

Patterns, Permutations, and Placements

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Dartmouth College

Williams College - February 2014

Permutations

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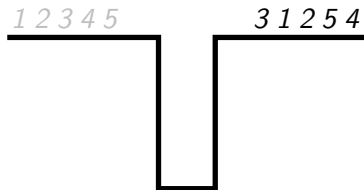
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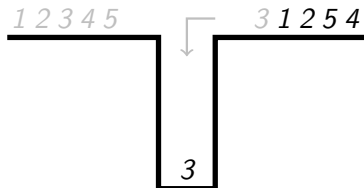


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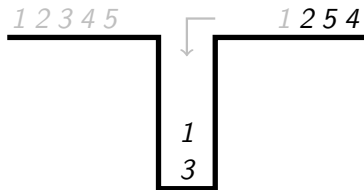


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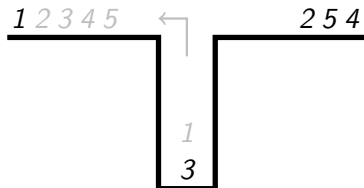


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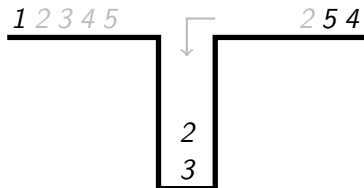


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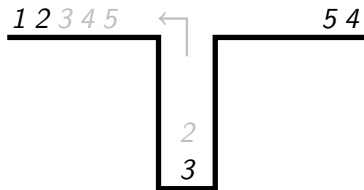


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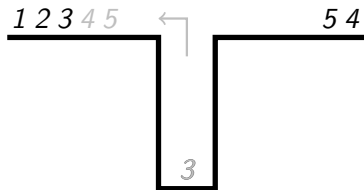


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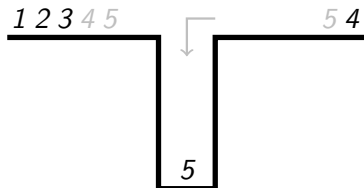


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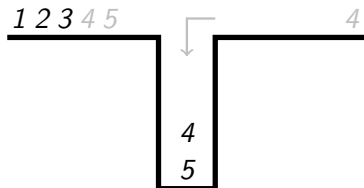


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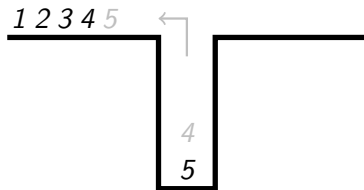


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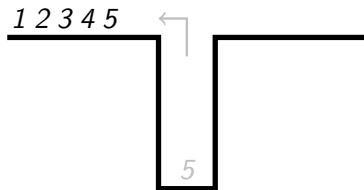


