline tax rate τ^0, and the Leeper tax τ^1 will be used to calibrate a dynamically efficient steady state with positive bonds. Notationally, a tax rate *without* a superscript (τ_t) will refer to the entire pay-roll tax $\tau_t = \tau_t^0 + \tau_t^1(B_t/L_t)$.[9]

Social security benefits z_t^j are paid according to a benefit earning rule:

$$z_t^j = \phi_t AIPE_t^j$$

where z_t^j represents the benefit paid to a retiree of age j in time t who retired at age T. The benefit earning rule shows how much of a retiree's average indexed period earnings ($AIPE_t^j$) are replaced by social security. The parameter ϕ_t is the replacement rate. The average indexed period earnings of a retiree who is currently age j in time t (who retired exogenously at age T) is given by:

$$AIPE_t^j = \sum_{i=1}^{T} \frac{I_t^{i,T} w_{t-T+i}}{T} \quad \text{for } j \geq T$$

[9]The Leeper tax leaves open the possibility of a total tax rate greater than one hundred percent if bonds levels are high or if τ^1 is large (i.e., $\tau_t^0 + \tau_t^1(B_t/L_t) > 1$, for large B_t/L_t and/or large τ^1). I avoid this by imposing the restriction τ^1 is either the exogenous parameter value chosen by the economist, or the (smaller) parameter value such that $\tau_t^0 + \tau_t^1(B_t/L_t) = 1$.

where $I_t^{i,T}$ is used to wage-index period earnings. $I_t^{i,T}$ is calculated by dividing wages from the year that the agent retired by average wages from the year when the agent was age i. This corresponds closely to the United States social security system where benefits are based on the average indexed monthly earnings (AIME) from the highest 35 years of wage earnings.[10] I simplify slightly, and assume that benefits depend on average indexed earnings from all working years. Social security benefits are constant during the retirement phase of life for any particular cohort.[11]

If social security benefits are greater than payroll taxes, the government issues one-period bonds to finance the difference. Additionally, the initial conditions of the economy can include positive government debt (or negative debt, which corresponds to the government holding assets). Government bonds B_t evolve according to the following rule:

$$B_{t+1} = R_t B_t + \sum_{j=T}^{J} N_{t+1-j} \phi_t AIPE_t^j - L_t(\tau_t^0 + \tau_t^1 (B_t/L_t)) w_t. \tag{5}$$

Debt issued in period $t+1$ covers the interest and principle from debt in period t and the deficit (if any) from the social security system.

2.5 Markets

Prices adjust in equilibrium to clear all markets: labor, assets (capital and bonds), and goods.

Equilibrium in the labor market requires the labor exogenously supplied by households to equal the labor used by the representative firm:

$$L_t = \sum_{j=1}^{T-1} N_{t+1-j}. \tag{6}$$

The asset market clears when aggregate capital and bonds are equal to the total savings of each cohort from the previous period:

$$K_{t+1} + B_{t+1} = \sum_{j=1}^{J} N_{t+1-j} a_t^j. \tag{7}$$

[10] For more information on how social security benefits are calculated see https://www.ssa.gov/oact/cola/Benefits.html.

[11] In the United States, social security benefits grow according to a cost of living adjustment, that ensures retirees purchasing power is not eroded by inflation or wage growth. There is no inflation or wage growth (i.e. technology or productivity growth) in this model, so constant benefits are consistent with the U.S. cost of living adjustment.

Households are indifferent between holding capital or bonds, since the earn the same real rate of return R_t. Thus, the stock of bonds is pinned down by the government's flow budget constraint equation (5).

The goods market clears when output is equal to consumption plus investment:

$$F(K_t, L_t) = \sum_{j=1}^{J} N_{t+1-j} c_t^j + K_{t+1} + (1-\delta) K_t. \tag{8}$$

2.6 Rational Expectations Equilibrium

Along the balanced growth path, cohort size (N_t), labor force (L_t), output (Y_t), capital (K_t), and bonds (B_t) all grow at rate n. The equilibrium is defined in per-worker (or "efficient") terms by diving variables by the labor force L_t. Lower case letters are used to indicate efficient variables.

Definition 1 *Given initial conditions k_0, b_0, $a_{-1}^1, \cdots a_{-1}^{J-1}$, and an initial population $\sum_{j=1}^{J}(1+n)^{1-j} N_0$ (where N_0 is the the initial cohort of young and n is the population growth rate), a competitive equilibrium is a sequence of functions for the household savings $\left\{a_t^1, a_t^2, \cdots, a_t^J\right\}_{t=0}^{\infty}$, production plans for the firm, $\{k_t\}_{t=1}^{\infty}$, government bonds $\{b_t\}_{t=1}^{\infty}$, factor prices $\{R_t, w_t\}_{t=0}^{\infty}$, and government policy variables $\{\tau_t^0, \tau_t^1, \phi_t\}_{t=0}^{\infty}$, that satisfy the following conditions:*

1. *Given factor prices and government policy variables, individuals' decisions solve the household optimization problem (1) and (2)*
2. *Factor prices are derived competitively according to (3) and (4),*
3. *government debt is given by according to (5), and*
4. *All markets clear according to (6), (7), and (8).*

The equilibrium equations are printed in appendix A.

The steady state is a collection $\{k, b, a^1, \cdots, a^J\}$ that solves the equilibrium equations. The steady state is printed in appendix A.

The equilibrium of the model is not unique; for many parameter combinations there are two steady states. The number of steady states depends on the model parameters and can be characterized by a saddle-node bifurcation. See appendix B for a brief discussion.

2.7 Finite Horizon Life-cycle Learning

This section relaxes the assumption of rational expectations and builds a new model of bounded rationality called finite horizon life-cycle learning. Under the assumption of finite horizon life-cycle learning, agents combine limited knowledge about the structure of the economy with adaptive forecasts for future macroeconomic aggregates. Agents look forward over a finite horizon (potentially shorter than the length of the life-cycle) and use an adaptive rule to forecast future endogenous variables, specifically, wages, interest rates, and government debt. Agents take these forecasts and embed them into the Euler-equations that correspond to their planning horizon and use these equations to make decisions. They update their choices each period as their planning horizon moves forward and they receive new information.

In the baseline model, agents have full knowledge of government policy. Agents know the future path of social taxes and the benefit replacement rate. I relax this assumption and have agents learn about future government policy adaptively in section 5.2.1.

Agents in the learning model forecast wages and interest rates using adaptive expectations:

$$w^e_{t+1} = \gamma w_t + (1-\gamma) w^e_t \tag{9}$$

$$R^e_{t+1} = \gamma R_t + (1-\gamma) R^e_t \tag{10}$$

with $\gamma \in (0,1)$. Here w^e_t indicates expected wage, and R^e_t indicates the expected interest rate. Agents also form expectations at time t of prices in future periods $t+j$ for $j > 1$:

$$w^e_{t+j} = w^e_{t+1} \tag{11}$$

$$R^e_{t+j} = R^e_{t+1} \tag{12}$$

That is, an agent's expectation for next period's wage (interest rate) is her best guess for all future wages (interest rates).

When the Leeper tax is non-zero ($\tau^1 > 0$), taxes depend on government debt, and agents need to know future bond levels to estimate their after-tax pay. I assume that agents forecast bonds using the same adaptive learning rule they use to forecast prices:

$$b^e_{t+1} = \gamma b_t + (1-\gamma) b^e_t \tag{13}$$

$$b^e_{t+j} = b^e_{t+1} \quad \text{for } j > 1 \tag{14}$$

I consider alternative way for agents to forecast their income net of taxes in section 5.2.1.

A young agent in the finite horizon life-cycle learning model chooses first period savings and consumption and plans future savings and consumption for the other periods that are within her planning horizon. If her planning horizon is $H < J - 1$, she forecasts H periods into the future and plans her savings and consumption to satisfy her $H - 1$ first order equations. The young agent's time t plan of second period savings is denoted $a^2_{t,t+1}$. In the following period, her time $t + 1$ actual choice of second period savings is given by a^2_{t+1}. Her time $t + 1$ plan does not have to be consistent with her time t choice. She can update her savings decision based on the new information she receives in period $t + 1$.

In order to solve the finite horizon life-cycle learning model model, it is necessary to specify a terminal condition for asset holdings at the end of the planning horizon. When the planning horizon equals the remaining life-cycle ($H = J - 1$), the terminal condition is simply that the agent dies without debt as in the baseline rational expectations model (the agent chooses optimally to die with zero assets). When the planning horizon is shorter than the remaining life-cycle, I assume that agents plan to hold the same amount of wealth at the end of the planning horizon that older cohorts held at the same age. Specifically, agents adaptively forecast the steady state value of age-specific wealth at the end of their planning horizon.

I assume the agents forecast the following terminal wealth holding for the end of the planning horizon

$$a^{j,e}_{t,terminal} = \gamma a^{j}_{t-1} + (1-\gamma) a^{j,e}_{t-1,terminal} \qquad (15)$$
$$\text{for } j = H, \cdots, J-1$$

where, $a^{j,e}_{t,terminal}$ is amount of assets the agent expects to hold at the end of the period when she is age j. This asset amount is based on the observed asset holding of age j agents from last period, and the forecast from last period.

The expectations in the model are homogeneous. Agents enter the model with knowledge of the previous period's prices, bonds, age-specific asset holdings, and expectations. This assumption is equivalent to assuming agents inherit expectations from the previous generation. At any moment in time, all agents in the model have the same expectation for future prices, bonds, and terminal conditions.

The finite horizon life-cycle learning model can be written as a recursive system. The system includes the first order equations for the households, asset market clearing, government bonds, and the expectation equations (9) - (15). For a life-cycle length of J and planning horizon of length H, there will be $J - H$ terminal conditions and $H(J - H) + (1/2)H(H - 1)$ household first order equations.

2.8 Finite Horizon Life-cycle Learning Discussion

2.8.1 Adaptive Expectations and Gain Parameter

Adaptive expectations of the form given in equation (9) are a special case of adaptive learning suitable for non-stochastic models. Using adaptive expectations in a non-stochastic model is equivalent to using constant gain learning in a model with shocks.

Adaptive expectations are optimal if agents think that the variable they are forecasting follows an IMA(1,1) process. Using adaptive expectations to forecast a variable is rational if the change in variable x has the following form $\Delta x_t = \epsilon_t + \theta \epsilon_{t-1}$ for shocks ϵ, and parameter $\theta \in (0, 1)$. Thus, in this context, modeling adaptive expectations is the same as modeling agents who believe the variables they are forecasting have a mixture of permanent and transitory shocks (See Muth (1960)).

The magnitude of the gain parameter γ has economic significance in this set-up. If all shocks are permanent, the optimal forecasting rule is a random walk: $x^e_{t+1} = x_t$, that is, $\gamma = 1$. If all shocks are temporary, the best forecasting rule is constant: $x^e_{t+1} = x^e_t$, that is, $\gamma = 0$. In this model, demographic and policy changes are permanent. Recessions will be modeled with temporary reductions in the TFP factor A. Agents in the model will not know which shocks are temporary or permanent.

In the baseline simulations, I will use a relatively large gain parameter that I selected to minimize the welfare cost to agents of making forecast errors along a transition path with a demographic change and a social security tax increase. I present results for alternative gain parameters as well.[12]

2.8.2 Planning Horizon

When the planning horizon is less than the length of the life-cycle, the agent must choose how much savings (assets) to hold at the end of their planning horizon. I have assumed that agents choose to hold the same level of assets that previous generations held at the same age.

As an example, suppose the planning horizon length is three. That is, agents look forward three periods when making decisions. A young agent chooses first period savings and consumption, and plans saving and consumption for period two and

[12]Evans and Ramey show that by appropriately tuning the free parameters of the forecast rule agents can obtain the best forecast rule within a given class of under-parameterized learning rules (Evans and Ramey (2006)).

three, using three first order equations and a terminal condition:

$$(y_t^1 - a_t^1)^{-\sigma} = \beta R_{t+1}^e (R_{t+1}^e a_t^1 + y_{t+1}^{e,2} - a_{t,t+1}^2)^{-\sigma}$$
$$(y_{t+1}^{e,2} - a_{t,t+1}^2)^{-\sigma} = \beta R_{t+2}^e (R_{t+2}^e a_{t,t+1}^2 + y_{t+2}^{e,3} - a_{t,t+2}^3)^{-\sigma}$$
$$(y_{t+2}^{e,3} - a_{t,t+2}^3)^{-\sigma} = \beta R_{t+3}^e (R_{t+3}^e a_{t,t+2}^3 + y_{t+2}^{e,4} - a_{t,terminal}^4)^{-\sigma}.$$

Here $a_{t,terminal}^4$ represents the amount of assets the young agent plans to hold at the end of period $t+3$ when she is model age 4. This planned asset holding is based on what previous generations held when they were model age 4. Older agents follow a similar pattern to plan their savings and consumption. Agents who have three or fewer periods of life remaining using the standard terminal condition $a^J = 0$.

An alternative assumption about the end of the planning horizon is to assume that agents do not hold assets or debt at the end of their planning horizon. This is equivalent to the agent treating the end of the planning horizon as the end of her life-cycle; it is as if the agent does not understand she will exist after the end of her planning horizon. This assumption drives the main results in the short-planning horizon literature (see Caliendo and Aadland (2007) and Park and Feigenbaum (2018)). This assumption generates a steady state that is different than the rational expectations steady state, because agents under-save relative to the rational expectations baseline. Although interesting, this type of myopia is beyond the scope of this paper.

When the planning horizon is equal to the length of the life-cycle, the agent does not need to forecast an asset-holding condition for the end of her planning horizon. The agent simply forecasts endogenous variables (wages, interest rates, and bonds) according to the adaptive rule, and plans saving and consumption for each period of life. In a companion paper, I call this type of learning "life-cycle horizon learning" and use the learning model to calculate the welfare effects of social security policy uncertainty (Cottle Hunt (2019)).

3 Expectational-Stability

The equilibrium of the underlying rational expectations model is not unique. One common equilibria selection criteria is determinacy. I examine the determinacy of the steady states of the rational expectations model in appendix B and find the high-capital steady state is determinate, while the low-capital steady-state is not. A complementary equilibrium selection criteria is stability under learning, or "expectational-stability." If the dynamics in a model with learning converge to a rational expectations equilibrium (REE), the REE is said to be stable under learning (see Evans and Honkapohja (2001)). Branch et al. (2013) show that the unique REE in an infinite-horizon Ramsey model is stable under finite horizon learning. I present a similar result in this paper. Specifically, I numerically verify that determinate REE in a multi-period overlapping generation models with capital and debt is stable under finite horizon life-cycle learning.

Expectational-stability can be verified numerically using a T-map which maps beliefs into actual prices. Given constant (potentially incorrect) expectations $p^e = (R^e, w^e, b^e, a^{j,e}_{terminal})'$ for $j = \{H, \cdots, J-1\}$, the learning dynamics of the finite horizon life-cycle learning model asymptotically converge to $p = (R, w, b, a^j)'$. This convergence is called a T-map. Given a life-cycle length of J and a finite planning horizon of length $H \leq J$, the T-map is:

$$T : \mathbb{R}^{3+J-H} \to \mathbb{R}^{3+J-H} \tag{16}$$

A fixed point of the T map is stable under learning if it locally stable under the ordinary differential equation

$$\frac{dp}{d\tau} = T(p) - p. \tag{17}$$

Stability under learning requires the real parts the eigenvalues of the derivative matrix $dT < 1$. I have numerically verified the determinate (high capital) steady state is Exceptionally-stable under finite horizon life-cycle learning (at all planning horizons). I have also numerically verified that the low capital steady state (which is a saddle) is *not* stable under learning.[13]

The stability under learning of the determinate REE in this model can be illustrated graphically. The dynamics of the learning model converge to the REE given arbitrary initial conditions near the steady state. Figure 1 illustrates this convergence in the learning model, calibrated with $J = 6$ (six period lives) and parameters calibrated as detailed in section 4.1. The example begins with capital, bonds, assets, and expectations below their steady state values. Agents update their expectations as they receive more information and eventually learn the steady state values, which are indicated with a dashed red line. Agents overshoot the steady state initially. This pattern is observed in Evans et al. (2009).

