Patterns, Permutations, and Placements

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Dartmouth College

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Permutations

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$$1, 2, \ldots, n.$$

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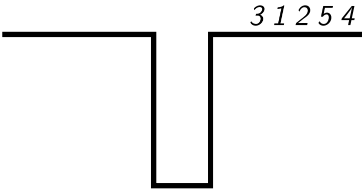

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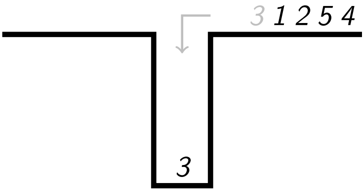

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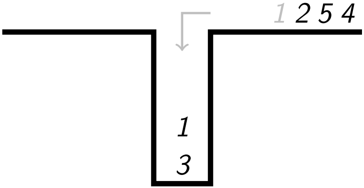

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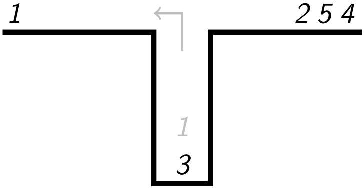

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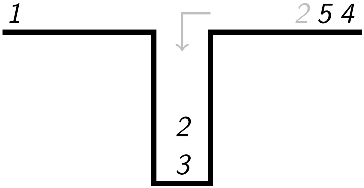

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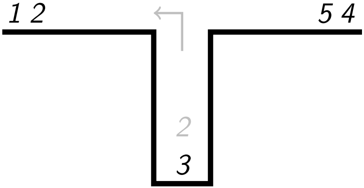

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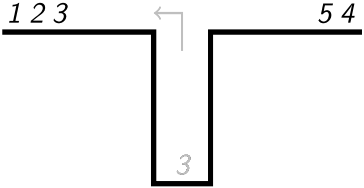

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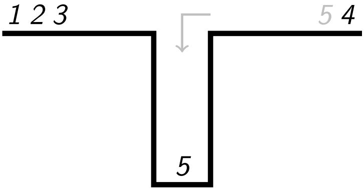

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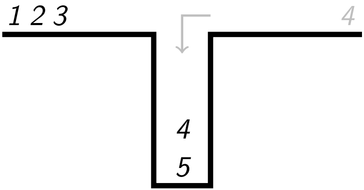

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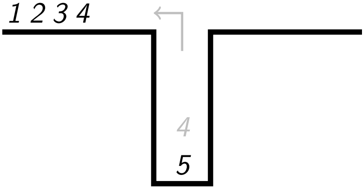

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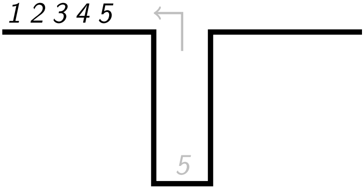

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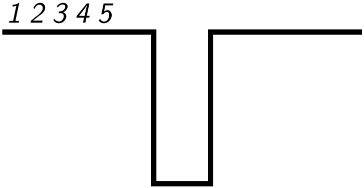

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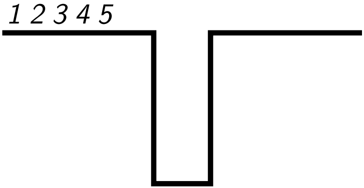

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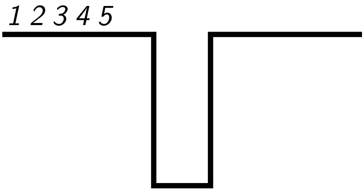

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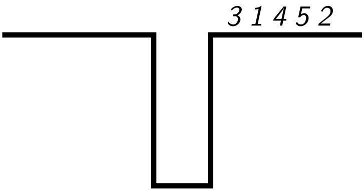

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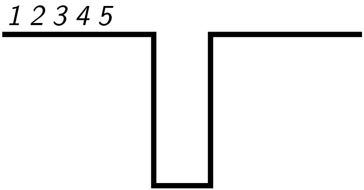

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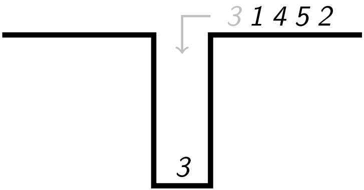