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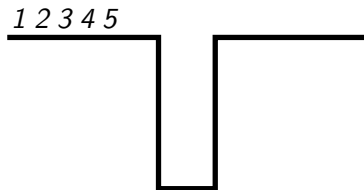
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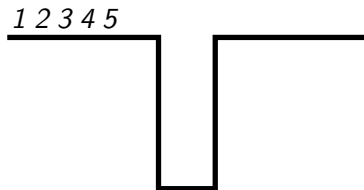
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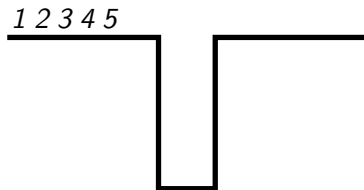
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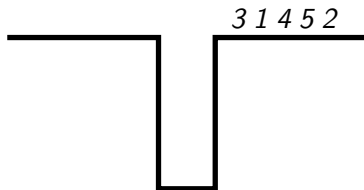
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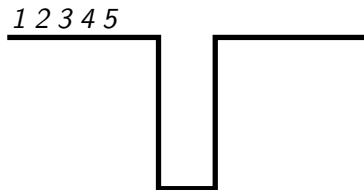
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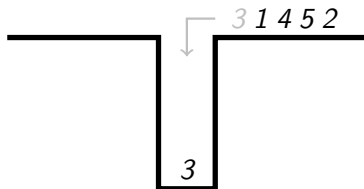
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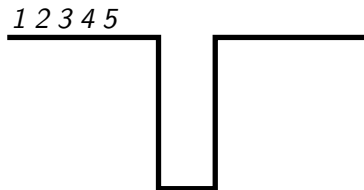
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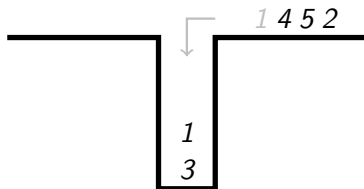
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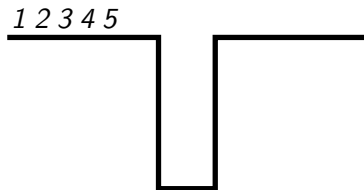
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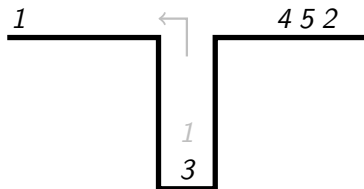
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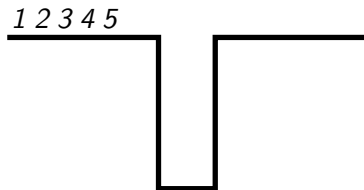
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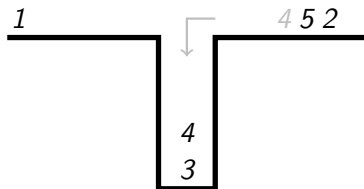
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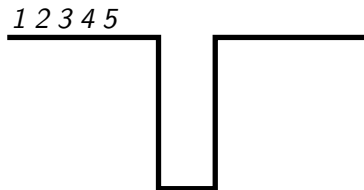
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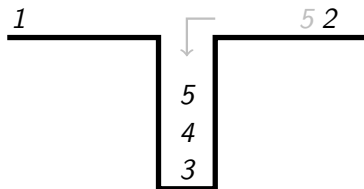
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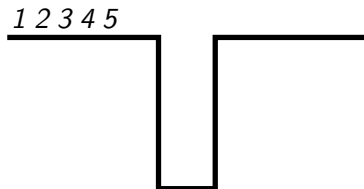
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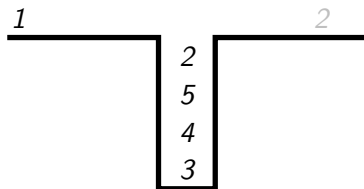
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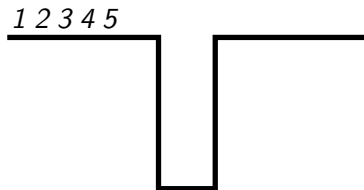
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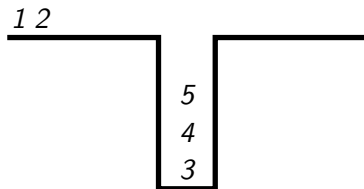
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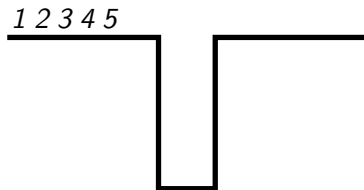
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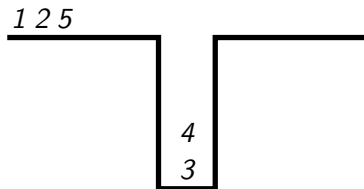
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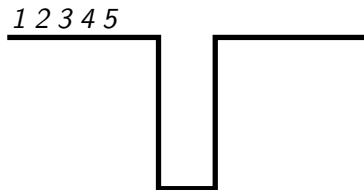
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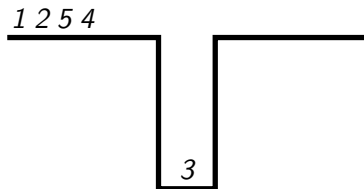
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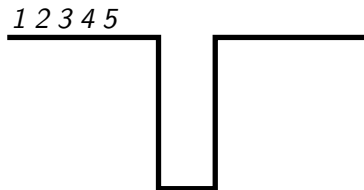
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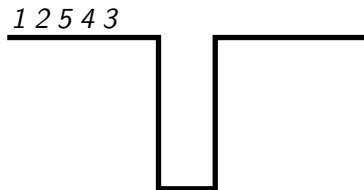
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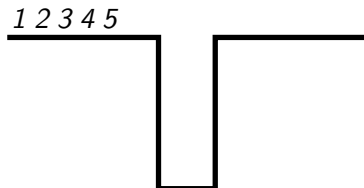
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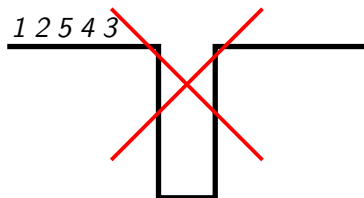
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Let $\pi = 3\ 1\ 4\ 5\ 2$