Branch et al. (2013) find that shorter planning horizons converge to the REE more quickly and that agents make larger errors (and there are larger aggregate fluctuations) when the planning horizon is longer. I find a similar result; the fluctuations in the economy are greatest when the planning horizon is equal to the remaining life-cycle ($H = J - 1$). The fluctuations decrease as the planning horizon decreases.

[13] The underlying model does not always have two steady states. The number of steady states depends on the parametrization of the model. I verified the E-stability of the steady states associated with many different parameterizations of the model. I searched across the parameter space $\beta \in (0, 1], \alpha \in (0, 1), \sigma \in (0.5, 4), \delta \in [0, 1], n \in [0, .5)$ and across different social security parameterizations (such as no social security, lower taxes, higher taxes, large deficits, small deficits, and so on).

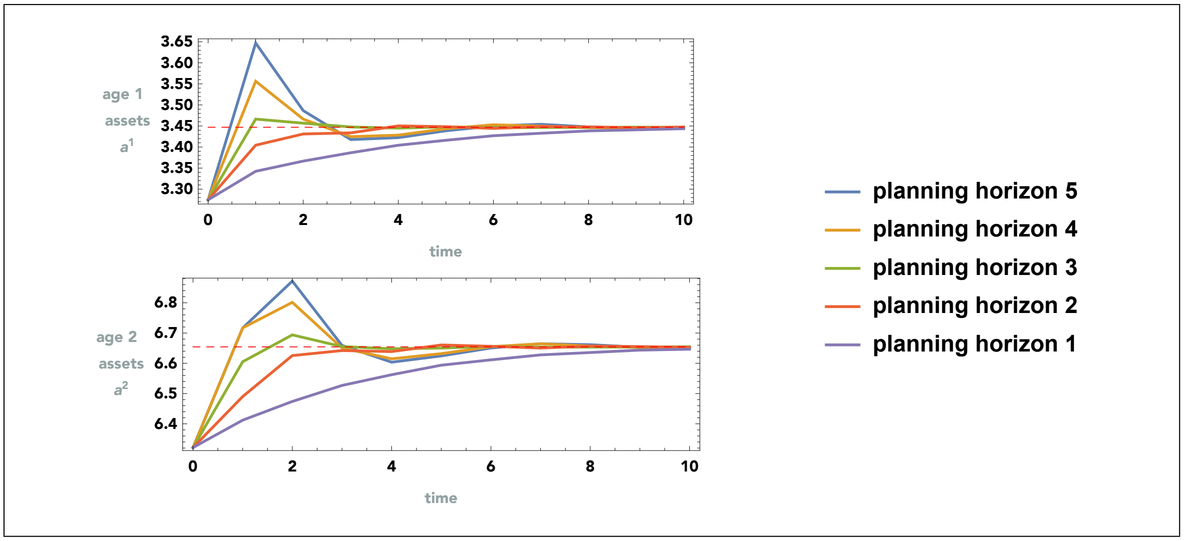

Figure 1: This graphs shows convergence to a steady state for the Finite Horizon Life-cycle Learning model with different planning horizons. The red dashed line indicates the steady state. The initial values were chosen to be near the steady state. The initial expectations were all set equal to the initial values. The top graph shows the path of age 1 assets; the bottom graph shows the path of age 2 assets. The converge of the other asset choices $(a^3, \ldots, a^5)$, capital, and bonds, follow a similar oscillating pattern.

4 Quantitative Examples

The analysis of the previous section shows that the finite horizon life-cycle learning model converges to the stable rational expectations equilibrium. If the economy begins in a steady state and there are no shocks, the economy simply remains in the steady state under rational expectations and also under learning. If the economy begins outside a steady state, or if there is some kind of shock, like a change in the population growth rate or a change in government policy, then the economy will converge to the (new) rational expectations equilibrium. The transition dynamics that lead to the equilibrium are different in the learning model compared to rational expectations. Additionally, the transition dynamics differ when a different planning horizon is used.

These transition dynamics are particularly important in an overlapping generations model, since agents in the model do not live forever and may only be alive during a transition period. The transition dynamics implied by the learning model change the life-time consumption (and therefore utility) of households alive during a transition period compared to a rational expectations model. Understanding these dynamics could be important for welfare analysis, especially relating to policy changes that impact different generations in different ways, such as changes to public pensions. As illustrations of the learning dynamics, I provide a simple introduction and then two main examples. The simple introduction highlights the role of the forecasting horizon. The first main example explores reforming social security in response to an aging population. The second main example depicts the transition dynamics following a recession that is caused by a temporary, surprise reduction in the total factor productivity parameter A.

4.1 Parameterization

Preferences are given as the standard constant elasticity of substitution function:

$$\begin{aligned} u(c) =& \frac{c^{1-\sigma}}{1-\sigma} \quad \text{if } \sigma \neq 1 \\ u(c) =& \ln(c) \quad \text{if } \sigma = 1. \end{aligned}$$

A model period represents 10 years. The life-cycle is six periods ($J = 6$); agents enter the model at age 25 and pass away at age 85. The total factor productivity parameter A is set to $A = 10$ (the size of this parameter simply scales the steady-state). The inverse elasticity of inter-temporal substitution to $\sigma = 1$ (which is in the range of empirical estimates in Gourinchas and Parker (2002)).[14] The remaining parameters are set at standard values.[15] The parameter values are listed in Table 1.

Parameter		Value
α	Capital share of income	$1/3$
β	Discount factor	$(1/1.005)^{10}$
σ	Inverse elasticity of substitution	1
δ	Depreciation	$1-(1-0.10)^{10}$
n	Steady state population growth rate	0.01
A	TFP factor	10

Table 1: Baseline parameters for 6 period model

The growth rate of the population, n is set such that the ratio of retirees to workers matches the ratio of social security beneficiaries to workers. This ratio is currently 0.35 in the U.S., and is expected to increase to 0.5 by 2095 (SSA (2017)). I will simulate the aging of the population with a one-time decrease in the growth rate of the population, as illustrated in Figure 2.

[14] See also Bullard and Feigenbaum (2007) and Feigenbaum (2008).

[15] See, for example, Branch et al. (2013). My discount factor β is slightly closer to one and the depreciation rate is slightly higher. I choose the higher discount factor and depreciation rate to better match the size of tax increase necessary to ensure long-run solvency of social security.

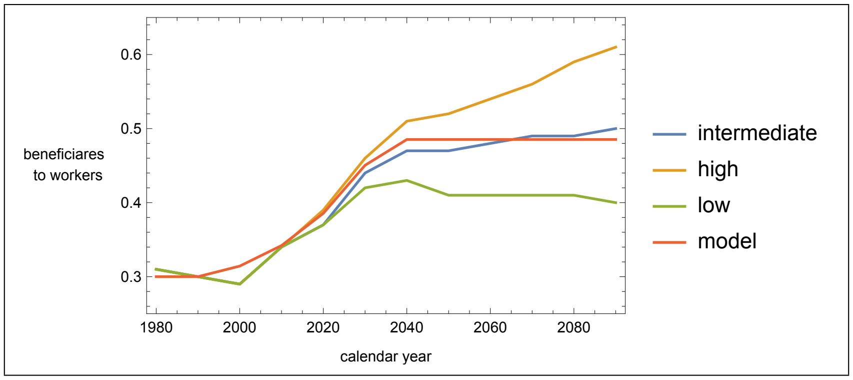

Figure 2: Ratio of beneficiaries to retirees. The red link in this graph illustrates the ratio of beneficiaries to retirees in the calibrated, six period model. The demographic change is modeled as a one-time reduction in the population growth rate n from 0.1802 to 0.01. The yellow, blue, and green lines show the high, intermediate, and low cost projections of the ratio from the 2017 SSA Trustees' Report. The model calibration closely tracks the intermediate assumptions from the SSA.

I set the gain parameter $\gamma = 0.93$ as a baseline. This value minimizes the maximum welfare cost to an agent of using adaptive expectations with longest possible planning horizon relative to an "infinitesimally rational" agent along the transition path that includes a demographic change and a social security tax increase. This is discussed further in section 5.1.1. The simulation results are sensitive to the gain parameter. I explore that relationship in section 5.1.

4.2 Simple example

Before jumping into a more policy relevant example, I want to highlight the role of the planning horizon. Agents only respond to policy changes when the announced policy enters their planning horizon. Agents making one step ahead forecasts will not respond to announced policy until the period right before the change. This is illustrated by a simple example in Figure 3. The economy starts off in a dynamically efficient steady state with government debt and a social security system that runs a small surplus. Payroll taxes are increased and benefits are cut in period $t = 6$. This policy change burdens all generations, as take-home pay and social security benefits are lower. The response of young agents is visible in the path of age one savings a^1 on the top of Figure 3. Young agents with a planning horizon of $H = 5$, increase their savings in period $t = 1$, since they will experience the benefit cut in their old age. Agents with a planning horizon of $H = 4$ do not respond until period $t = 2$, when the policy change is four periods away. Young agents with a planning horizon of $H = 1$ don't increase their savings until period $t = 5$, despite the fact that the policy change was announced prior. The savings choices of agents at model ages two through five are also included in the figure. Agents of model age five (which is calibrated to be age 75) do not respond to the announced policy until one period ahead, regardless of the planning horizon. In the graph, the small deviations from the initial steady state that appear multiple periods before the reform are simply due to the general equilibrium changes in prices from the changes made by younger agents. The older agents (mode age five) do not change their old age savings earlier because they will not be alive to be bothered by the policy change.

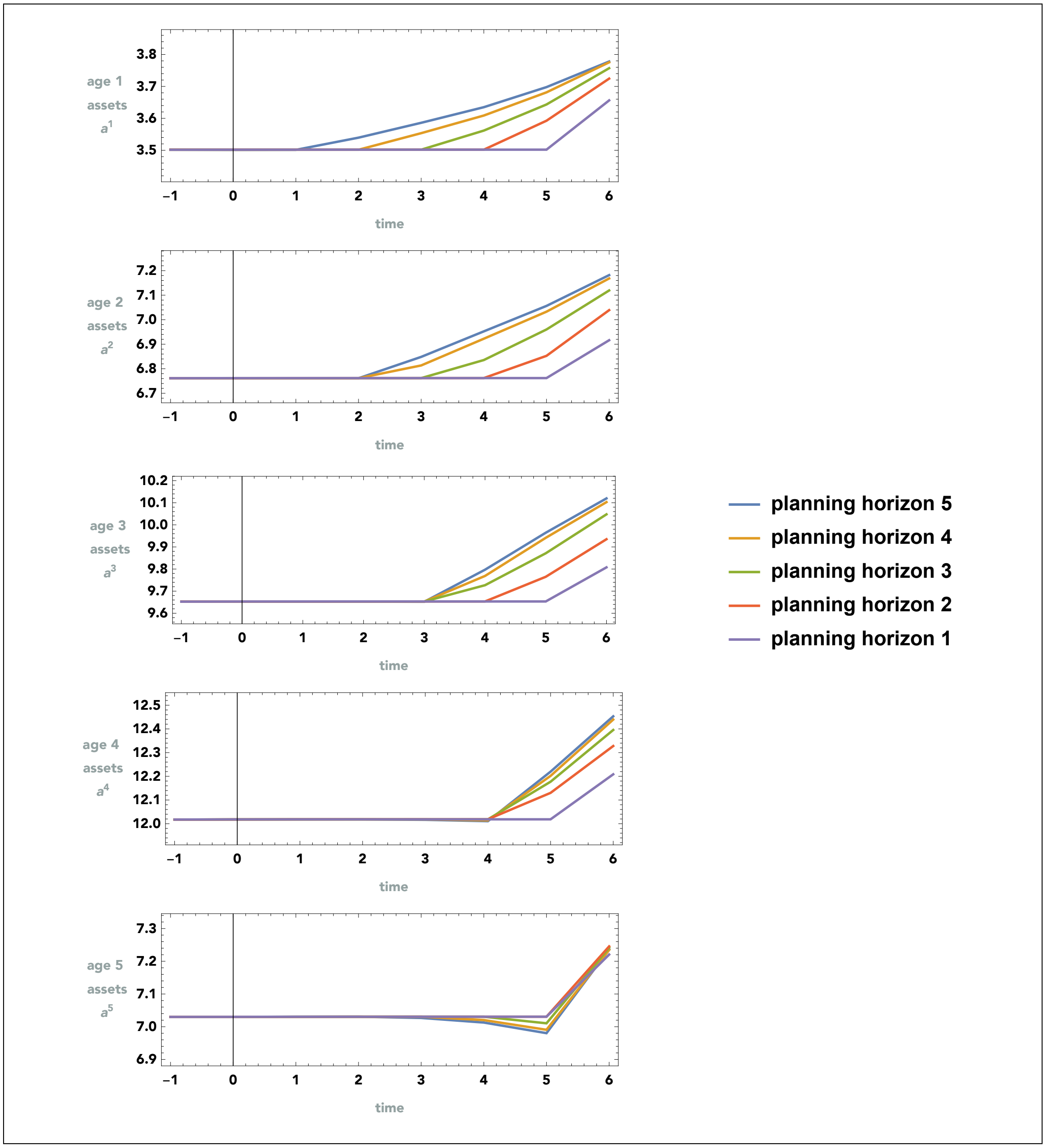

Figure 3: Transition paths for asset holdings for the finite horizon life-cycle learning model with different panning horizons. The economy starts off in a steady state; in period $t = 6$ taxes are raised *and* benefits are cut. Agents do not respond to the announced policy change until it enters their planning horizon. The figure only includes the first few time periods, to show the earlier responses of longer planning horizons.

4.3 Announced Social Security Reform

The U.S. population is aging (as depicted in Figure 2). This aging puts pressure on the pay-as-you-go social security system. In the absence of tax increases or benefits cuts, the government would only be able to fund benefits by issuing debt.[16] [17] However, in a dynamically efficient economy, there is a limit to how much the government can borrow.[18] I illustrate the demographic change and social security policy response in the following example.

The government runs a social security system that generates a small surplus. As the population ages, the surplus changes to a deficit. The growing social security deficits deplete the government's assets and cause the government to accumulate debt. After a number of periods, the government reforms social security by raising taxes (or in a separate example, by cutting benefits).

I calibrate this example to correspond to U.S. The growth rate of the population falls in 1980. This creates a smooth six-period transition in the ratio of retirees to workers that closely matches social security projections. Social security taxes are $\tau^0 = 0.124$, the full employer and employee Old-Age, Survivors, and Disability Insurance (OASDI) tax rate since 1990. This includes the disability component, which allows me to better match the size of the social security benefit. The benefit replacement rate $\phi = 0.4$ which corresponds closely the average replacement rate in the US.[19] Reform is modeled as a tax rate increase to $\tau^0 = 0.1516$. The SSA estimates that this tax increase would eliminate the long-run funding short-fall for the social security system. The Leeper tax rate is set to $\tau^1 = 0.045$ before and after reform to ensure dynamic efficiency and the existence of two steady states.

Figure 4 plots the path of capital and bonds for this tax increase in the rational expectations model. Government bonds increase due to the social security deficits that arise as the population ages. Debt would increase to infinity were it not for the social security tax increase. Capital per worker increases mechanically as the population growth rate falls (and their are fewer workers relative to retirees). Capital also increases as savings increases in anticipation of the tax increase. Figure 5 adds the paths for the finite horizon life-cycle learning model (for all five possible planning horizons).

[16] The government could also fund social security benefits using tax revenue from some other existing tax. I have abstracted away from other taxes, which means that is not possible in this model.

[17] Social security tax revenue exceeded social security benefit payments for most of the 1980s through the mid 2000s. The surplus tax revenue was lent to the Treasury and is called the trust fund. The SSA could use the trust fund assets to pay benefits. The SSA estimates the trust fund will be depleted by around 2034.

[18] Even in a dynamically inefficient economy, the government cannot borrow infinitely. See Chalk (2000) for a discussion of maximum government deficits in a dynamically inefficient economy.

[19] For example, Olovsson (2010) uses $\phi = 0.437$ and Nelson (2017) uses $\phi = 0.364$.

