An Introduction to Wedderburn Theory & Group Representations

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Contents

1 Basic definitions and preliminaries

Throughout we let F be an arbitrary field.

Definition. An F-algebra A is a vector space over F with a multiplication on A satisfying

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

where $\lambda \in F$ and $a, b \in A$. We say that A is finite dimensional if A as an F-vector space is *finite dimensional.* Further, we say A has *unity* if there exists a multiplicative identity in A.

Definitions. Let M be an A-module. We say M is *simple* if M has no nonzero proper submodules. On the other hand we say M is *semisimple* if $M \neq 0$ and

$$M \cong S_1 \oplus S_2 \oplus \ldots \oplus S_r$$

where the S_i's are simple R-modules.

In the following we will let A be a finite dimensional F-algebra with unity and M an A-module. Observe that since $1 \in A$ then M is an F-vector space by defining

$$\lambda \cdot a = \lambda 1 \cdot a$$

for $a \in A$ and $\lambda \in F$.

Lemma 1.1. *An A-module M is finitely generated if and only if M is a finite dimensional F-vector space.*

Proof. First, assume M is finitely generated and let $\{m_1, \ldots, m_r\}$ be its generating set. As A is finite dimensional then let $\{e_1, \ldots, e_n\}$ be its basis over F. It is now clear that the finite set $\{e_i m_j \mid 1 \leq i \leq n, 1 \leq j \leq r\}$ spans M as an F-vector space. The converse is trivial. $\square$

In the following we will further assume all A-modules are finitely generating, or in light of the previous result, finite dimensional as a F-vector space.

Lemma 1.2 (Schur)**.** *Let S and T be simple R-modules. If $\varphi \colon S \to T$ is a module homomorphism then either $\varphi = 0$ or φ is an isomorphism.*

Proof. As $\ker \varphi$ is a submodule of S then either $\ker \varphi = 0$ or $\ker \varphi = S$. If the latter occurs then $\varphi = 0$. In the case of the former we see that φ is injective. As the image of φ is a submodule of T we see that this must either be all of T, in which case φ is an isomorphism, or 0, in which case $\varphi = 0$. $\square$

Lemma 1.3. *Let M be an A-module. The following are equivalent:*

a) M is semisimple

b) M is the sum (not necessarily direct) of finitely many simple modules

c) Any submodule of M is a direct summand of M.

Proof. (a $\Rightarrow$ b) Clear.

(b $\Rightarrow$ a) First note that if S and T are simple submodules with $S \neq T$ then $S \cap T = \{0\}$. If $S_1, \cdots \oplus S_r$ are the distinct simple submodules that sum to M it now follows that

$$M \cong S_1 \oplus \ldots \oplus S_r.$$

(c $\Rightarrow$ a) Observe that if c) holds for M then it must hold for any submodule of M. To see this let U be a submodule of M. Thus $M \cong U \oplus V$ for some submodule V. Now let U_0 be a submodule of U. Thus $U_0 \oplus V_0 \cong M$. Letting $U_1 = U_0 \cap V_0$, we see that $U_0 \oplus U_1 = U$. By Lemma 1.1 we know that $\dim_F(M) < \infty$. We no proceed by induction. Let $S \subset M$ be simple. So $S \oplus V \cong M$ for some V. As $\dim_F(V) < \dim_F(M)$ then by induction V is semisimple and we are done.

(c $\Rightarrow$ a) Let U be a submodule of M. Let us define V to be the maximal submodule of M so that $U \cap V = \{0\}$. Thus $U \oplus V$ is a submodule of M. If all the simple modules of M sit inside $U \oplus V$ then $M \subset U \oplus V$ and we are done. For a contradiction we may assume this is not the case. Let S be a simple module with $S \not\subset U \oplus V$. As S is simple we immediately see that $S \cap U \oplus V = \{0\}$. This means $U \oplus V \oplus S \subset M$ which contradicts the maximality of V.

□

Lemma 1.4. *Submodules and homomorphic images of semisimple modules are semisimple.*

Proof. Let N and M be A-modules where $M = S_1 \oplus \cdots \oplus S_r$ is semisimple. If $\varphi \colon M \to N$ is an onto A-module homomorphism then (by Schur's Lemma) $\varphi|_{S_i}$ is either the zero map or an isomorphism. Thus N is the sum of simple modules and by Lemma 1.3 it must be simple.