▶ π is **NOT** stack-sortable



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$\alpha = 3\ 1\ 2\ 5\ 4$ is stack-sortable

$\Rightarrow \alpha$ **avoids** the pattern 231

Permutation Patterns

It's easier with pictures!

$$\pi = 3\ 1\ 4\ 5\ 2$$

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			×	
		×		
×				
				×
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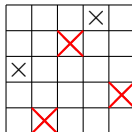
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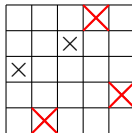


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- ▶ π contains 132
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▶ $S_n(\tau) =$ the set of permutations of length n that avoid τ .

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Definition

Two patterns σ and τ are **Wilf-equivalent** if for all n ,

$$|S_n(\tau)| = |S_n(\sigma)|.$$

Permutation Patterns

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\Rightarrow 12 is Wilf-equivalent to 21

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If τ is any pattern of length 3, then

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$$|S_6(\tau)| = 132$$

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Permutation Patterns

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$n =$	5	6	7	8	9
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2314 1234 1324

- ▶ I. Gessel (1990) gave a formula for $|S_n(1234)|$
- ▶ M. Bóna (1997) gave a formula for $|S_n(2314)|$

Open Problem

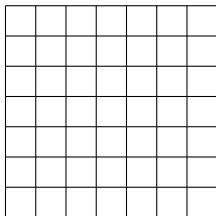
Find a formula for $|S_n(1324)|$.

Rook Placements

A **Ferrers Board** F is a square array of boxes with a “bite” taken out of the northeast corner.

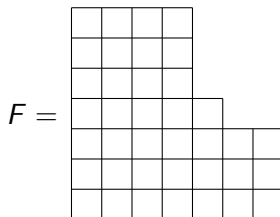
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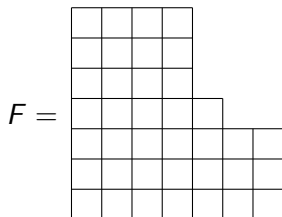
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Rook Placements

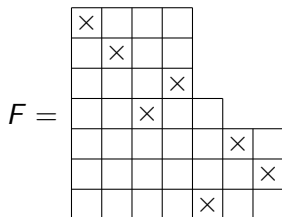
A **Ferrers Board** F is a square array of boxes with a “bite” taken out of the northeast corner.



A **full rook placement** (f.r.p.) on F is a placement of markers with **EXACTLY** one in each row and column.

Rook Placements

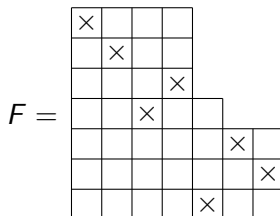
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Notation

- ▶ $\mathcal{R}_F =$ set of all f.r.p.'s on the Ferrers board F

Rook Placements

Patterns?

×						
	×					
			×			
		×				
					×	
						×
				×		

Rook Placements

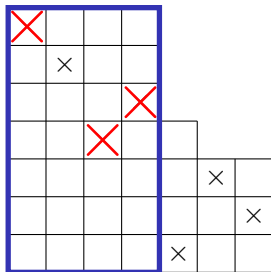
Patterns?

×						
	×					
			×			
		×				
					×	
						×
				×		

► **contains** the pattern 312

Rook Placements

Patterns?



► **contains** the pattern 312

Rook Placements

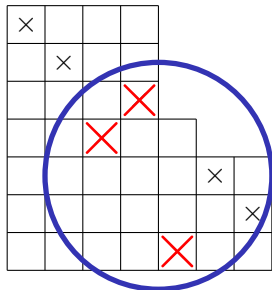
Patterns?