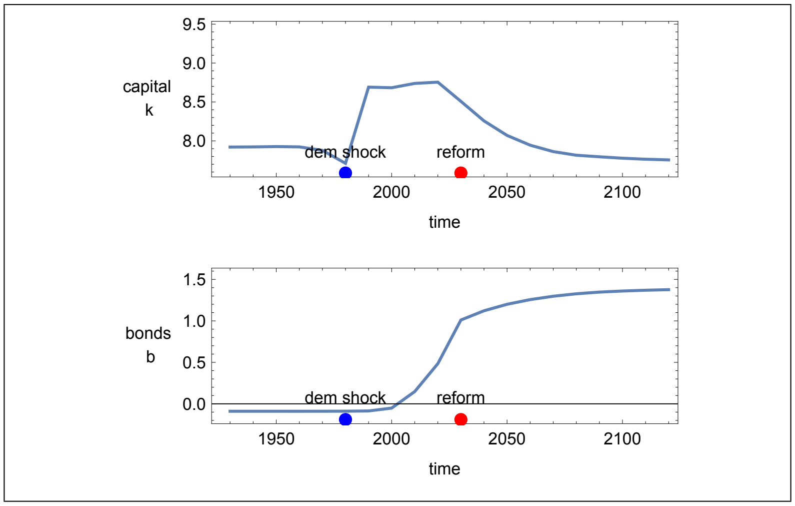

Figure 4: Time paths for capital and bonds in the rational expectations model for an announced tax increase. The initial population growth rate n is 0.1802 and falls to 0.01 in 1980. Demographic changes drive increasing debt until tax increase in 2030. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. The gain parameter $\gamma = 0.93$ for this example.

Agents in the learning model overestimate the interest rate and underestimate the wage in the first few periods following the change in the population growth rate (which moves the economy away from the initial steady state). Because agents are expecting a higher interest rate and a lower wage, they save (relatively) more. This increases the capital stock relative to the rational expectations model. The increased capital stock decreases the interest rate and increases wages, which causes agents to save less which eventually drives down the capital stock. This oscillatory pattern continues along the convergence path to the new steady state.

The learning dynamics in the finite horizon life-cycle learning depend on the forecasting or planning horizon. Agents make decisions based on their forecasts over their short planning horizon. Agents forecast prices, bonds, and the assets they expect to hold at the end of their planning horizon. Over time, agents learn the steady state values of each of these quantities. If agents' estimates for the assets they'll need at the end of the planning horizon are close to the rational expectations (steady state) values, they make relatively better decisions using a shorter planning horizon than using a longer horizon. This is because agents using a longer horizon forecast over more periods and can make larger mistakes. Additionally, the assets held at the end of the planning horizon are conceptually similar to a value function for the remaining life-cycle. If an agent holds the optimal amount of assets at the end of the planning horizon, this is similar to planning explicitly for the periods beyond the end of the planning horizon. Thus, agents using a short planning horizon who hold (close to) the optimal amount of assets for the end of their planning horizons may outperform

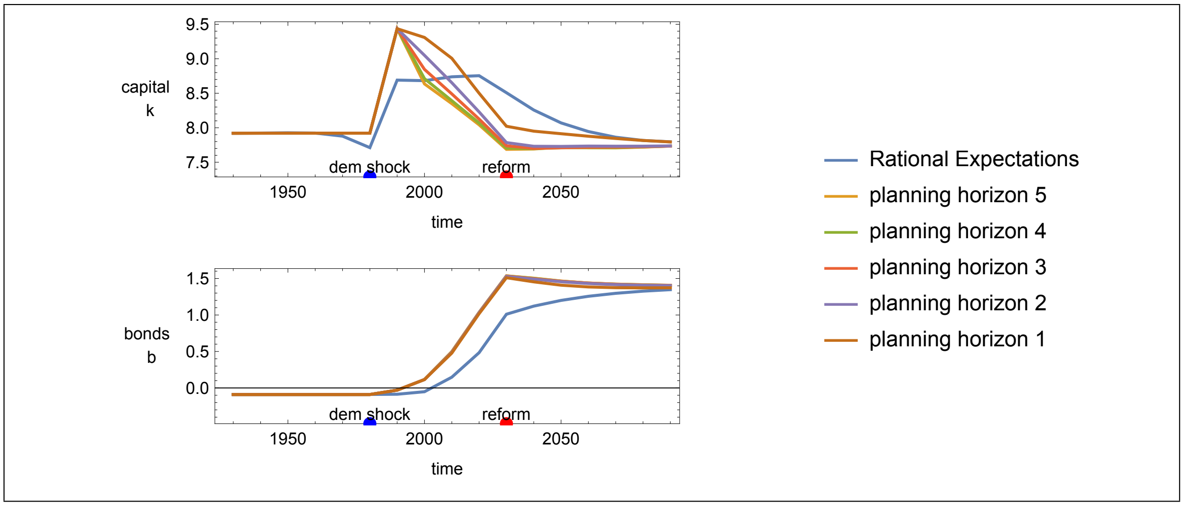

Figure 5: Time paths for an announced tax increase. Paths of capital and bonds for RE (blue), Finite horizon life-cycle learning planning horizon 5 (yellow), 4 (green), 3 (red), 2 (purple), 1 (brown). Demographic changes drive increasing debt until tax increase in the year 2030. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. γ=0.93.

agents who forecast over a longer horizon.

The impact of the planning horizon is evident by comparing the transition dynamics of different planning horizon lengths for the a social security tax increase in response to a demographic shock. The fluctuations are larger for longer planning horizons and smaller for shorter planning horizons, as depicted in Figure 6 which shows the asset holdings (savings choices) at each age in the learning model with each possible forecasting horizon. Although the shorter planning horizons generate smaller fluctuations, agents using a short planning horizon fail to respond to announced policy until it enters the planning window. Agents using a longer planning horizon respond in advance to announced policy changes. This is evident by noting the path of savings of the young is closer to the rational expectations baseline with a longer planning horizon, as detailed in Figure 7. Note that the agents using learning do not understand how demographic change will impact wages, interest rates, and government debt, so they only respond adaptively to the aging of the population. The savings choices of the households, combined with the government policy, lead to the paths of capital and bonds depicted in Figure 5.

The government could also address growing social security deficits by reducing benefits, rather than increasing taxes. Figure 8 depicted the time paths of capital and bonds for an announced social security benefit cut in 2030 (following the demographic transition that begins in 1980). The time paths from the benefit cut look somewhat similar to the times paths for the tax increase. The main difference is that the benefit cut policy change induces the older generations to save more leading up to the reform,

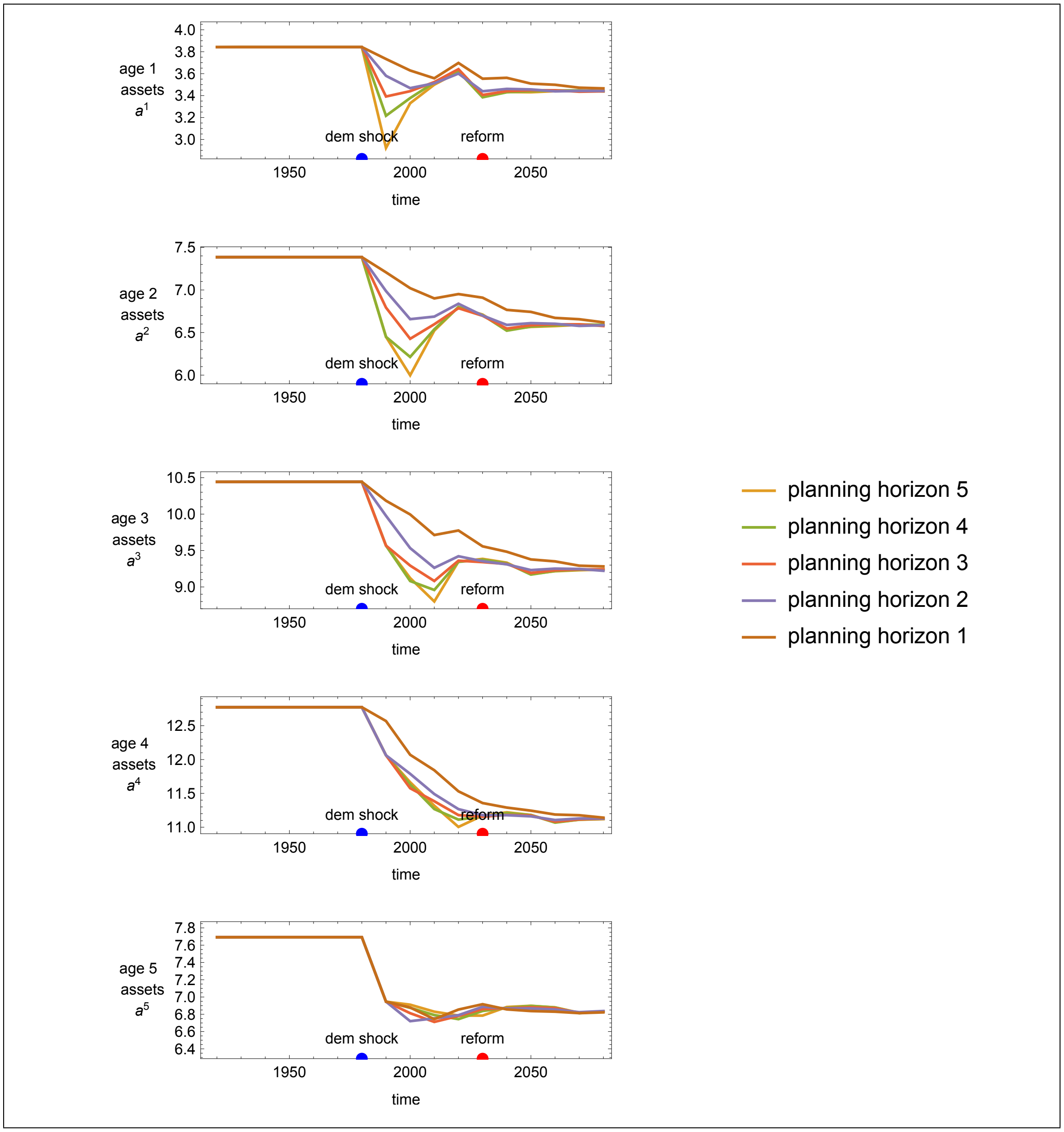

Figure 6: Time paths for an announced tax increase. Paths of assets, Finite horizon life-cycle learning planning horizon 5 (yellow), 4 (green), 3 (red), 2 (purple), 1 (brown). Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. γ=0.93. The initial population growth rate is 0.1802 and falls to 0.01 in 1980.

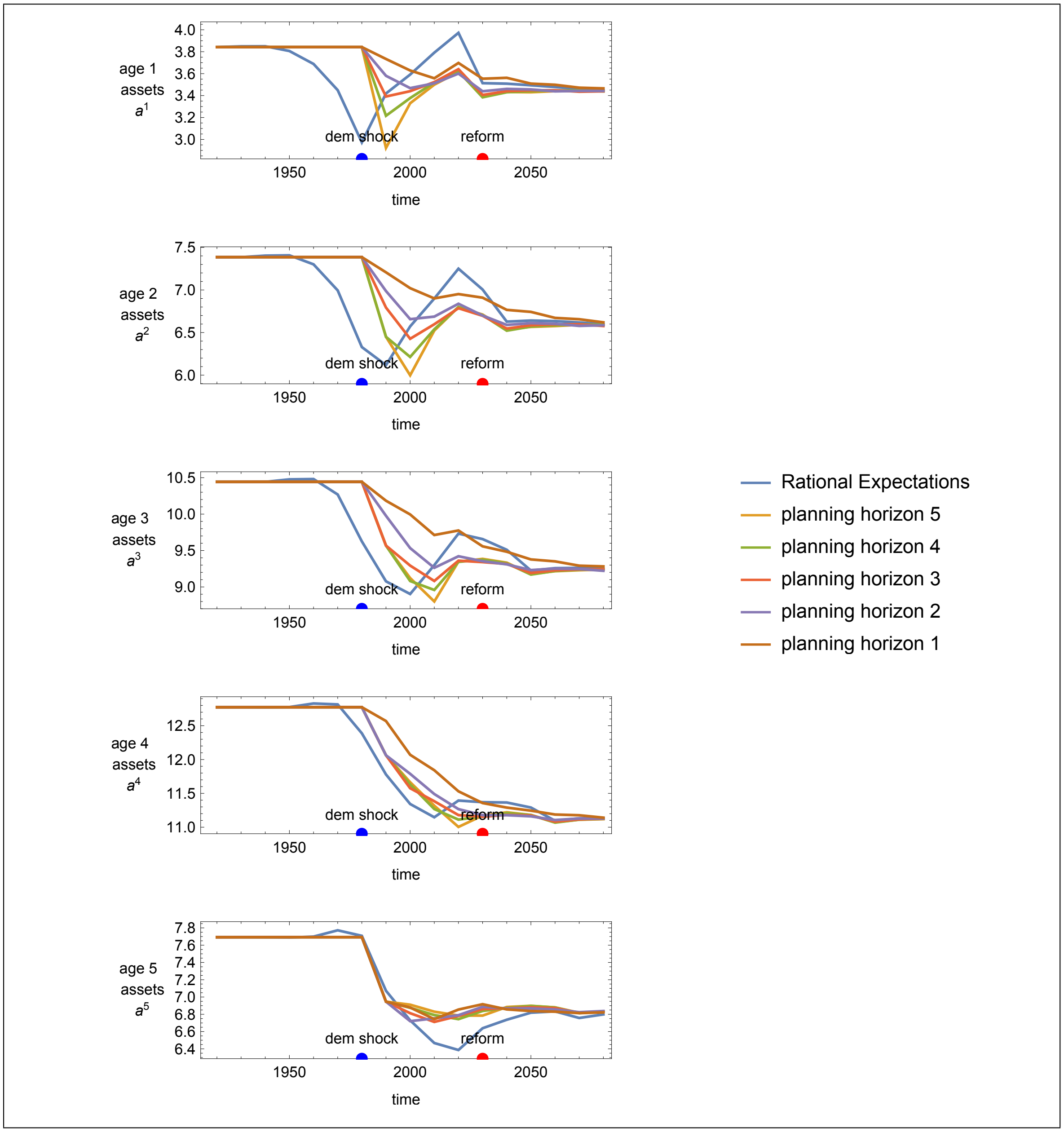

Figure 7: Time paths for an announced tax increase. Paths of assets, rational expectations (blue), finite horizon life-cycle learning planning horizon 5 (yellow), 4 (green), 3 (red), 2 (purple), 1 (brown). Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. γ=0.93. The initial population growth rate is 0.1802 and falls to 0.01 in 1980.

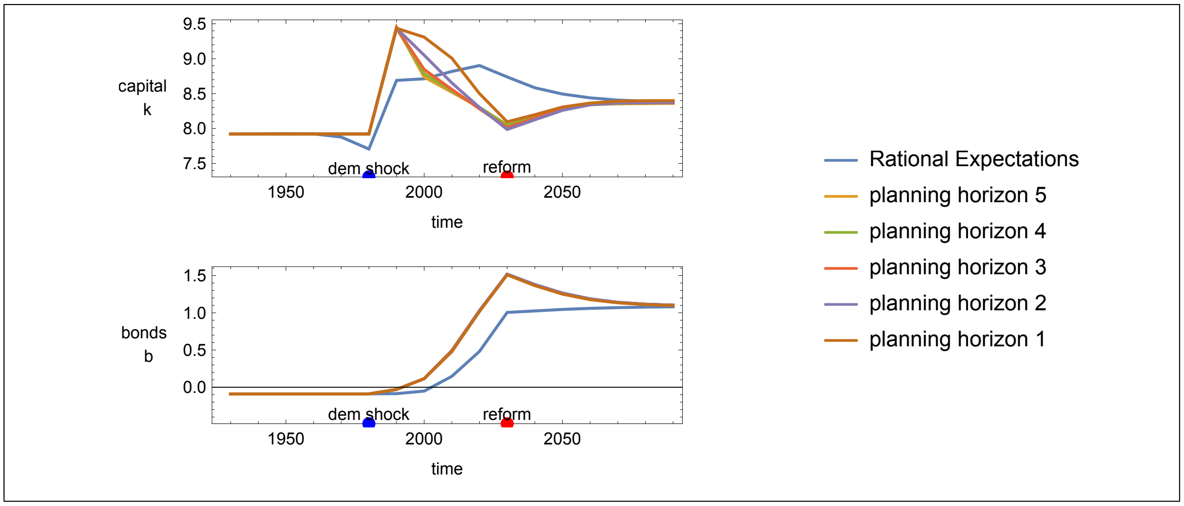

Figure 8: Time paths for an announced benefit cut. Paths of capital and bonds for rational expectations (blue) and finite horizon life-cycle learning planning horizon 5 (yellow), 4 (green), 3 (red), 2 (purple), 1 (brown). Demographic changes drive increasing debt until benefit cut in the year 2030. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change decreases ϕ to 0.332. γ=0.93.

so aggregate capital is higher along the transition path.