Now to prove the first claim let U be a submodule of M. By Lemma 1.3 we have $M = U \oplus V$ for some submodule V. The result now follows by the first part and the fact that $M/V \cong U$ is the homomorphic image of M. □

Definition. We say an algebra A is *semisimple* if all modules over A are semisimple.

Lemma 1.5. *The algebra A is semisimple if and only if the A as an A-module is semisimple.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let M be an A-module. As M is finitely generated, take $\{m_1, \ldots, m_r\}$ to be a generating set. Now define the surjective module homomorphism $\varphi \colon A^r \to M$ given by $(a_1, \ldots, a_r) \mapsto a_1 m_1 + \cdots + a_r m_r$. So M is the homomorphic image of a semisimple module, namely A^r. By Lemma 1.3 we see that M is semisimple as well. □

The following beautiful theorem tell us that for any semisimple module A there are only finitely many distinct simple A-modules. We then show in Theorem 1.2 that the decomposition of any A-module in terms of simple modules is unique.

Theorem 1.1. *Let A be semisimple and assume $A = S_1 \oplus \ldots \oplus S_r$ where the S_i are simple A-modules. Then any simple A-module S is isomorphic to some S_i.*

Proof. As $S \neq 0$ then we may choose some nonzero $v \in S$. Define $\varphi \colon A \to S$ by $a \mapsto av$. As S is simple then φ must be surjective and nonzero. Now consider the restriction of φ to each of A's summands. As $\varphi \neq 0$ is follows that at least one of these restrictions, $\varphi|_{S_i}$ is nonzero. By Schur's Lemma the map

$$\varphi_{S_i} \colon S_i \to S$$

must be an isomorphism. □

Theorem 1.2. *Let $S_1, \ldots, S_r$ be the distinct simple A-modules. If $M \cong n_1 S_1 \oplus \cdots \oplus n_r S_r$ then the n_i are uniquely determined.*

Proof. By Schur's Lemma it follows that we have the following F-vector space isomorphisms

$$\mathrm{Hom}_A(M, S_i) \cong \mathrm{Hom}_A(n_i S_i, S_i) \cong n_i \,\mathrm{Hom}(S_i, S_i)$$

As M is finite dimensional over F (all our modules are assumed finitely generated) we see that $\dim_F(\mathrm{Hom}_A(M, S_i)) < \infty$. Thus n_i is given by a ratio of dimensions. □

2 The semisimplicy of matrix algebras

Definitions. Define $A^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in A \right\}$ and $M_n(A)$ be the algebra of $n \times n$ matrices with entries in A.

Lemma 2.1. *Let D be a division algebra. Then D^n is a simple $M_n(D)$-module. Further*

$$M_n(D) \cong nD^n$$

Proof. Let $x \in D^n$ be nonzero. Without loss of generality, assume $x_1 \neq 0$. So $E_{1,1}(x_1^{-1})x = e_1$, where the E_{ij} are the elementary matrices and $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. The theorem now follows immediately.

□

Convention. Let M be an A-module and N be a B-module. Then $M \oplus N$ is an $A \oplus B$-module by defining $(a, b) \cdot (m, n) := (am, bn)$.

Theorem 2.1. *Let D_i be division algebras for $1 \leq i \leq r$. Then the algebra*

$$A = \bigoplus_{i=1}^{r} M_n(D_i)$$

is semisimple and has precisely r distinct simple modules (up to isomorphism).

Proof. By our convention A has a canonical A-module structure. Moreover, since $M_n(D_i) \cong nD_i^n$ (as modules) it follows that

$$A \cong \bigoplus_{i=1}^{r} nD_i^n.$$

Since the D_i^n are simple it now follows that A is semisimple. The r distinct simple modules of A are (easily) seen to be $D_1^n, \ldots, D_r^n$. □

In the next section we show that every semisimple algebra A is essential a direct sum of matrix algebras over division rings.

3 Wedderburn's theorem

The purpose of this section is to prove the following classification of semisimple algebras due to Wedderburn.

Theorem 3.1 (Wedderburn). *The algebra A is semisimple if and only if it is isomorphic with a direct sum of matrix algebras over division rings.*

First, observe that Theorem 2.1 proves the reverse direction. Next, we prove an important corollary of Wedderburn's Theorem. We start with the following lemma.