×						
	×					
			×			
		×				
					×	
						×
				×		

- ▶ **contains** the pattern 312
- ▶ **avoids** the pattern 231

Rook Placements

Patterns?



- ▶ **contains** the pattern 312
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Rook Placements

Patterns?

×						
	×					
			×			
		×				
					×	
						×
				×		

- ▶ **contains** the pattern 312
- ▶ **avoids** the pattern 231

Notation

- ▶ $\mathcal{R}_F(\tau)$ = set of all f.r.p. on F that avoid τ .

Rook Placements

Definition

Two patterns σ and τ are **shape-Wilf-equivalent** if for **every** F ,

$$|\mathcal{R}_F(\sigma)| = |\mathcal{R}_F(\tau)|.$$

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- ▶ $123 \dots k \sim k \dots 321$ (J. Backlin, J. West, and G. Xin, 2000)

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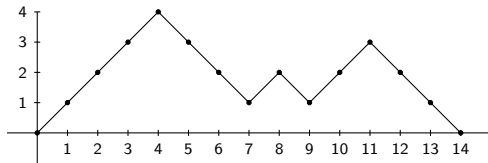
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What's Known?

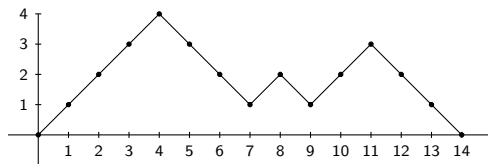
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 - Complicated proof \Rightarrow can't count things
- ▶ We give a simple proof that $231 \sim 312$
 - Can count things!

Dyck Paths

Dyck Paths

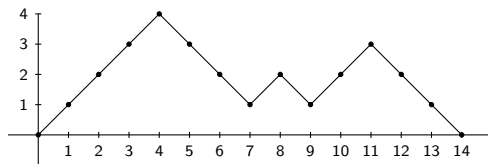


Dyck Paths



A **Dyck path** of size n is a path that:

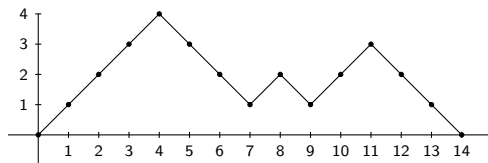
Dyck Paths



A **Dyck path** of size n is a path that:

- ▶ starts at the origin

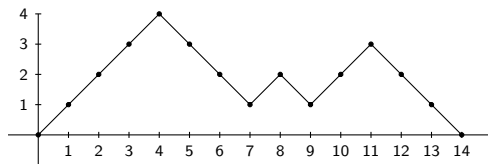
Dyck Paths



A **Dyck path** of size n is a path that:

- ▶ starts at the origin
- ▶ ends at the point $(2n, 0)$

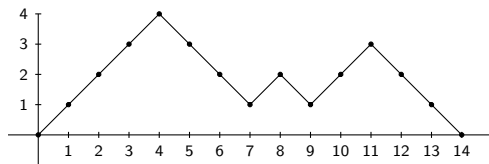
Dyck Paths



A **Dyck path** of size n is a path that:

- ▶ starts at the origin
- ▶ ends at the point $(2n, 0)$
- ▶ never goes below the x-axis

Dyck Paths



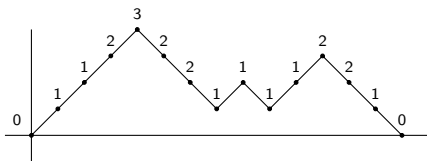
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- ▶ never goes below the x-axis

It is well known that

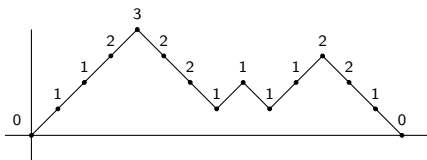
$$\# \text{ Dyck paths of size } n = \frac{1}{n+1} \binom{2n}{n}$$

Labeled Dyck paths



We label the Dyck path so that:

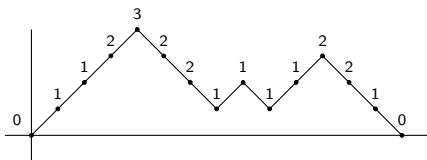
Labeled Dyck paths



We label the Dyck path so that:

- ▶ Monotonicity
 - $+1/0$ up step and $-1/0$ down step

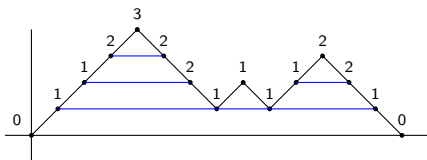
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- ▶ Zero Condition
 - All zeros lie precisely on the x -axis

Labeled Dyck paths



We label the Dyck path so that:

- ▶ Monotonicity
 - $+1/0$ up step and $-1/0$ down step
- ▶ Zero Condition
 - All zeros lie precisely on the x -axis
- ▶ Tunnel Property
 - “Left” \leq “Right”

Our proof of $231 \sim 312$

An outline

Our proof of $231 \sim 312$

An outline

1. 231-avoiding rook placement \mapsto Tunnel property

Our proof of $231 \sim 312$

An outline

1. 231-avoiding rook placement \mapsto Tunnel property
2. Tunnel Property \mapsto Reverse Tunnel Property

Our proof of $231 \sim 312$

An outline

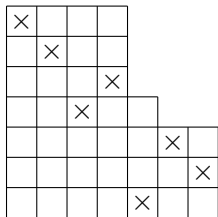
1. 231-avoiding rook placement \mapsto Tunnel property
2. Tunnel Property \mapsto Reverse Tunnel Property
3. Reverse Tunnel Property \mapsto 312-avoiding rook placement

Our proof of $231 \sim 312$

1. 231-avoiding f.r.p. \mapsto Tunnel property

Our proof of $231 \sim 312$

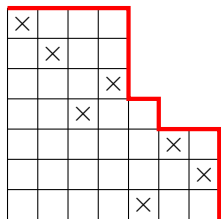
1. 231-avoiding f.r.p. \mapsto Tunnel property



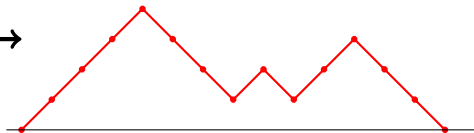
$\mathcal{R}_F(231)$

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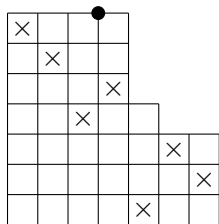


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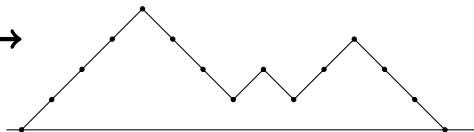


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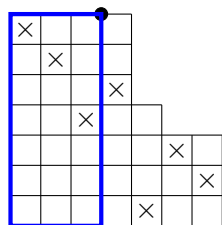


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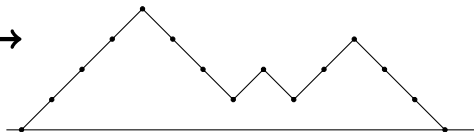


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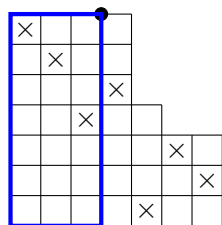


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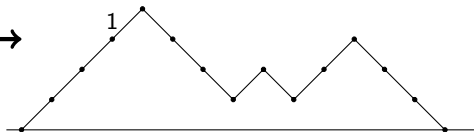


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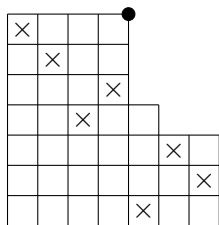


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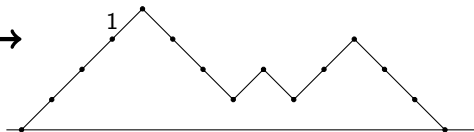


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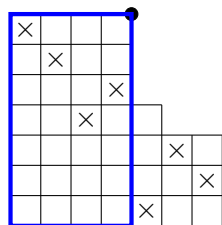


$\mathcal{R}_F(231)$

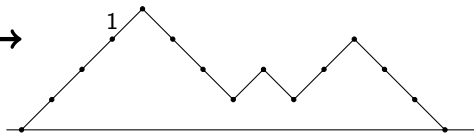


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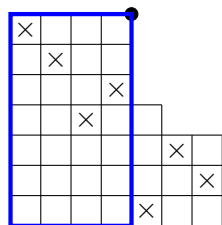