4.4 Recessions

The examples in the previous section illustrate the dynamics in the finite horizon life-cycle learning model with social security policy changes. An alternative framework to highlight the differences between the learning model and the rational expectations model is to simulate a recession and compare the transitions back to the steady state.

I simulate a recession as a surprise decrease in the total factor productivity A. The recession is depicted in Figures 9, 10, and 11. I model the recession as a one period reduction in the total factor productivity. The recession reduces output per worker and aggregate consumption by 5% in the rational expectations model.[20] Agents in the rational expectations model know that the recession is only one period. Agents in the learning models do not know the relationship between total factor productivity (A), prices (w and R) and bonds (b); they have no way to incorporate changes in the value of A into their forecasts. They continue to forecast prices and bonds adaptively. The recession generates cycles in capital and bonds, as agents overshoot the steady state before converging. Convergence back to the steady state is non-monotonic in

[20]During the Great Recession (2007-2009) output fell by 7.2% and consumption fell by 5.4%. The average post-war recession saw a decline in output and consumption of 4.4% and 2.1%, respectively. See Christiano (2017).

the rational and learning models.[21] The oscillatory dynamics are larger and more persistent in the learning model. The decrease in consumption is also larger in the learning models relative to the rational expectations baseline.[22]

Throughout the recession examples, the growth rate of the population is held constant (in contrast to the social security section), and thus each life-cycle period t is simply referred to by a number $t = 1, 2, 3 \cdots$, rather than to a specific decade (such as $t = 1930, 1940, 1950, \cdots$). The life-cycle is still calibrated to $J = 6$; agents make economic choices for six ten-year periods.

The path for consumption at each age is depicted in Figure 9. The consumption of the young (c^1) initially decreases following a recession in the rational expectations model, and also for planning horizons of length 1-4. Surprisingly, young consumption increases during and immediately following a recession when the planning horizon is 5 periods. This is through an interest rate effect, that is driven by the large gain parameter γ. The young agent (mistakenly) forecast that future wages, interest rates, and social security benefits will be very close to what she observes in the current period. Both the wage and the interest rate fall during the recession, but the interest rate falls by relatively less (since the capital stock falls during the recession). If a smaller gain parameter is used for forecast future endogenous variables, agents place less weight on the current state of the economy and more weight on their previous expectations, and consumption falls during the recession for all planning horizons.

The path of assets (savings) at each age is depicted in Figure 10. Agents at all planning horizons and all ages reduce their saving during the recession period ($t = 10$). This is true both in the learning model and in rational expectations. This is because wages and interest rates fall during the recession; agents offset the reduction in income by saving less. In the learning model, older agents (choosing saving for the third, forth, or fifth period of life) reduce their savings by less than rational agents. The learning agents do not anticipate that wages and interest rates will rebound, and so they are relatively more cautious than the fully rational agents. Young agents with shorter planning horizons also reduce their savings by less than the fully rational agents. Only young agents will long forecasting horizons decrease their savings by more than rational agents during the period of the recession; these agents also increase their consumption in the recession, as explained above.

The welfare effects of the recession are illustrated using a consumption equivalent variation technique in Figure 11. This consumption equivalent variation calculates by how much an agent's period consumption would need to be increased in the initial steady state to make the agent indifferent between living in the initial steady state,

[21] See Azariadis et al. (2004) for a discussion of cycles in rational, multi-period OLG economies.

[22] Eusepi and Preston (2011) show that adaptive learning in an RBC model creates dynamics that more closely match that data than a standard, rational expectations model. They also show that the shocks required to generate a realistic recession are smaller in the model with learning.

and living in some other time period, as given by the following equation:

$$\sum_{j=1}^{J} \beta^{j-1} u(c_{ss}^{j}(1+\Delta)) = \sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j}). \tag{18}$$

Here Δ is the consumption equivalent variation, c_{t+j-1}^{j} is the consumption of an agent age j in time period $t+j-1$, and c_{ss}^{j} is the consumption in the initial steady state. A negative consumption equivalent variation implies that the agent prefers the initial steady state to being born in a different period.

The one-period recession takes place in period $t = 10$. Agents live for six periods, and thus the recession directly harms cohorts born in periods $t = 5$ through $t = 10$. Agents born in period $t = 5$ to $t = 10$ are alive during the recession and experience lower life-time consumption. Agents born after $t = 10$ may also be harmed (or benefited) if the capital stock is lower (or higher) than in the initial steady state. The cohorts of agents born near the recession are worse off in the learning model than in the rational expectations model. The most harmed cohort in the learning model (with the longest planning horizon) would need to have 1.99% add to their period consumption to be indifferent between the recession and the initial steady state. The most harmed cohort in the rational expectations model would only need 1.41% added to their period period consumption to be indifferent between the recession and the initial steady state.

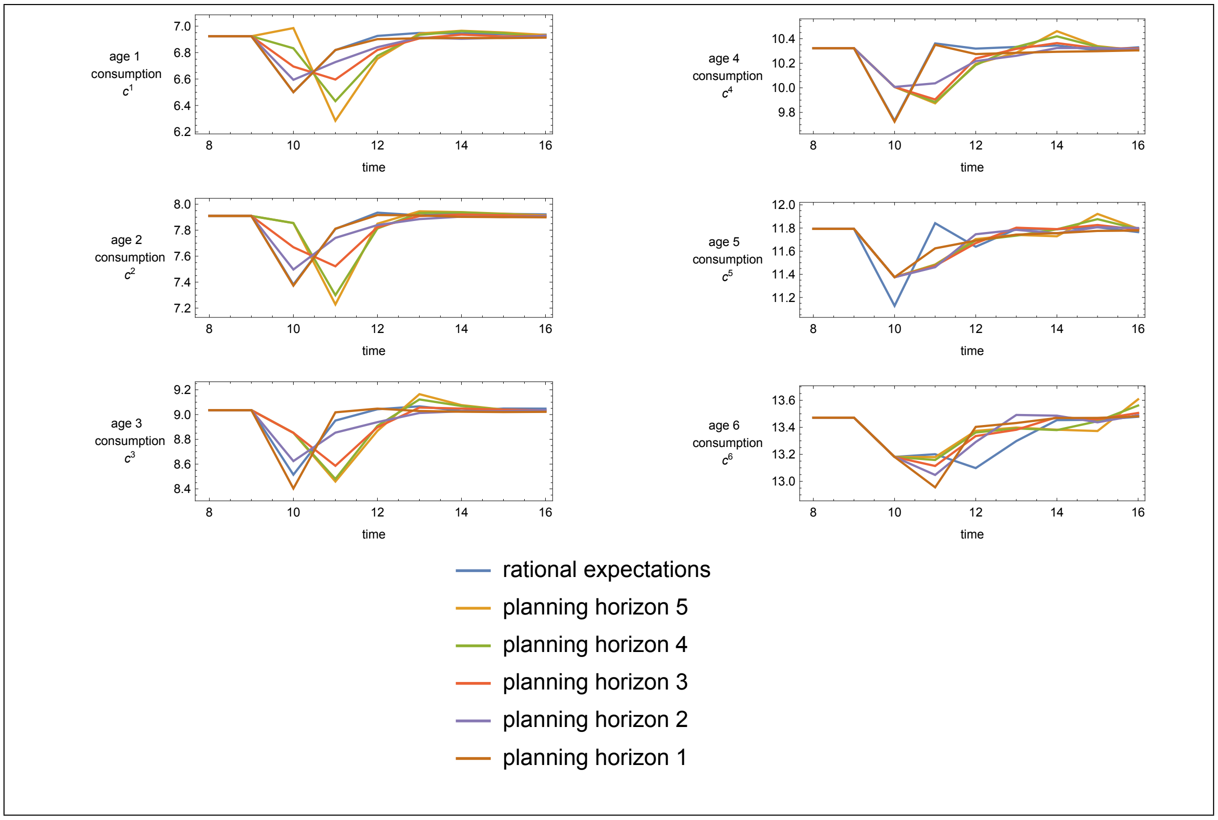

Figure 9: Path of consumption for the rational and learning model for a recession in period $t = 10$. The recession is a surprise reduction in the TFP factor from $A = 10$ to $A = 9.5$ in period $t = 10$. The TPF factor is depressed for a single period, and the returns to the pre-recession level. The population growth rate is constant at $n = 0.01$, and social security policy is constant $\tau^0 = 0.1516$,$\tau^1 = 0.045$, $\phi = 0.4$. In the learning model, the gain parameter $\gamma = 0.93$.

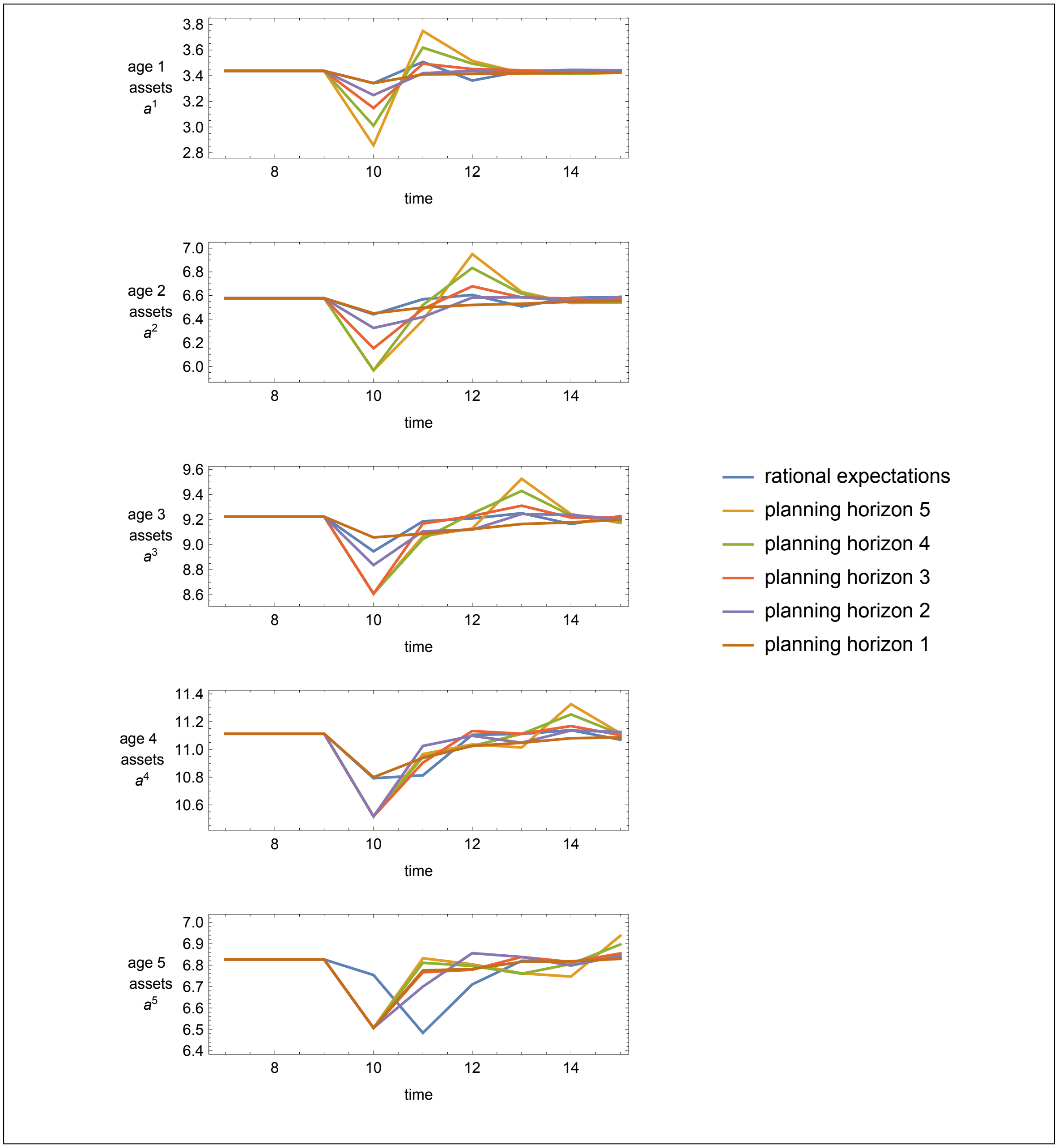

Figure 10: Path of assets (savings) for the rational and learning model for a recession in period $t = 10$. The recession is a surprise reduction in the TFP factor from $A = 10$ to $A = 9.5$ in period $t = 10$. The TPF factor is depressed for a single period, and the returns to the pre-recession level. The population growth rate is constant at $n = 0.01$, and social security policy is constant $\tau^0 = 0.1516$,$\tau^1 = 0.045$, $\phi = 0.4$. In the learning model, the gain parameter $\gamma = 0.93$.

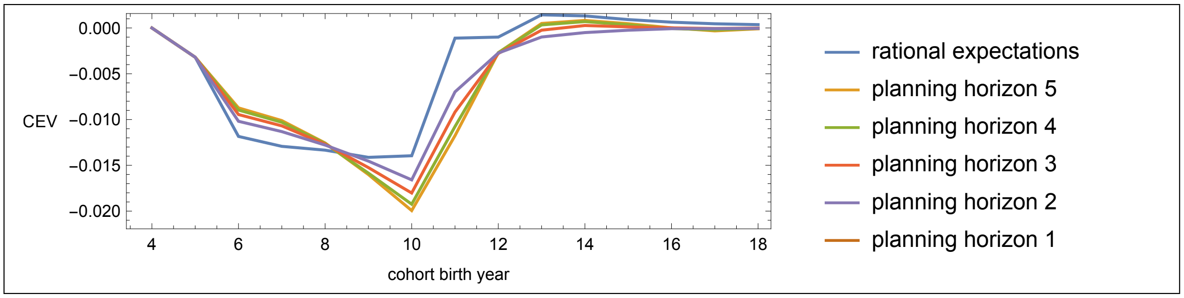

Figure 11: Consumption equivalent variation for an economy with a recession in period $t = 10$ that lasts for 1 period. The CEV compares utility for each cohort to the utility of living in the initial steady state. The population growth rate is constant at $n = 0.01$, and social security policy is constant $\tau^0 = 0.1516$,$\tau^1 = 0.045$, $\phi = 0.4$. The recession is a surprise reduction in the TFP factor from $A = 10$ to $A = 9.5$ in period $t = 10$. The TPF factor is depressed for a single period, and the returns to the pre-recession level. In the learning models, the gain parameter $\gamma = 0.93$.

The welfare cost of the learning depends on the depth of the recession. If the recession reduces output by 10% rather than by 5%, the most harmed cohort in the rational expectations model would need 2.88% added to their period consumption to be indifferent between the recession and the initial steady state (compared to 1.41% for a recession which reduces output by 5%). Similarly, the most harmed cohort in the learning model (with the longest planning horizon) would need 4.30% added to their period consumption (compared to 1.99%) to be indifferent between the recession and the initial steady state. Regardless of the depth of the recession, the largest welfare cost occurs for the agents in the learning model with the longest planning horizon who are born in the period of the recession. The welfare cost is smaller when a shorter planning horizon is used, since agents do not forecast low output as far into the future. The welfare effects of the recession also depend on the choice of gain parameter γ and are discussed in section 5.1.

5 Robustness and Discussion

5.1 Gain Parameter Sensitivity

The transitions dynamics of the learning model depend both on the planning horizon and also on the gain parameter, γ. Recall from equations (9) -(15) that the agents' expectations about the future value of a variable depends on the current value of the variable and the agents' expectation from the previous period. That is, $x^e_{t+1} = \gamma x_t + (1-\gamma)x^e_t$ for various x. Agents place a weight of γ on the current information and a weight $1-\gamma$ on their previous expectation. When the gain parameter γ is high (as is the case in the baseline parameterization of the model), agents place more weight on current information relative to their previous expectations. When γ

is high, their expectations adjust quickly as they incorporate new information into their expectations. In contrast, when γ is low, agents place relatively less weight on current observations and place more weight on their old expectations. When γ is low, expectations are slow to adjust.