Lemma 3.1. *If F is algebraically closed, then $\mathrm{End}_A(S) \cong F$, where S is a simple A-module.*

Proof. Let $\varphi \in \mathrm{End}_A(S)$. As S is a vector space over our algebraically closed field F then φ must have an eigenpair (x, λ). So $x \in \ker(\varphi - \lambda)$ and by Schur's Lemma we must have $\varphi - \lambda = 0$. In other words, $\varphi = \lambda$. The result now follows. □

Since we will predominately be working over $\mathbb{C}$ the following Corollary of Wedderburn's result will be especially useful for us.

Corollary 3.1. *Let F be algebraically closed. If A is semisimple then it is isomorphic to a direct sum of matrix algebras over F.*

We now work toward a proof of (the forward direction of) Wedderburn's result. The following definitions and lemmas will be needed.

Definition. If A is an algebra, then A^{op}, called the *opposite algebra*, is the algebra with the same underlying vector space as A but where multiplication is given by $a \cdot b := ba$ where the right side is the given multiplication in A.

Lemma 3.2. $A^{op} \cong \mathrm{End}_A(A)$.

Proof. Let $\varphi \in \mathrm{End}_A(A)$. Observe that $\varphi(b) = b\varphi(1)$ for all $b \in A$. This means that φ is completely determined by where it maps 1. Now define

$$\rho\colon A^{op} \to \mathrm{End}_A(A)$$

by $a \mapsto \varphi_a$, where $\varphi_a(1) = a$. In light of our first observation this is clearly a vector space isomorphism. Lastly,

$$\rho(a \cdot b) = \rho(ba) = \varphi_{ba} = \varphi_a \circ \varphi_b = \rho(a) \circ \rho(b).$$

□

Lemma 3.3. *We have $M_n(A^{op}) \cong M_n(A)^{op}$.*

Proof. Let tr be the transpose map. It will suffice to show that it preserves multiplication. Observe

$$tr(\alpha E_{ij} \beta E_{lk}) = tr(\beta\alpha E_{ij} E_{lk}) = \begin{cases} \beta\alpha & \text{if } l = i \\ 0 & \text{else.} \end{cases}$$

Similarly,

$$tr(\alpha E_{ij}) \cdot tr(\beta E_{lk}) = \alpha E_{ji} \cdot \beta E_{lk}) = \begin{cases} \beta\alpha & \text{if } l = i \\ 0 & \text{else.} \end{cases}$$

We see that the map tr extends by linearity to be an algebra isomorphism.

□

Lemma 3.4. *Let M and N be A-modules. If*

$$\mathrm{Hom}_A(M,N) = 0 = \mathrm{Hom}_A(N,M)$$

then

$$\mathrm{End}_(\,M \oplus N) \cong \mathrm{End}_(\,M) \oplus \mathrm{End}_(\,N)$$

as algebras.

Proof. Define $\rho_M \colon M \oplus N \to M$ and $\rho_N \colon M \oplus N \to N$ be projection maps. Now take $\gamma \in \mathrm{End}_A(M \oplus N)$ and observe that

$$\rho_M \circ \gamma|_M \in \mathrm{Hom}(M,N) \qquad \text{and} \qquad \rho_N \circ \gamma|_N \in \mathrm{Hom}(N,M).$$

By our assumption these are both zero. Thus

$$\gamma = \rho_M \circ \gamma|_M \oplus \rho_N \circ \gamma|_N$$

where $\rho_M \circ \gamma|_M \in \mathrm{End}_A(M)$ and $\rho_N \circ \gamma|_N \in \mathrm{End}_A(N)$. In other words, every element in $\mathrm{End}_A(M \oplus N)$ looks like the sum $\alpha + \beta$ where $\alpha \in \mathrm{End}_A(M)$ and $\beta \in \mathrm{End}_A(N)$. As every map of this form is and endomorphism of $M \oplus N$ the result now follows. □

Lemma 3.5. *Let $S_1, \ldots, S_r$ be the distinct simple A-modules. Define*

$$U = \bigoplus_{i=1}^{r} n_i S_i$$

then

$$\mathrm{End}_A(U) \cong \bigoplus_{i=1}^{r} \mathrm{End}_A(n_i S_i)$$

as algebras.