$\mathcal{R}_F(231)$

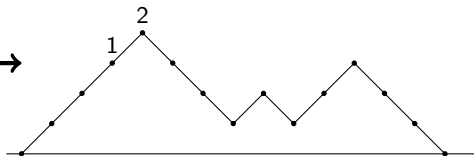


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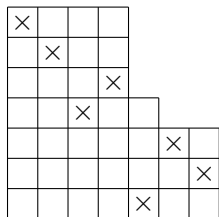


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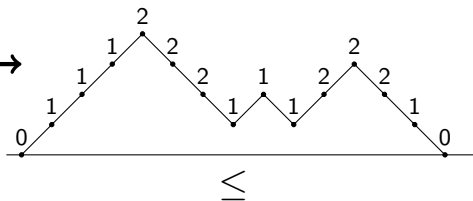


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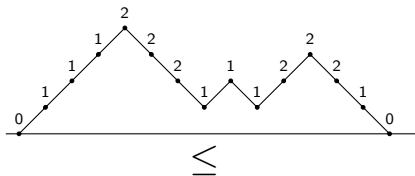


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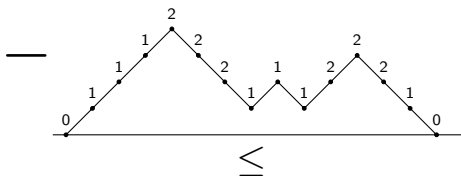
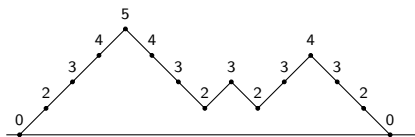
Our proof of $231 \sim 312$

2. Tunnel property \mapsto Reverse tunnel property



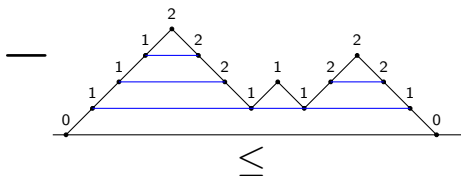
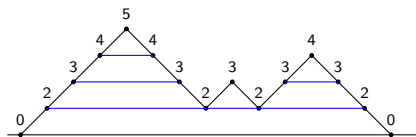
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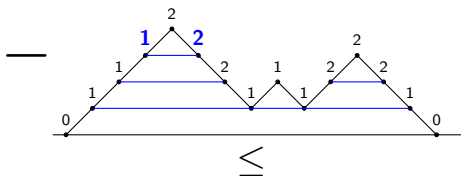
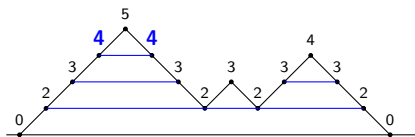
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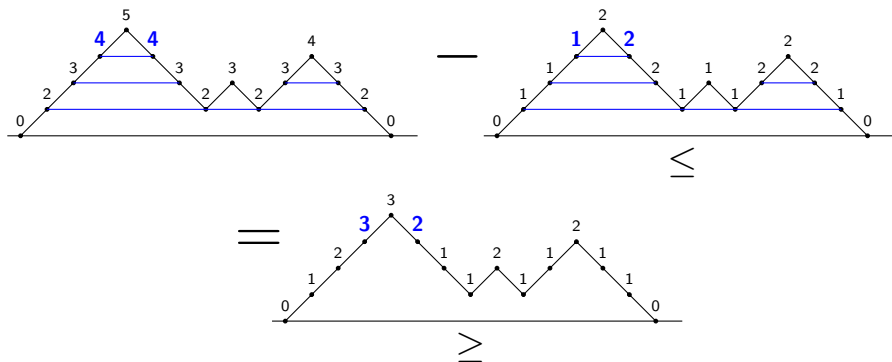
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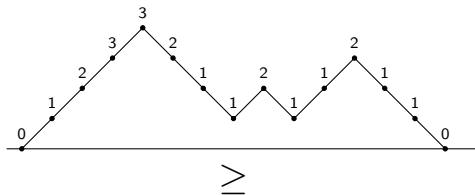
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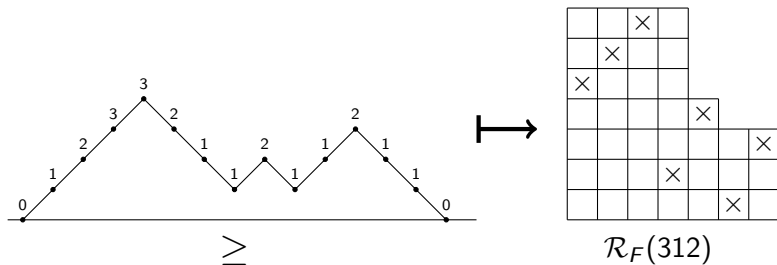
Our proof of $231 \sim 312$