Figure 12 illustrates the time path for the economy under rational expectations and under finite horizon life-cycle learning (with the planning horizon $H = 5$) and different gain parameters γ, for a tax increase following a reduction in the population growth rate (as in figure 4). When the gain parameter is large, the time path of the economy is closest to rational expectations. The oscillations introduced by the learning feedback are relatively small and the economy convergences to the new steady state more quickly. When the gain parameter is small, agents don't place very much weight on new observations and hang onto their old expectations. This introduces larger cycles that are more persistent. It takes the economy longer to converge to the new steady state with a small gain parameter γ.

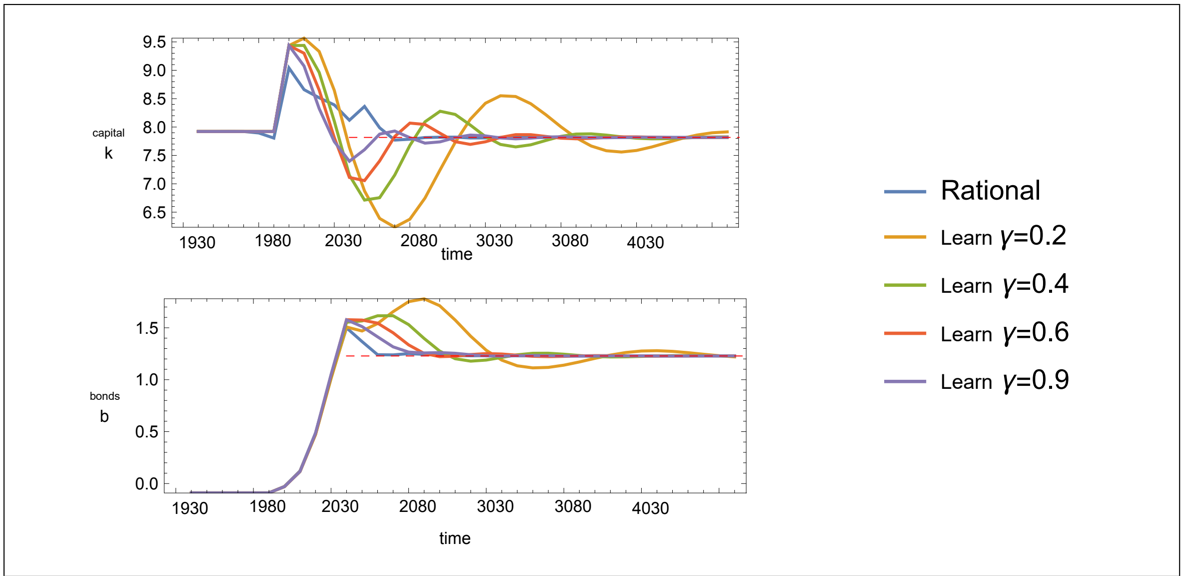

Figure 12: Time paths for capital and bonds for an economy that experiences a demographic change in 1980 and a social security tax increase in 2030 under rational expectations (blue line) and under finite horizon life-cycle learning with different gain parameters γ (other lines). For the learning paths, the forecast horizon is equal to the length of the life-cycle. As the gain parameter increases the amplitude of endogenous cycles also increases.

The welfare effects of recessions depend on the gain parameter γ. When the gain parameter is larger, the welfare cost is also larger. This is because agents place more weight on recent data when the gain parameter is high, and so they expect prices to be more depressed following a one period recession than agents using a smaller gain. The minimum consumption equivalent variation that calculated how much would need to added to period consumption such that agents would be indifferent between the recession and the initial steady state is 1.13% when $\gamma = 0$ and 1.99% when $\gamma = 1$ for learning agents using the longest planning horizon. Although the most harmed cohort is better off when the gain parameter γ is smaller, this is not the case for all cohorts. The economy recovers and converges back to the steady state more slowly when the gain parameter is small, thus more cohorts are born into a world with depressed capital (and thus lower consumption) when the gain parameter is small. One way to illustrate this cost to calculate the cumulative consumption equivalent variation across many generations. This cumulative consumption equivalent variation is 6.55% in the rational expectations model, 8.39% in the learning model (with the longest planning horizon) when $\gamma = 1$, and 40% when $\gamma = 0$ (summing over 100 generations). This contrast is presented in Figure 13 which plots the consumption equivalent variations for a one period recession (A falls by 5%) with $\gamma = 1$ and $\gamma = 0$. Note that the CEV eventually converges back to zero as the economy converges back to the steady state.

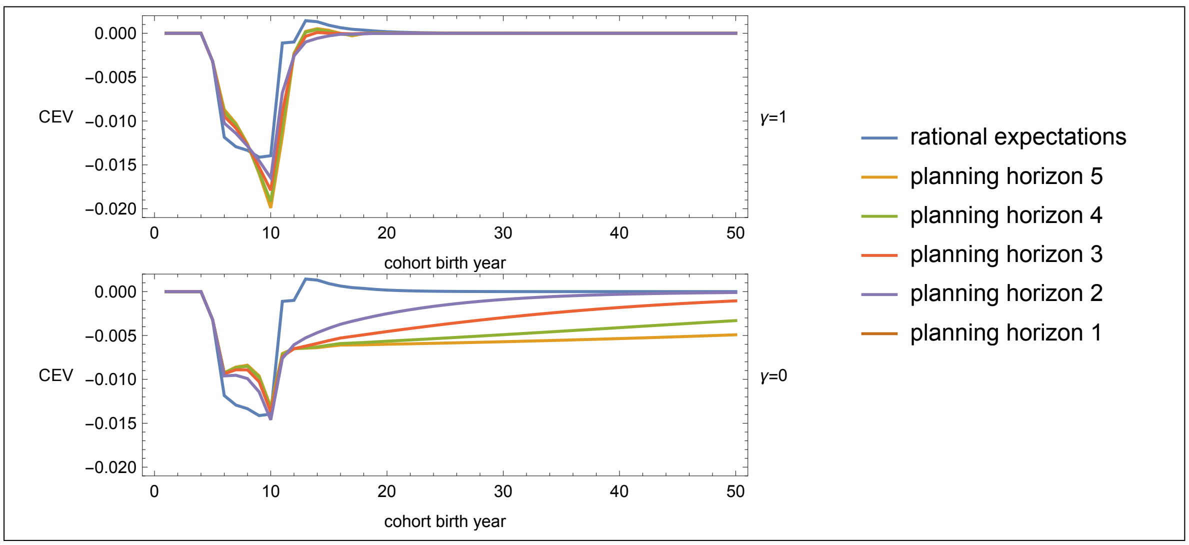

Figure 13: Consumption equivalent variation for an economy with a recession in period $t = 10$ that lasts for 1 period, for different learning parameters: $\gamma = 1$ in the top figure, $\gamma = 0$ in the lower figure. The CEV compares utility for each cohort to the utility of living in the initial steady state. The population growth rate is constant at $n = 0.01$, and social security policy is constant $\tau^0 = 0.1516, \tau^1 = 0.045$, $\phi = 0.4$. The recession is a surprise reduction in the TFP factor from $A = 10$ to $A = 9.5$ in period $t = 10$. The TPF factor is depressed for a single period, and the returns to the pre-recession level.

In the learning model, the ability of households to respond to changes in government debt depends partially on the gain parameter γ. Specifically, if the gain parameter is zero ($\gamma = 0$) households do not update their forecasts, and thus cannot respond directly to changes in government debt. This is important because when the Leeper tax is positive and government debt is non-zero, the after-tax take-home pay of households depends on government debt. The interaction of the gain parameter γ and the Leeper tax τ^1 is explored in three detailed examples in Appendix C.

5.1.1 Choice of Gain Parameter

In the baseline parameterization of the model, I choose a relatively large gain parameter $\gamma = 0.93$. This gain parameter minimizes the cost to agents of making forecast errors. To make this selection, I calculate the consumption of an "infinitesimally rational" agent in the finite horizon life-cycle learning model. An infinitesimally rational agent is a single agent in the learning model who is able to forecast the endogenous variables perfectly–that is, she can see the entire path of the economy with perfect foresight. She is such a small part of the economy that she does not change prices. Thus she experiences the dynamics induced by the adaptive learning, but she does not make any forecast errors. I construct a consumption equivalent variation that equates the utility of consumption of the infinitesimally rational agent with the utility of consumption of a finite horizon learning agent using the longest possible planning horizon. The consumption equivalent variation is always negative because the agents using adaptive learning have lower utility than the infinitesimally rational agent. I select the gain parameter γ that minimizes the welfare cost of learning for the most harmed cohort. The gain parameter of $\gamma = 0.93$ minimizes the welfare cost of making forecast errors in the particular scenario that is depicted in figure 5 (a demographic transition followed by a social security tax increase.) The welfare cost for the most harmed cohort is relatively small (less than 0.3% of consumption), suggesting that agents are not overly harmed by their forecast errors. Table 2 shows the minimum consumption equivalent variation equating the utility of consumption of the infinitesimally rational agent to the learning agent for various gain parameters γ for a social security tax increase and also for a social security benefit cut.

Infinitesimally rational agent comparisons

Tax increase			Benefit cut		
γ	min CEV	cohort	γ	min CEV	cohort
<0.1	-3.49%	2270	0	-0.76%	1990
0.1	-2.56%	2056	0.1	-0.52%	1990
0.2	-1.57%	2040	0.2	-0.44%	2030
0.3	-1.12%	2030	0.3	-0.44%	2030
0.4	-8.44%	2020	0.4	-0.42%	2020
0.5	-0.64%	2020	0.5	-0.34%	2020
0.6	-0.48%	2010	0.6	-0.28%	2010
0.7	-0.40%	2010	0.7	-0.24%	2010
0.8	-0.31%	2010	0.8	-0.19%	2010
0.9	**-0.29%**	**2000**	**0.9**	**-0.18%**	**2000**
1	-0.33%	2000	1	-0.23%	2000

Table 2: Minimum consumption equivalent variation equating the utility of the infinitesimally rational agent with the learning agent (using the longest possible planning horizon). In both cases a demographic change begins in 1980, social security taxes are increases in 2030 (on the left of the table), or social security benefit are cut in 2030 (on the right). The year refers to the birth year of the cohort experiencing the minimum CEV (the largest welfare cost). Note that for the tax increase example, the minimum gain parameter γ that will converge following a demographic shock with is 0.083. The policy parameters correspond to figure 5 and 8.

It is unsurprising that the optimal gain parameter is large for this example, since the demographic change and policy change are both permanent. Recall, when $\gamma = 1$ agents fully update their expectations each period. This forecasting rule is rational when all shocks are permanent. The optimal gain parameter of $\gamma = 0.93$ is not equal to one because the selection criteria favors a gain parameter than minimizes the welfare cost for the most harmed cohort, without considering other cohorts. The optimal gain parameter is equal to one if the selection criteria is minimizing the cumulative welfare cost across all generations.

The optimal gain parameter is smaller in the context of a recession. The TFP shock that causes the recession is temporary, thus a smaller gain parameter is better for households. The optimal gain parameter is not $\gamma = 0$, however, since the effects of the recession linger for more than one period. The gain parameter that minimizes the welfare cost of making forecast errors (as measured by the infinitesimally rational consumption equivalent variation) is close to $\gamma = 0.3$ for a variety of policy parameterizations. This welfare cost is presented in Table 3 for three different recessions: the baseline parameterization (with social security and a Leeper tax), a social security system with no Leeper tax ($\tau^1 = 0$), and no social security.

Infinitesimally rational agent comparisons

Recessions					
Baseline		No Leeper Tax		No Social Security	
γ	min CEV	γ	min CEV	γ	min CEV
0	-0.10%	0	-0.05%	0	-0.33%
0.1	-0.07%	0.1	-0.38%	0.1	-0.03%
0.2	-0.05%	0.2	-0.04%	0.2	-0.25%
0.3	**-0.05%**	**0.3**	**-0.37%**	**0.3**	**-0.02%**
0.4	-0.05%	0.4	-0.04%	0.4	-0.02%
0.5	-0.06%	0.5	0.05%	0.5	-0.24%
0.6	-0.61%	0.6	-0.06%	0.6	-0.25%
0.7	-0.70%	0.7	-0.07%	0.7	-0.27%
0.8	-0.80%	0.8	-0.08%	0.8	-0.29%
0.9	-0.92%	0.9	-0.09%	0.9	-0.03%
1	-0.11%	1	-0.11%	1	-0.04%

Table 3: Minimum consumption equivalent variation equating the utility of the infinitesimally rational agent with the learning agent (using the longest possible planning horizon) for a one period recession. The left columns show the baseline parameterization with social security and a positive Leeper tax (as in figure 9). The middle two columns show an alternative policy parameterization with no Leeper tax (that is $\tau^1 = 0$ and taxes do not respond directly to debt). In the right columns, there is no social security system ($\tau^0 = \tau^1 = \phi = 0$). In all three, the welfare cost is minimized for $\gamma = 0.3$.

5.2 Alternative Information Assumptions

In this section, I consider robustness exercises that change the information set of agents in the subsections that follow. In section 5.2.1, I model an agent who does not understand the details of the tax system and uses an adaptive rule to forecast future taxes. In section 5.2.2, I model an agent who forecasts taxes and social security benefits adaptively; that is, the agent does not respond to announcements, but instead waits to observe policy changes before updating her expectations.

5.2.1 Forecast tax burden

In main specification, agents fully understand the Leeper-tax and forecast government debt levels in order to estimate their upcoming taxes. I back away from that assumption in this section and assume that agents do not fully understand the tax system. Agents observe the total tax burden (rate) they face in the current period, and they use adaptive learning to forecast their future tax burden. They do not incorporate knowledge about the structure of the tax system into their forecasts. Agents continue to forecast social security benefits as the replacement rate ϕ times the expected wage at the time of retirement.

Agents in this framework understand that a fraction of their wage is taken by the government every period, but they do not distinguish between the Leeper tax and the baseline payroll tax. Agents are not forward looking with regard to tax changes. They are, however, forward looking with regard to social security benefit changes. This assumption will be relaxed in the next section. Under this specification, agents forecast:

$$
\begin{aligned}
w_{t+1}^e &= \gamma w_t + (1-\gamma) w_t^e \\
R_{t+1}^e &= \gamma R_t + (1-\gamma) R_t^e \\
\tau_{t+1}^e &= \gamma(\tau_t^0 + \tau_t^1 b_t) + (1-\gamma)\tau_t^e
\end{aligned}
$$

with $\gamma \in (0,1)$. Forecasts many periods ahead are equal to the one-step-ahead forecasts: $x_{t+j}^e = x_{t,t+j}^e = x_{t+1}^e$, for $x = R, w, \tau$ and $j > 1$.

Agents do not respond to an announced tax change until the change is implemented. The households do not know how to incorporate the announced tax change into their forecasts, and so they wait until the change has been implemented and update their forecasts adaptively. The swings in capital stock that follow the policy change are larger than in the baseline learning model when agents anticipate tax changes.

This information assumption does not change the behavior of households in response to announced benefit changes relative to the baseline model. Households still respond to benefit changes as the policy change enters their planning horizon.

5.2.2 Forecast Policy Adaptively

As a final robustness check, suppose that agents do not anticipate any policy changes, but rather agents learn all policy adaptively. Agents forecast their tax burden $(1-\tau_t)w_t$ and social security benefit, z_t adaptively. Agents expect to receive the same social security benefit in both periods of retirement (which is consistent with the actual policy process). They forecast social security benefits for the period in which they retire. Agents forecast:

$$\begin{aligned} w^e_{t+1} &= \gamma w_t + (1-\gamma) w^e_t \\ R^e_{t+1} &= \gamma R_t + (1-\gamma) R^e_t \\ \tau^e_{t+1} &= \gamma(\tau^0_t + \tau^1_t b_t) + (1-\gamma)\tau^e_t \\ z^e_{t+1} &= \gamma \phi_t w_t + (1-\gamma) z^e_t \end{aligned}$$

with $\gamma \in (0,1)$. Forecasts many periods ahead are equal to the one-step-ahead forecasts: $x^e_{t+j} = x^e_{t,t+j} = x^e_{t+1}$, for $x = R, w, \tau, z$ and $j > 1$.