Proof. By Schur's Lemma observe that $\mathrm{Hom}(\bigoplus_{i \in I} n_i S_i, \bigoplus_{j \in J} n_j S_j) = 0$ provided $I \cap J = \emptyset$. The result now follows by repeated applications of Lemma 3.4. □

Lemma 3.6. *If S is a simple A-module then*

$$\mathrm{End}_A(nS) \cong M_n(\mathrm{End}_A(S)).$$

Proof. For notational ease let $D = \mathrm{End}_A(S)$. By Schur's Lemma D is a division ring so it makes sense to talk about a n-dimensional D-module V. Let $e_1, \ldots, e_n$ be a basis for V. For $\varphi \in \mathrm{End}_A(nS)$ define $\overline{\varphi} \in \mathrm{End}_D(V)$ by

$$\overline{\varphi}(\bigoplus_{i=1}^{n} \alpha_i e_i) = \bigoplus_{i=0}^{n} \rho_i \varphi(\alpha_1, \ldots, \alpha_n) e_i$$

where $\alpha_i \in D$ and ρ_i is the projection map onto the ith coordinate. Now if $\varphi, \psi \in \mathrm{End}_A(nS)$ then we have

$$\begin{aligned}
\overline{\varphi} \circ \overline{\psi}(\alpha_1 e_1 + \cdots + \alpha_n e_n) &= \overline{\varphi}\left(\bigoplus_{i=1}^{n} \rho_i \psi(\alpha_1, \ldots, \alpha_n) e_i\right) \\
&= \bigoplus_{i=1}^{n} \rho_i \varphi(\rho_1 \psi(\alpha_1, \ldots, \alpha_n), \ldots, \rho_n \psi(\alpha_1, \ldots, \alpha_n)) e_i \\
&= \bigoplus_{i=1}^{n} \rho_i \varphi\psi(\alpha_1, \ldots, \alpha_n) e_i \\
&= \overline{\varphi\psi}
\end{aligned}$$

It now follows that the map $\varphi \mapsto \overline{\varphi}$ is an injective algebra homomorphism. As this map is between two vector spaces with the same dimension, namely $n^2 \dim(D)$, it must also be surjective.

□

We are now ready to prove Wederburn's main result.

Proof of Wedderburn's Theorem. As we mentioned above it only remains to prove the forward direction. To do first let $S_1, \ldots, S_r$ be a complete list of distinct simple A-modules. Thus $A = U_1 \oplus \cdots \oplus U_r$ where $U_i = n_i S_i$. Now

$$\begin{aligned}
A^{op} \cong \mathrm{End}_A(A) &\cong \bigoplus_{i=1}^{n} \mathrm{End}_A(U_i) && \text{(Lemmas 3.2, 3.5)} \\
&\cong \bigoplus_{i=1}^{n} M_{n_i}(\mathrm{End}_A(S_i)) && \text{(Lemmas 3.6)}
\end{aligned}$$

Therefore Lemma 3.3 gives us that

$$A \cong \bigoplus_{i=1}^{n} M_{n_i}(\mathrm{End}_A(S_i)^{op}).$$

It only remains to show that $\mathrm{End}_A(S_i)$ is a division ring but this easily follows from Schur's Lemma.

□

4 Representation theory of groups

In this section let G be a finite group and V a vector space over an algebraically closed field F.

Definition. A *representation* of G (on V) is a homomorphism $\rho \colon G \to \mathrm{End}(V)$.

Definition. Assume G acts on V. We say this action is *linear* if

$$g(v + w) = gv + gw$$

and

$$g(\lambda v) = \lambda gv$$

where $g \in G$, $w, v \in V$ and $\lambda \in F$.

Observe that every representation ρ of G on V gives rise to a linear action as follows

$$g \cdot v := \rho(g)v.$$

Likewise, if G acts linearly on V then this defines a representation $\rho\colon G \to \mathrm{End}(V)$ by setting

$$\rho(g)v := gv.$$

ρ is indeed a homomorphism since $\rho(gh)v = (gh)v = g(hv) = \rho(g)\rho(h)v$ for all $v \in V$. As these constructions are inverses of one another it follows that linear actions and representations are equivalent.

Definition. Define the set FG to be the set of all formal linear combinations of G by elements in F. We call FG the *group algebra.*

It is clear that FG is a an F-vector space with dimension $|G|$. To see that it also has a canonical algebra structure, and hence is deserving of its name, note that we can define

$$\left(\sum_{g \in G} \alpha_g g\right)\left(\sum_{h \in G} \beta_h h\right) := \sum_{g,h \in G} \alpha_g \beta_h gh$$

where gh is computed in the group and $\alpha_g \beta_h$ is computed in the ground field. Lastly, FG is also an algebra with unity since $e \in G$ acts as a multiplicative identity.