3. Reverse tunnel property \mapsto 312-avoiding f.r.p.



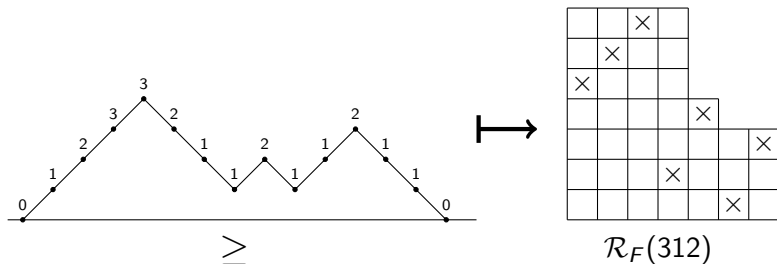
Our proof of $231 \sim 312$

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Theorem (Bloom–Saracino '11)

*This mapping is a bijection between $\mathcal{R}_F(231)$ and $\mathcal{R}_F(312)$.
 \Rightarrow 231 and 312 are shape-Wilf-equivalent.*

Enumerative Results: 2314-Avoiding Permutations

In 1997, Bóna proved the celebrated result:

$$|S_n(2314)| = (-1)^n \left[\frac{-7n^2 + 3n + 2}{2} + 6 \sum_{i=2}^n (-2)^i \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2} \right]$$

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Our Proof

6257413 \mapsto

$S_n(2314)$

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Our Proof

6257413 \mapsto

			×			
×						
		×				
				×		
						×
	×					
					×	

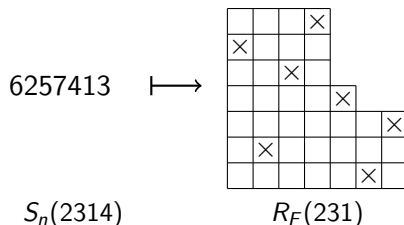
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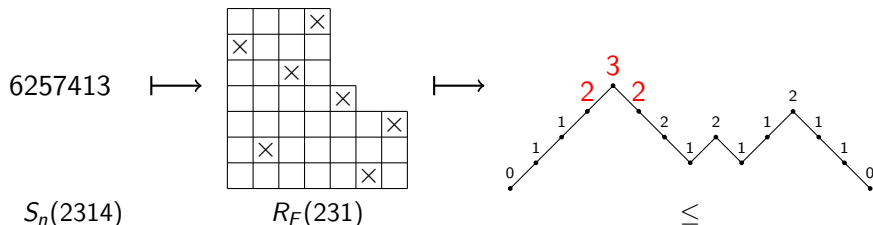
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Our Proof



Thank you!

Appendix: New Enumerative Results

- ▶ In 2012, D. Callan and V. Kotesovec conjectured that

$$\begin{aligned}\sum_{n=0}^{\infty} |S_n(2314, 1234)| z^n &= \frac{1}{1 - C(zC(z))} \\ &= 1 + z + 2z + 6z^2 + 22z^3 + \dots\end{aligned}$$

where $C(z)$ is the generating function for the Catalan numbers.

Appendix: New Enumerative Results

- ▶ In 2012, D. Callan and V. Kotesovec conjectured that

$$\begin{aligned}\sum_{n=0}^{\infty} |S_n(2314, 1234)| z^n &= \frac{1}{1 - C(zC(z))} \\ &= 1 + z + 2z^2 + 6z^3 + 22z^4 + \dots\end{aligned}$$

where $C(z)$ is the generating function for the Catalan numbers.

- ▶ All 231-avoiding f.r.p. are counted by

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- ▶ New enumerative results in the theory of perfect matchings and set partitions.