Under this specification, agents in learning models do not respond to announced policy changes. They only respond to tax or benefit changes when they have been implemented. The transition dynamics following a tax change are identical to the previous section. However, the transition dynamics for benefit changes differ. Using the set-up from the previous section, suppose the population growth rate is initially high, and then falls in 1980, leading to an accumulation of government debt until social security benefits are cut in 2030. Agents for forecast policy adaptively will not respond to the reduction in benefits until after the change is implemented. Thus, the consumption of the older generations as the reform is implemented is much lower.

5.3 Sensitivity Analysis

The qualitative results of this paper are sensitive to parameter choice. As a robustness check, I calibrated the model to be dynamically *inefficient* and ran similar experiments (raising taxes or lowering social security benefits). I parameterize a dynamically inefficient model by setting the discount factor β greater than one. This increases agents desire to consume in old age and drives up savings, and thus the capital stock. The results of the model are similar under this dynamically inefficient specification.

6 Conclusion

The aging of the U.S. population (as well as the populations of many other countries) makes public pension reform likely. Standard, representative agent infinite horizon models cannot model pensions, rather life-cycle or overlapping generations models are required. The majority of papers analyzing public pension reforms, demographic changes, and life-cycle savings rely on rational expectations. I relax this assumption and model agents who forecast future interest rates and wages adaptively. I develop a model of finite horizon life-cycle learning, in which agents plan over a finite horizon (potentially shorter than their life-cycle) and make optimal choices conditional on their forecasts and the assets they plan to hold at the end of their planning horizon. This model embeds the main ideas from finite horizon optimal learning into an overlapping generations model with finitely lived agents.

Using the learning model, I demonstrate that the high capital steady state in the overlapping generations model with debt and capital is stable under finite horizon life-cycle learning and that the low capital steady state is not stable under learning. I also provide examples of transition dynamics for the economy using finite horizon life-cycle learning in response to social security policy changes, and also in response to a recession. The learning mechanism introduces cyclical dynamics into the model that leads the capital stock and government debt to oscillate before reaching a new steady state. The oscillations alter the path of the economy following a social security policy change and deepen recessions.

The learning model developed in this paper includes endogenous government debt and can accommodate demographic changes. Several interesting questions that are beyond the scope of this paper can be addressed in this framework. I plan to explore the relationship between delaying social security reform, growing deficits, and explosive government debt in future work. The learning model developed in this paper provide a framework to examine explosive debt that is not possible in a standard, rational expectations framework.

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A Rational Expectations Equilibrium

Written in efficient terms, the equilibrium equations include the asset market clearing equation, the government debt equation, and the $J-1$ household first order conditions:

$$(k_{t+1}+b_{t+1})(1+n_t) = \frac{\sum_{j=1}^{J} N_{t+1-j} a_t^j}{L_t}$$

$$(1+n_t)b_{t+1} = R_t b_t + \frac{\sum_{j=T}^{J} N_{t+1-j}\phi_t w_{t+T-j}}{L_t} - (\tau_t^0 + \tau_t^1 (B_t/L_t))w_t$$

$$(R_t a_{t-1}^{j-1} + y_t^j - a_t^j)^{-\sigma} = \beta E_t[R_{t+1}(R_{t+1}a_t^j + y_{t+1}^{j+1} - a_{t+1}^{j+1})^{-\sigma}]$$

$$\text{for } j = 1, \cdots, J-1.$$

Here y^j indicates period labor income during working life and social security income after retirement: $y_{t+j-1}^j = (1-\tau_{t+j-1})w_{t+j-1}$ for $j < T$ and $y_{t+j-1}^j = z_{t+j-1}$ for $j \geq T$. Note that $a^0 = 0$, and factor prices are given by (3) and (4).

There are three markets in this economy: assets, labor, and goods. The asset market clearing equation is included in the equilibrium equations above, the labor market clears when the labor used by the firm is equal to the exogenous labor supply, and the goods market clears by Walras law (and thus is not included in the equilibrium equations above).

The equilibrium definition above accommodates a time-varying population growth rate n_t. In a steady state, the growth rate of the population is constant. When the growth rate is constant, several terms can be written concisely by defining $\nu = N_t/L_t = (\sum_{j=1}^{T-1}(1+n)^{1-j})^{-1}$.

Definition 2 *The steady state is a collection* $\{k, b, a^1, \cdots, a^J\}$ *that solves:*

$$(k+b)(1+n) = \nu \sum_{j=1}^{J} (1+n)^{1-j} a^j$$

$$(1+n)b = R(k)b + \nu \sum_{j=T}^{J} (1+n)^{1-j}\phi_t w(k) - (\tau^0 + \tau^1 b)w(k)$$

$$(R(k)a^{j-1} + y^j - a^j)^{-\sigma} = \beta \left[R(k)(R(k)a^j + y^{j+1} - a^{j+1})^{-\sigma}\right]$$

$$\textit{for } j = 1, \cdots, J-1$$

with $y^j = (1-(\tau^0+\tau^1 b))w(k)$ *for* $j < T$, $y^j = \phi w(k)$ *for* $j \geq T$, $a^J = 0$, *and factor prices* $R(k)$ *and* $w(k)$ *given by (3) and (4). Note in steady state* $w_t = w(k)$ $\forall t$, *so the average indexed period earnings for all cohorts are simply equal to the wage thus steady state social security benefits are simply* $\phi w(k)$.

B Saddle Node bifurcation

The equilibrium of the model is not unique. There are at most two steady states, and at fewest, zero. The number and stability of the steady states depends on the parameters of the model. The change in the number of steady states is called a saddle-node bifurcation.

The intuition of the bifurcation in this model is that as a given parameter changes (like the payroll tax rate τ^0 or the population growth rate n), it impacts the social security deficit or surplus. Suppose the model is calibrated such that two steady states exist. As the deficit increases (endogenously in response to a parameter change), this increases government debt and crowds out capital, pushing the two steady states closer together. At a critical value of the parameter, only one steady state exists; beyond that, deficits are too large, government debt is explosive, and no steady states (and no equilibria) exist.

I analyze the determinacy of equilibria by linearizing the model around a steady state and computing the eigenvalues of the linearized system (see Laitner (1990)). There are three predetermined variables in the model (k, b, and a^{J-1}) and J-2 free variables (a^1,...,a^{J-2}). There are three possible cases. If three eigenvalues of the linearized system are less than one in modulus and the remaining $J-2$ have modulus greater than one, the system is determinate. If more than three eigenvalues lie inside the unit circle, the system is indeterminate. If more than $J-2$ eigenvalues lie outside the unit circle, the system is explosive. I have confirmed numerically that when two steady states exist, the steady state with higher capital is stable (determinate), and the lower capital steady state is explosive. Chalk (2000) also finds that the high capital steady state is determinate and the low capital steady state is explosive in a many-period OLG model with debt and capital. See also Azariadis (1993) (Chapter 8 and Chapter 20) for a discussion of saddle-node bifurcations in planar OLG models.

C Gain Parameter and Leeper Tax

Social security is funded in the model using a payroll tax with two components: $\tau_t = \tau_t^0 + \tau_t^1 b_t$, a baseline payroll tax rate τ^0, and a Leeper tax τ^1 that responds to government debt. If government debt and the Leeper tax are both non-zero, the payroll tax paid by the household is influenced by the government's debt.

The assumption of a Leeper tax in this paper is useful for two reasons: first it allows a calibration of the model that is dynamically efficient with positive government debt. Without a Leeper tax the government could only hold debt in a dynamically inefficient steady state. Second, the Leeper tax can help the rational expectations model and the learning algorithm converge by increasing the basin of attraction for the stable steady state. Convergence in the rational expectations and learning models depends on the size of the shock (or policy change), the initial government debt, the learning gain parameter γ, and the Leeper tax τ^1.

As discussed in section 3, the finite horizon life-cycle learning model is E-stable for all planning horizons. This means that the learning dynamics converge to the rational expectations stable steady-state, given arbitrary initial conditions near the steady state. E-stability does not, however, guarantee that the learning dynamics will converge to the stable steady-state from *any* initial condition. Thus, when a policy change, demographic change, or productivity shock occurs, it is possible that the state variables will be too far away from the (new) steady state and will not converge.[23] A small, positive Leeper tax is helpful in the model because it can slow the accumulation of government debt following a policy change or shock and thus keeps debt near the stable steady state. The gain parameter γ also interacts with the Leeper tax. When the gain parameter is larger, households place more weight on current observations and this can also slow the accumulation of government debt. I will illustrate this interaction with three examples. In the fist example, the gain parameter and Leeper tax are both greater than zero (as in the baseline parameterization of the paper). In the second example, the gain parameter is zero while the Leeper tax is positive. In the final example, the gain parameter is positive while the Leeper tax is zero.

C.1 Gain parameter and Leeper tax both positive

In main specification of this paper, the adaptive learning agents fully understand the policy process. They anticipate any announced policy changes that fall within their planning horizon and they also understand how changes to the tax rate impact their take-home pay. The agents understand the Leeper component of the tax. As long as the gain parameter γ is not equal to zero, the households will update their

[23] This is also true in the rational expectations model; however, it is possible to calibrate many examples where the rational expectations model converges and the learning model does not.

expectations about debt (and thus take-home pay via the Leeper tax) following a policy change or shock.

As an example, consider the announced benefit cut following the demographic shock, as in section 4.3. In the learning model, the agents have no way of incorporating the demographic change into their forecasts; they respond to the general equilibrium changes by updating their forecasts. The households incorporate the announced policy change as soon at it enters their planning horizon. Figure C.1 depicts the household's expectations along with the data in the learning model. Since the gain parameter is large, the households' expectations update quickly and closely track the changes in the actual data. The relationship between expectations and data is similar when a longer planning horizon is used. The paths of the interest rate, wage, capital, and bonds for this exercise for each of the five possible planning horizons are depicted in Figure C.2. Bonds increase more rapidly following the demographic in the learning models than in the rational expectations model, but the combination of the Leeper tax and the high gain parameter keep bonds from exploding.

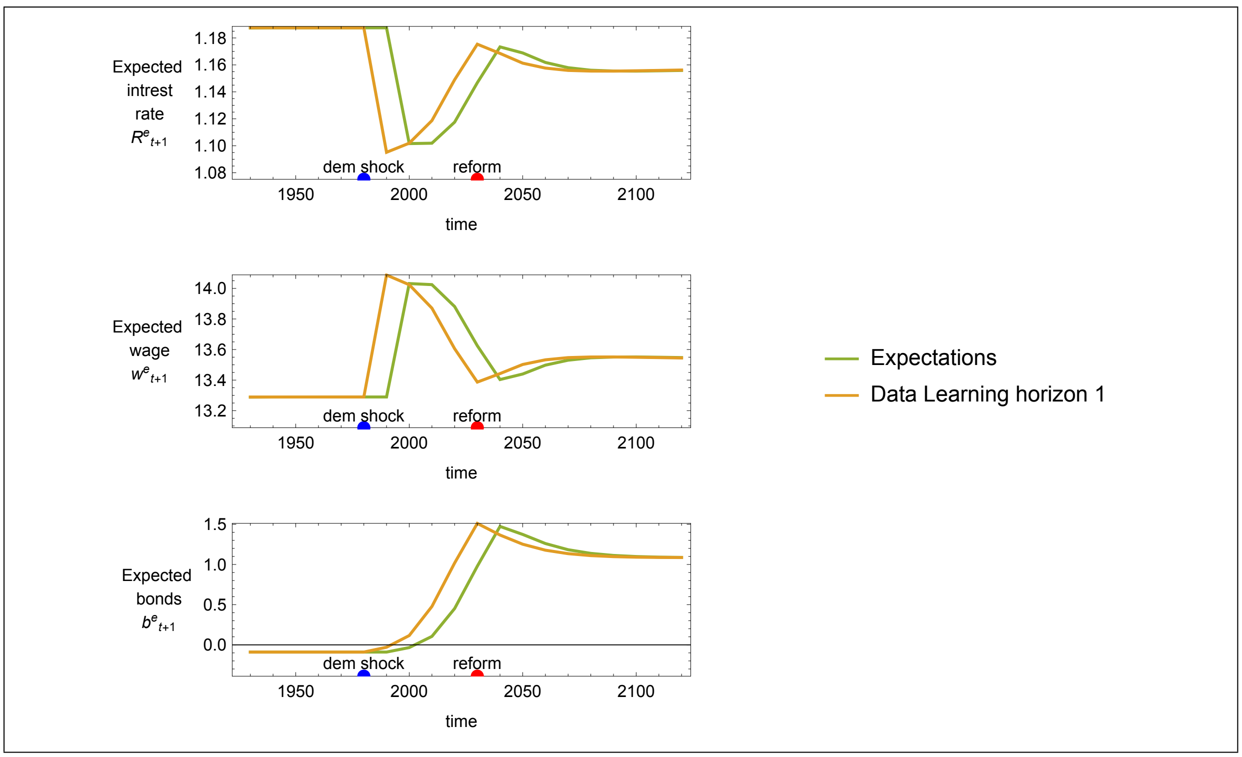

Figure C.1: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1. The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The Leeper tax is positive ($\tau^1 = 0.045$) and the gain parameter $\gamma = 0.93$.

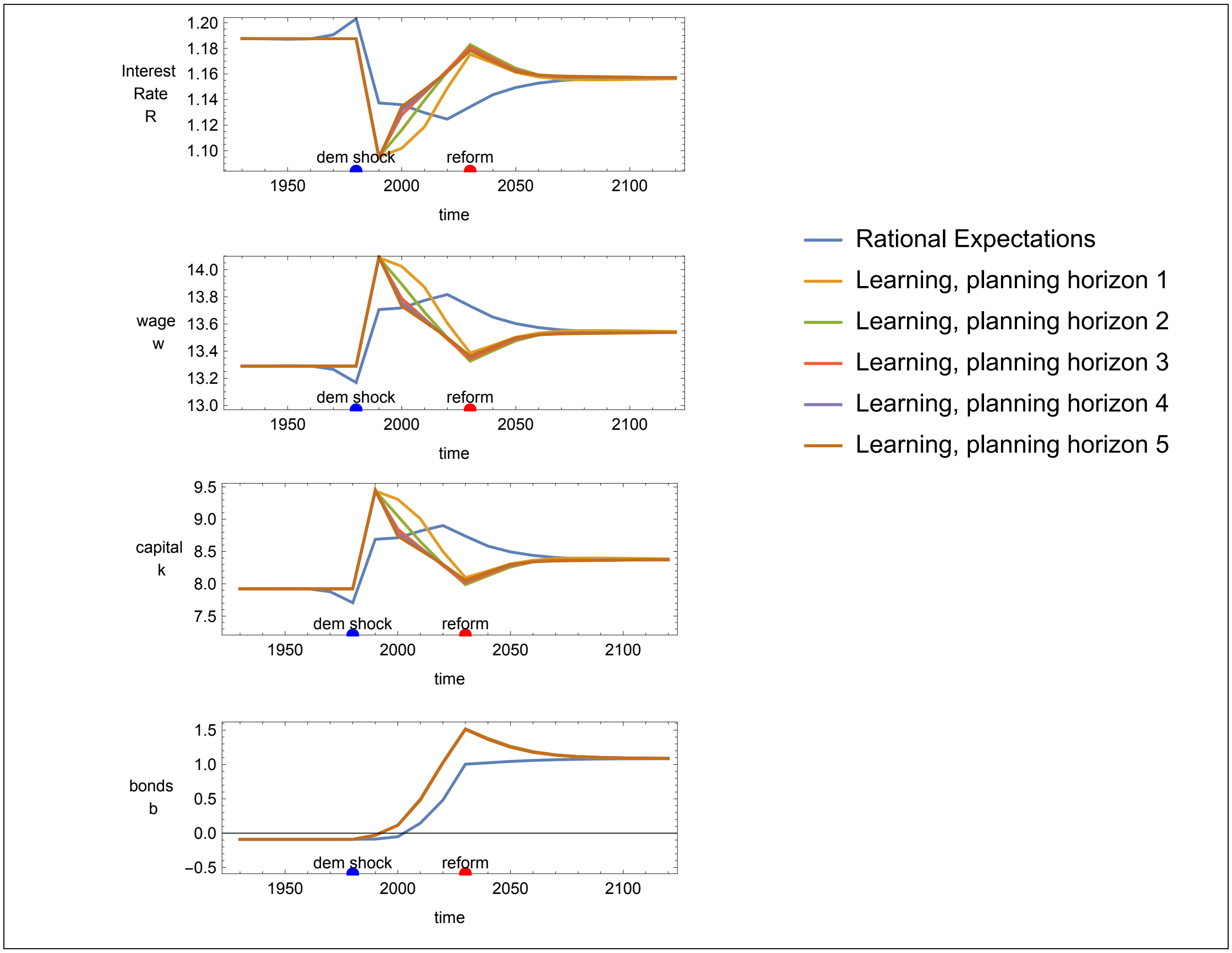

Figure C.2: Time paths for interest rate, wage, capital, and bonds in the finite horizon life-cycle model (for different planning horizons) and in the rational expectations model (blue line). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The Leeper tax is positive ($\tau^1 = 0.045$) and the gain parameter $\gamma = 0.93$.