Before continuing, let us observe that any FG-module V is equivalent to a linear action of G on V and, by the paragraph above, is equivalent to a representation of G on V.

Theorem 4.1 (Maschke)**.** *If* $\mathrm{Char}(F) = 0$ *or* $(\mathrm{Char}(F), |G|) = 1$*, then* FG *is a semisimple algebra.*

Theorem 4.2. *The group algebra* $\mathbb{C}\mathrm{G}$ *is isomorphic to*

$$M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}).$$

Further, $\mathbb{C}\mathrm{G}$ *has exactly* r *distinct simple modules* $S_1, \ldots, S_r$ *where* $\dim(S_i) = r_i$*.*

Proof. By Maschke's theorem we know that $\mathbb{C}\mathrm{G}$ is semisimple. Wedderburn's theorem, its corollary, and Theorem 2.1 give the reset. □

Corollary 4.1. $\displaystyle\sum_{i=1}^{r} n_i^2 = |G|$

Proof. Compare the dimensions in the decomposition in Theorem 4.2.

$$|G| = \dim(\mathbb{C}\mathrm{G}) = \dim\left(\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})\right) = \sum_{i=1}^{r} n_i^2.$$

□

Corollary 4.2. *The number of distinct simple modules over* $\mathbb{C}\mathrm{G}$ *is precisely the number of conjugacy classes in* G*.*

Proof. Here we use the decomposition given in Theorem 4.2 and compare the dimension of the centers. On the right hand side we have

$$Z\left(\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})\right) = \mathbb{C}\, I_{n_1} \oplus \cdots \oplus \mathbb{C}\, I_{n_r}$$

where I_n is the $n \times n$ identity matrix. Thus the dimension of the center is r. For the left hand side we have, for $h = \sum_{g \in G} \alpha_g g$ and $x \in G$:

$$\begin{aligned} h \in Z(\mathbb{C}\mathrm{G}) \text{ iff } & x\left(\sum_{g \in G} \alpha_g g\right) x^{-1} = h \\ \text{iff } & \sum_{g \in G} \alpha_g x g x^{-1} = h \\ \text{iff } & \sum_{g \in G} \alpha_{x^{-1} g x} g = h. \end{aligned}$$

It now follows that $h \in Z(\mathbb{C}\mathrm{G})$ if and only if $\alpha_g = \alpha_{x^{-1}gx}$, i.e., the coefficients α_g must be constant on conjugacy classes. Thus

$$\dim(Z(\mathbb{C}\mathrm{G})) = \text{number of conjugacy classes in } G.$$

The result now follows. □

5 Characters

In this section again let G be a finite group and V a $\mathbb{C}$G-module where $S_1, \ldots, S_r$ are the r irreducible $\mathbb{C}$G-modules. As usual let ρ be the corresponding representation.

Definition. Let $\rho : G \to \mathrm{End}(V)$ be a representation. We define a *character of V* to be the function

$$\chi_V : G \to \mathbb{C}$$

where $\chi_V(g)$ is the trace of $\rho(g)$.

Note that if the representation is irreducible then we usually say χ_V is an *irreducible character*. Further, we will often extend χ_V linearly so that

$$\chi_V : \mathbb{C}\mathrm{G} \to \mathbb{C}\,.$$

Last, we adopt the convention that $\chi_i = \chi_{S_i}$.

Definition. A *class function of* G is a function $f : G \to \mathbb{C}$ that is constant on conjugacy classes of G. We denote by $\mathcal{C}$ the $\mathbb{C}$-vector space of all class functions on G.

Observe that Corollary 4.2 says that $\dim(\mathcal{C}) = r$. Moreover since

$$\chi_U(xgx^{-1}) = \mathrm{Trace}(\rho(x)\rho(g)\rho(x)^{-1}) = \mathrm{Trace}(\rho(g)) = \chi_U(g)$$

we see that $\chi_i \in \mathcal{C}$. In fact, much more is true.