C.2 Gain parameter equal to zero

As a second example, consider when the gain parameter is set to zero ($\gamma = 0$). In this case, agents do not update their expectations. This is problematic if there has been a permanent change, such as a demographic shift or a policy change. The agents continue forecasting $x^e_{t+1} = x^e_t$ for $x \in \{R, w, b\}$, even though those variables have changed. The agents' persistent, incorrect expectations prevent the economy from converging to the rational expectations steady state. The learning models converge to a forecast-horizon specific fixed-point.

This is illustrated in the following example: the economy experiences a demographic change in 1980 and social security benefits are cut in 2030. The Leeper tax is positive. The learning gain parameter $\gamma = 0$, despite the permanent demographic and policy changes, and thus the learning models do not converge to the rational expectations equilibrium. Figure C.3 shows that the household expectations remain fixed at their initial values and do not update over time. Figure C.4 shows the time paths for the interest rate, wage, capital, and bonds for this exercise. In this particular example, bonds do not grow infinitely because the benefit cut and the Leeper tax generate sufficient revenue to pay for the debt. The households consistently over-estimate the interest rate and underestimate the wage, which leads to over-saving (relative to the rational expectations model) which also prevents bonds from exploding. It is, of course, possible to generate an example where the learning models do not converge to a fixed point when the gain parameter is zero following a permanent shock (either because the Leeper tax is too small to stabilize the debt, or because households under-save relative to the rational expectations model). Note finally, that the learning model can converge to the rational expectations steady state when the gain parameter is zero following a temporary shock (such as a one period recession).

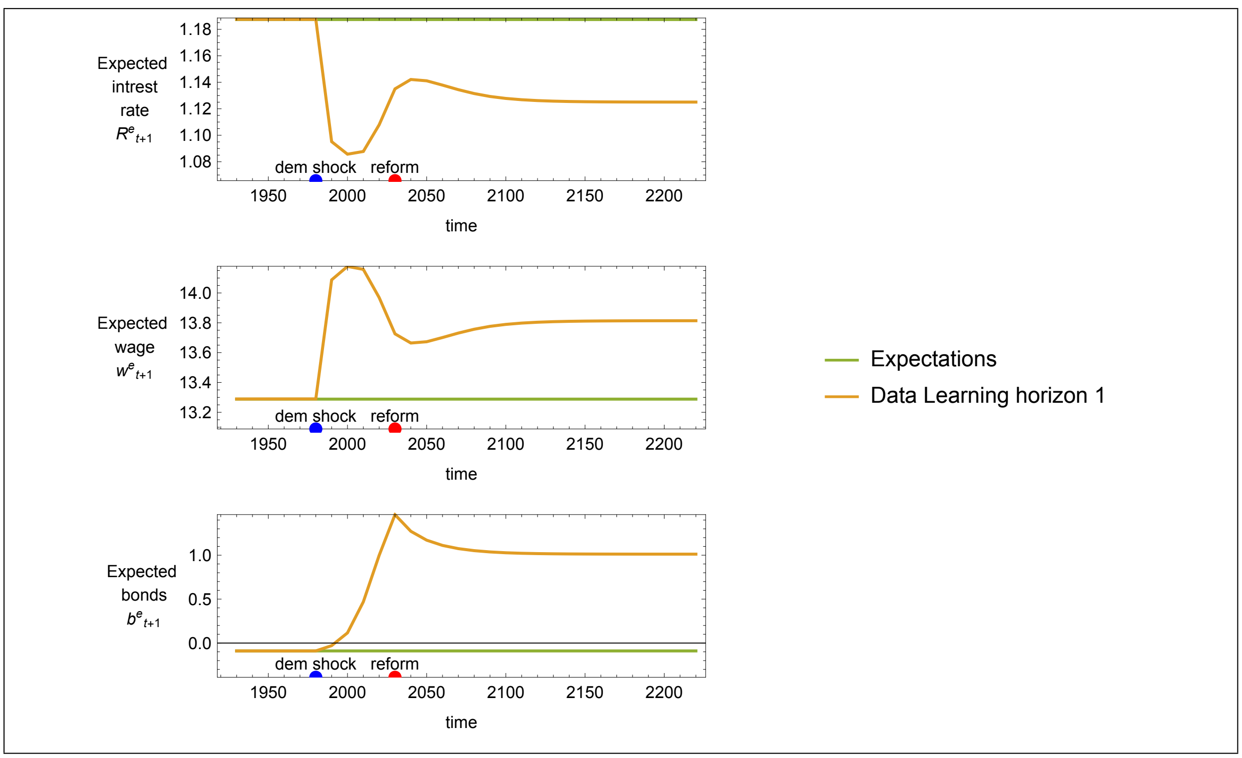

Figure C.3: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1. The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The Leeper tax is positive ($\tau^1 = 0.045$) and the gain parameter is zero $\gamma = 0$.

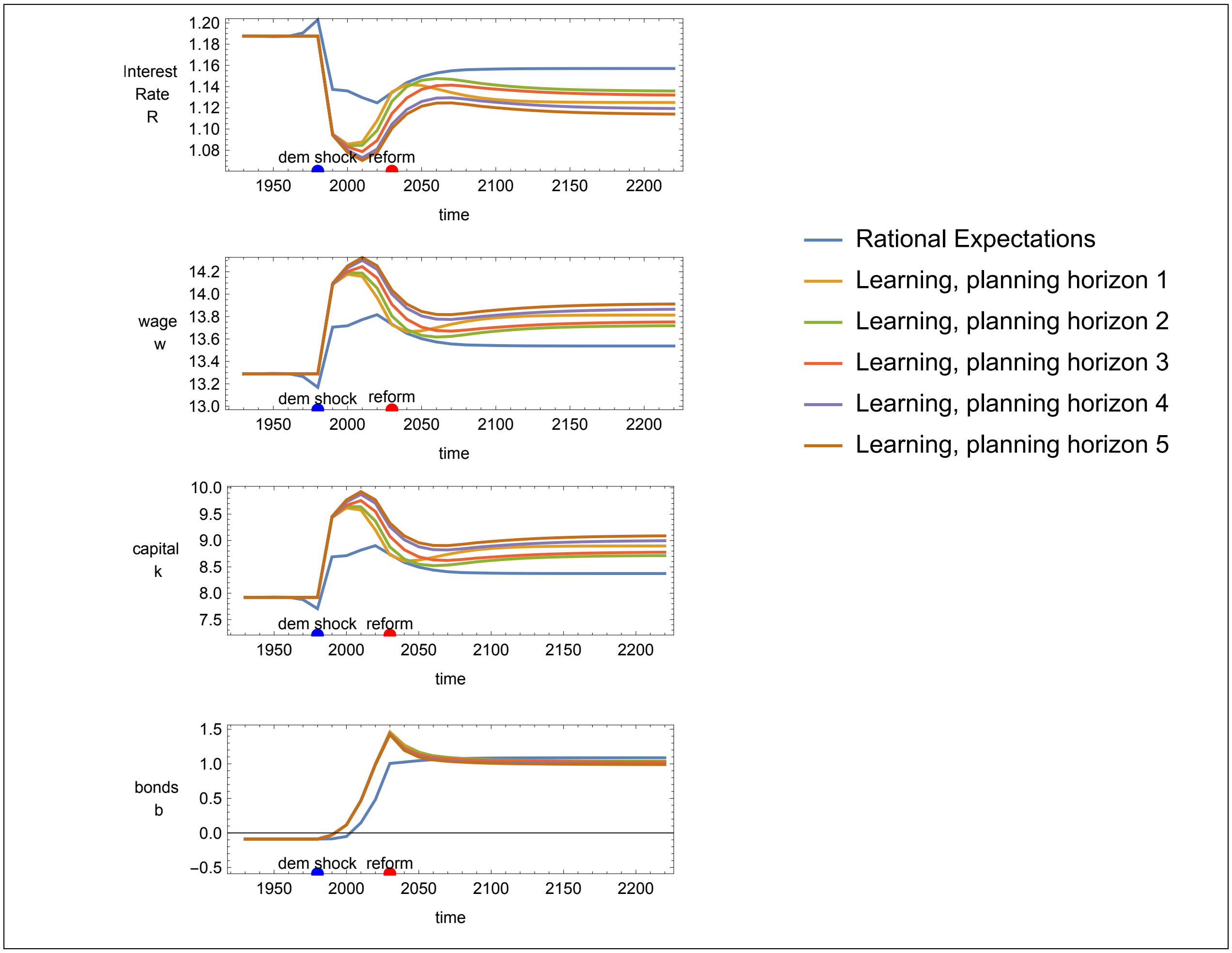

Figure C.4: Time paths for interest rate, wage, capital, and bonds in the finite horizon life-cycle model (for different planning horizons) and in the rational expectations model (blue line). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The Leeper tax is positive ($\tau^1 = 0.045$) and the gain parameter is zero $\gamma = 0$.

C.3 Leeper tax examples

As a final example of the interaction between the Leeper tax and the learning gain parameter γ, consider a case where the Leeper tax is used to stabilize explosive debt. In this example, the economy experiences a demographic shock in 1980, social security benefits are cut in 2030 (a larger benefit cut than in the previous examples), and the Leeper tax is zero. Bonds begin to grow after the demographic change and do converge following the policy change in the learning models. This is depicted in Figure C.5 which shows expectations and Figure C.6 which plots the path of the interest rate, wage, capital, and bonds. In these figures, the gain parameter is $\gamma = 0.93$.[24]

Introducing a positive Leeper tax to the example above can stabilize the debt. For example, in Figures C.7 and C.8, the Leeper tax $\tau^1 = 0.02$, which is sufficient to prevent bonds from exploding following the demographic change and benefit cut for any gain parameter.

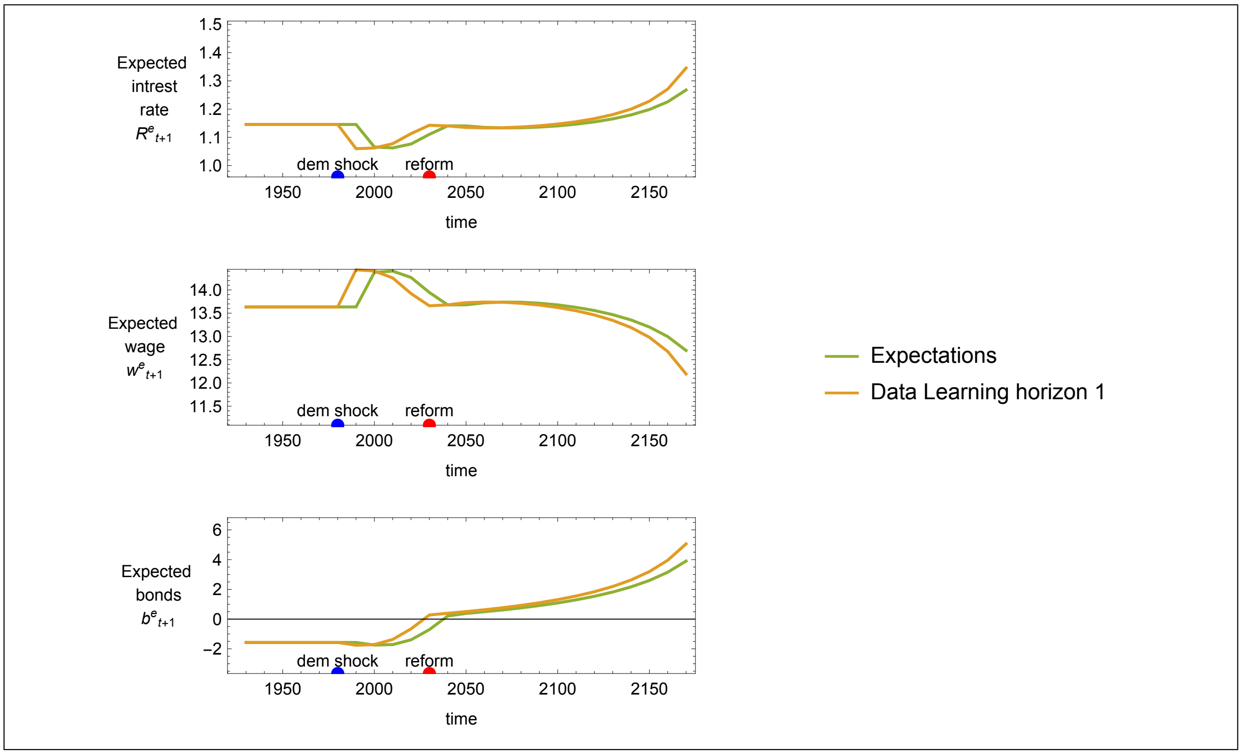

Figure C.5: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1, when the Leeper tax equals zero ($\tau^1 = 0$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Bonds grow infinitely (not an equilibrium path).

[24]Note that bonds explode for any gain parameter in this example.

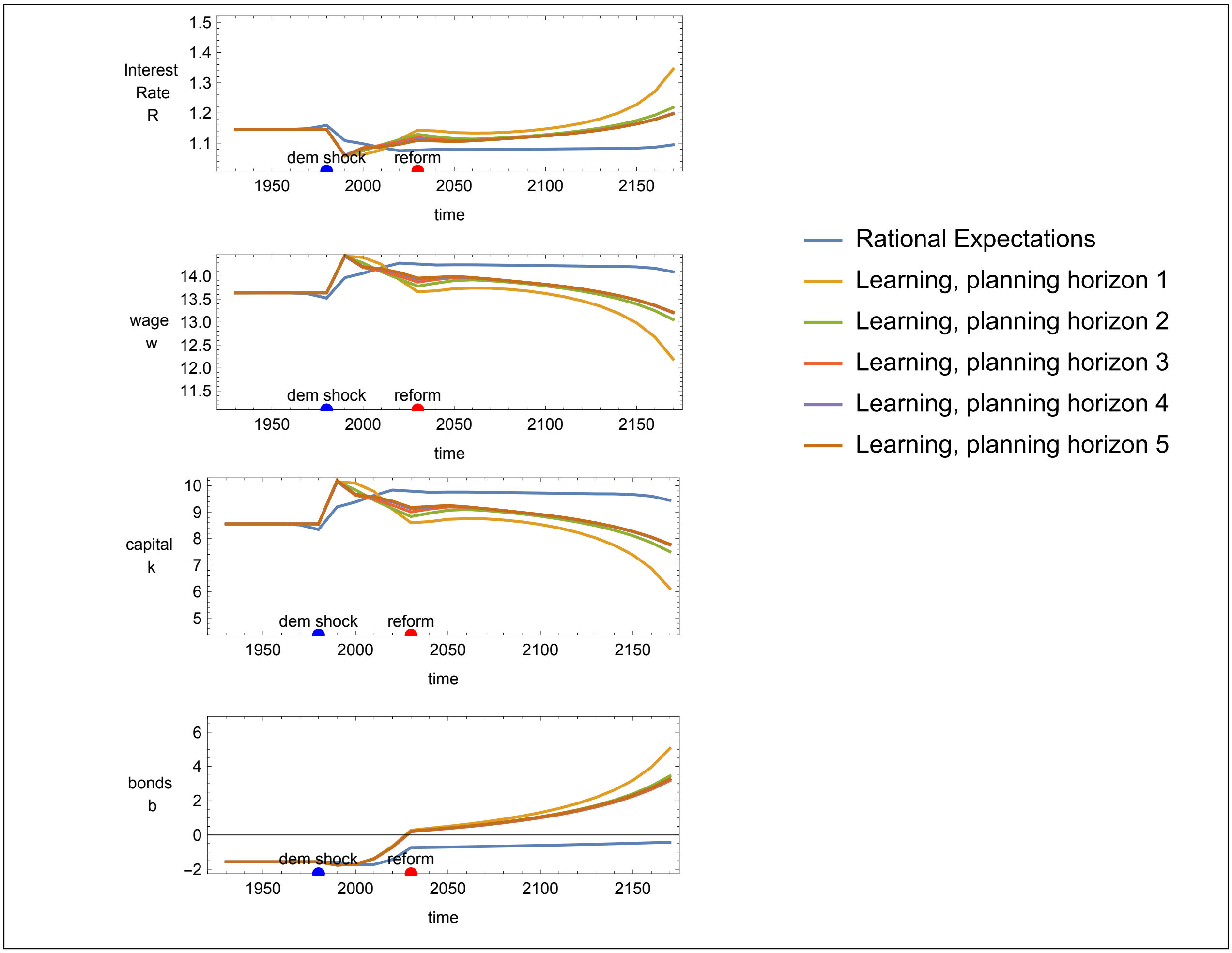

Figure C.6: Time paths for interest rate, wage, capital, and bonds in the finite horizon life-cycle model (for different planning horizons) and in the rational expectations model (blue line), when the Leeper tax equals zero ($\tau^1 = 0$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Bonds grow infinitely in the learning models.