Theorem 5.1. *The irreducible characters $\chi_1, \ldots, \chi_r$ form a basis for $\mathcal{C}$.*

Proof. Since $\dim(\mathcal{C}) = r$ it will suffice to show that the χ_i are linearly independent. Let e_i be the element in $\mathbb{C}$G that corresponds to the element in $\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$ that is zero in every coordinate except the ith coordinate where we place the identity matrix. Thus

$$\chi_i(e_j) = \delta_{ij} n_j.$$

Now, if $0 = \sum_{i=1}^r \alpha_i \chi_i$, then

$$0 = \sum_{i=1}^r \alpha_i \chi_i(e_j) = \alpha_j n_j.$$

We conclude that the α_i must all be zero and that the χ_i are linearly independent. □

Theorem 5.2. *Two $\mathbb{C}$G-modules U and V are isomorphic if and only if $\chi_U = \chi_V$.*

Proof. The forward direction is clear. For the other direction decompose U and V as

$$\bigoplus_{i=1}^r \alpha_i S_i \qquad \text{and} \qquad \bigoplus_{i=1}^r \beta_i S_i.$$

It now follows that

$$\sum_{i=1}^r \alpha_i \chi_i = \chi_U = \chi_V = \sum_{i=1}^r \beta_i \chi_i.$$

By the linear independence of the χ_i we see that $\alpha_i = \beta_i$. Thus U and V are isomorphic. □

Lemma 5.1. *Assume $g \in G$ of order n. Then*

a) $\rho(g)$ is a diagonalizable. All its eigenvalues are nth roots of unity

b) $\chi_V(g^{-1}) = \bar{\chi_V}(g)$

c) $|\chi_V(g)| \leq \chi_V(1)$. Further, we have equality if and only if $g \in \ker(\rho)$.

d) $\{g \in G \mid \chi_V(g) = \chi_V(1)\} \trianglelefteq G$

Proof. To prove a), observe that since $g^n = 1$, then the minimal polynomial of $\rho(g)$ must divide $x^n - 1$. Thus the minimal polynomial must have distinct roots all of which are nth roots of units. The claim now follows from basic linear algebra. Next we prove b). By part a) we first diagonalize $\rho(g)$ so that the diagonal entries are roots of unity. So $\rho(g)^{-1} = \bar{\rho(g)}$ and thus $\chi_V(g^{-1}) = \bar{\chi_V}(g)$. Lastly the proofs of c) follows directly from a). To see d) observe that the set in question is precisely $\ker(\rho)$. □

If U is another $\mathbb{C}$G-module then $\mathrm{Hom}_{\mathbb{C}}(V, U)$ inherits a natural $\mathbb{C}$G-module structure as follows. First define

$$g \cdot \varphi := g\varphi(g^{-1} \cdot)$$

for $g \in G$ and $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V, U)$. As usual we then extend this action to all of $\mathbb{C}$G. Consequently, $V^* = \mathrm{Hom}(V, \mathbb{C})$ has a $\mathbb{C}$G-module structure since $\mathbb{C}$ can always be thought of as a $\mathbb{C}$G-module with the trivial action. Therefore if $f \in V^*$ then

$$g \cdot f = gf(g^{-1} \cdot) = f(g^{-1} \cdot)$$

in this case. We may also think of $V \otimes_{\mathbb{C}} U$ as a $\mathbb{C}$G-module by defining the linear action of G on $V \otimes_{\mathbb{C}} U$ by

$$g \cdot v \otimes u := gv \otimes gu.$$

This is called the *diagonal* action. There is an important connection between these two definitions. To understand this connection first recall the vector space isomorphism

$$\Phi\colon V^* \otimes_{\mathbb{C}} U \to \operatorname{Hom}_{\mathbb{C}}(V, U)$$

given by mapping $f \otimes u \mapsto uf(\cdot)$. With the definitions given above Φ is actually a $\mathbb{C}$G-module isomorphism. Explicitly we have

$$\Phi(g \cdot \varphi \otimes u) = \Phi(g\varphi \otimes gu) = gu\varphi(g^{-1}\cdot) = g \cdot (u\varphi) = g \cdot \Phi(\varphi \otimes u).$$

Lemma 5.2. *Let U be another $\mathbb{C}$G-module. Then*

a) $\chi_{V \otimes U} = \chi_V \chi_U$.

b) $\chi_{V^*} = \bar{\chi_U}$.

c) $\chi_{\operatorname{Hom}(V,U)} = \bar{\chi_V} \chi_U$.

Proof. Part a) follows from basic linear algebra regarding trace and tensor products. □