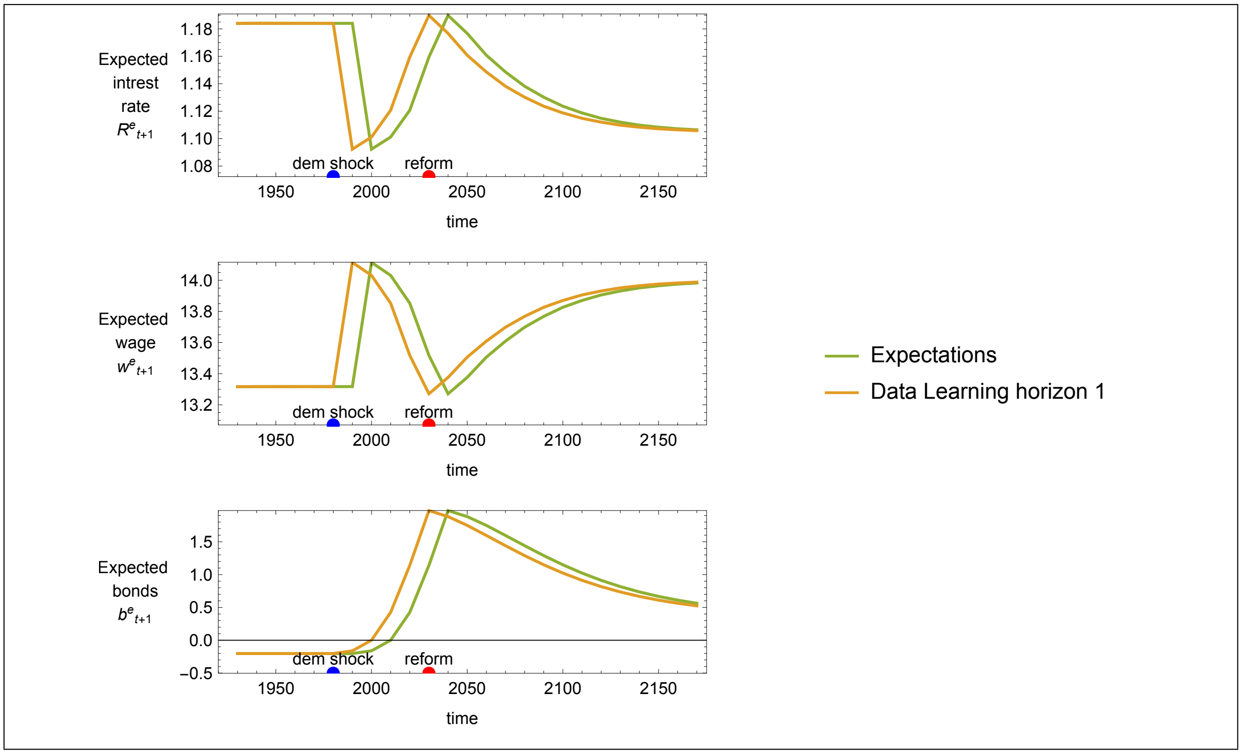

Figure C.7: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1, with a positive Leeper tax that stabilizes government bonds ($\tau^1 = 0.02$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Bonds would grow infinitely were it not for the Leeper tax.

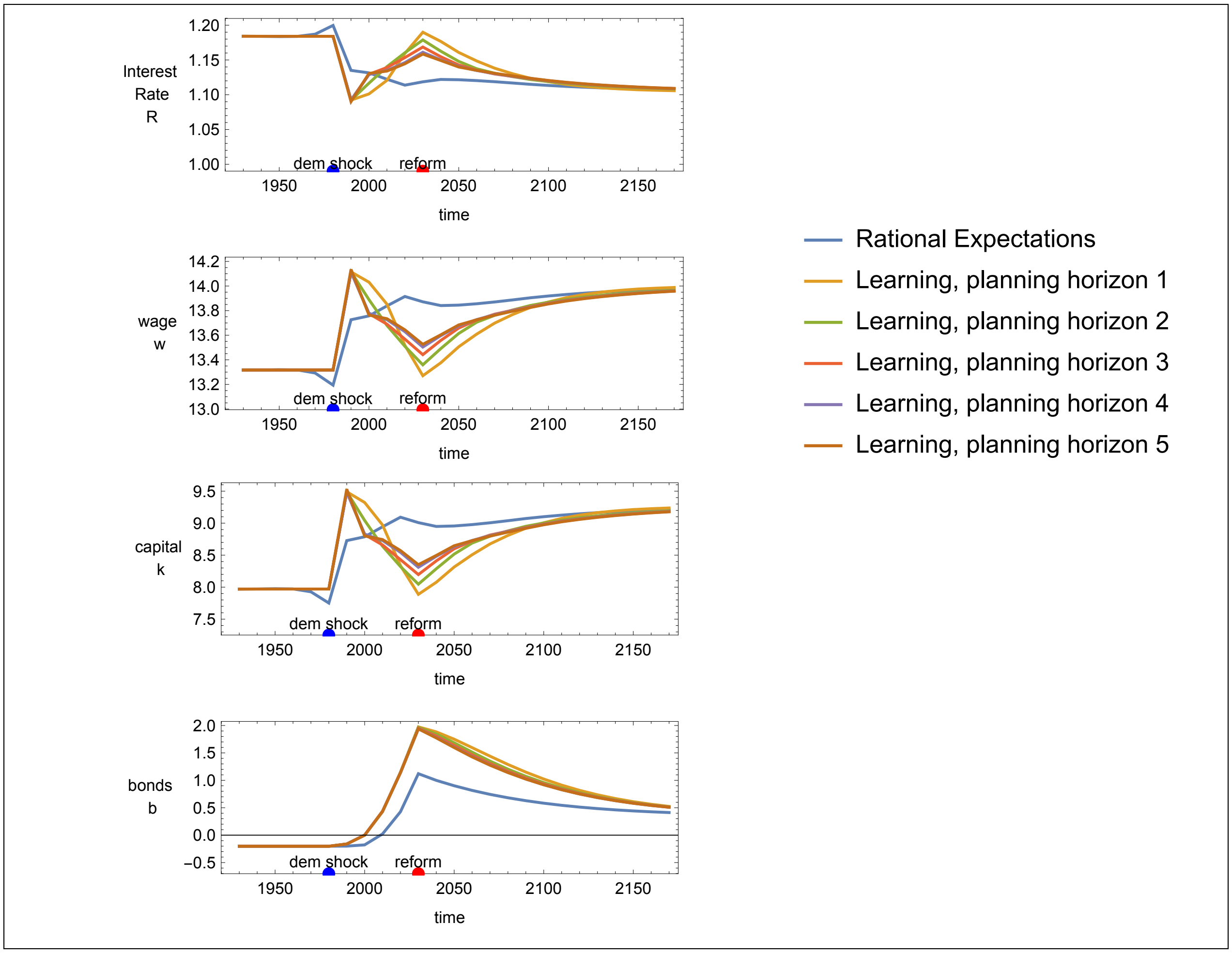

Figure C.8: Time paths for interest rate, wage, capital, and bonds in the finite horizon life-cycle model (for different planning horizons) and in the rational expectations model (blue line), with a positive Leeper tax that stabilizes bonds ($\tau^1 = 0.02$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Bonds would grow infinitely in the learning models, were it not for the Leeper tax.

If the Leeper tax is too large, it can lead to unstable oscillatory dynamics as depicted in figure C.10. In this is example, the Leeper tax is large enough that the endogenous government deficit oscillates between positive and negative values. The initial demographic change causes government debt to increase. When benefits are cut in 2030, this change combined with the Leeper tax, leads to a large endogenous surplus, which generates negative bonds. The following period, the total tax rate is lower (because the Leeper tax is multiplied by negative government bonds), which leads to a deficit and positive debt. This pattern continues with the debt oscillating further away from zero each period. This path is explosive.

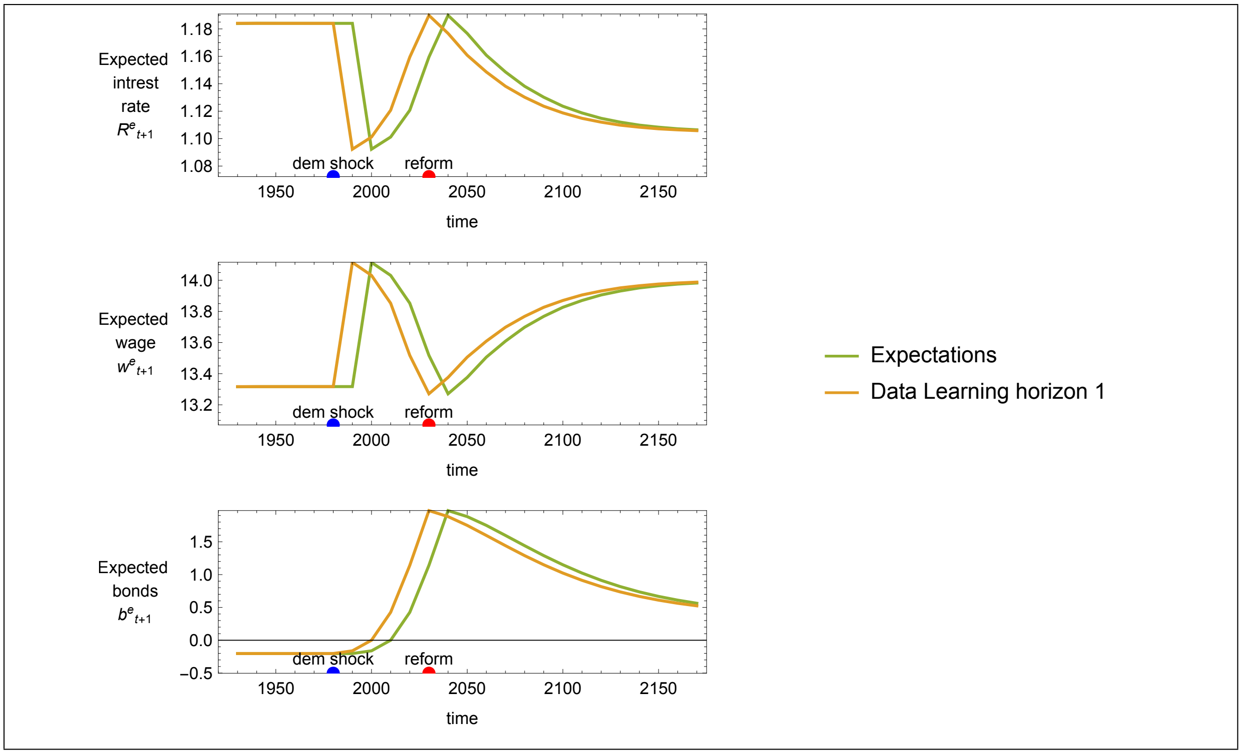

Figure C.9: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1, with a large positive Leeper tax that de-stabilizes government bonds ($\tau^1 = 0.16$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Explosive, oscillatory bond path.

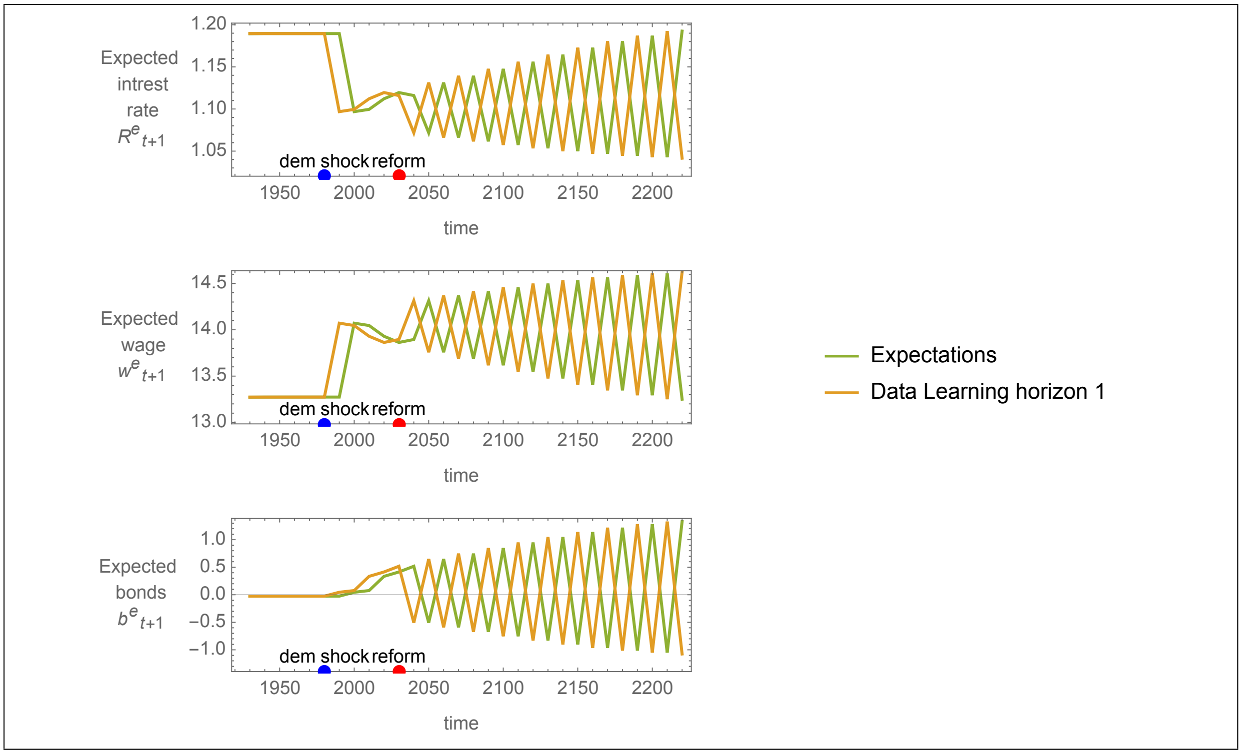

Figure C.10: Time paths for expected interest rate, wage, and bonds (green line) and actual interest rate, wage, and bonds (yellow line) in the finite horizon life-cycle model with a planning horizon of 1, with a large positive Leeper tax that de-stabilizes government bonds ($\tau^1 = 0.16$). The economy experiences a demographic change in 1980 and a social security benefit cut in 2030. The gain parameter is zero $\gamma = 0.93$. Explosive, oscillatory bond path.

The interaction of the learning gain parameter γ, the Leeper tax, and the planning horizon could be explored further in future work.