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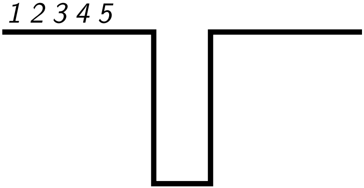

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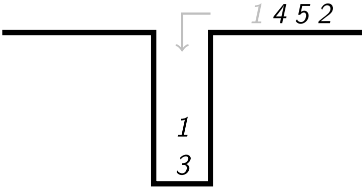

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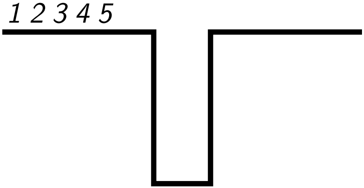

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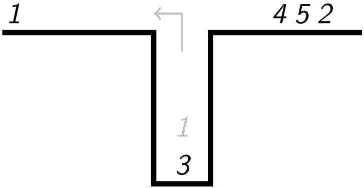

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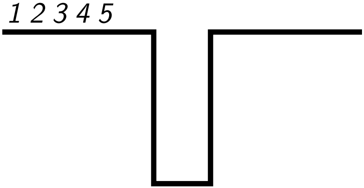

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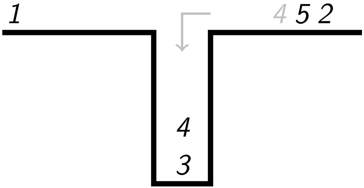

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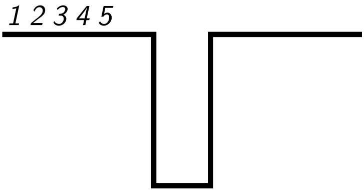

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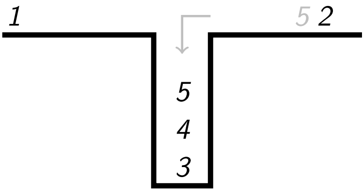

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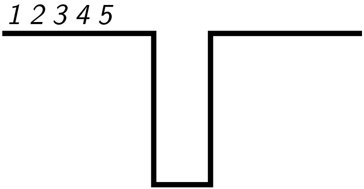

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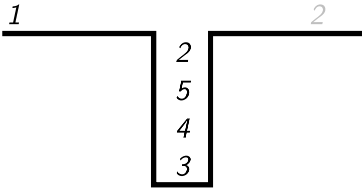

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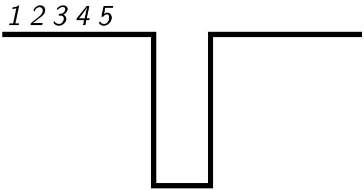

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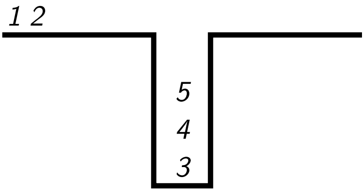

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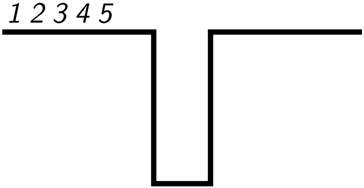

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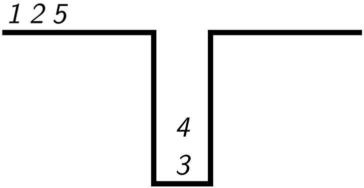

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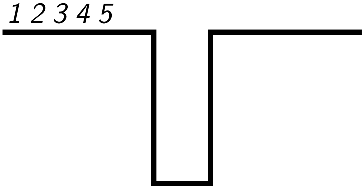

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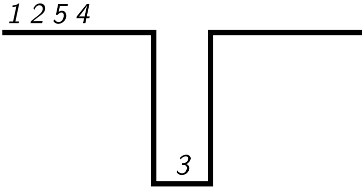

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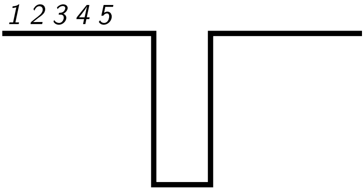

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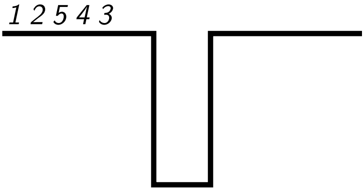

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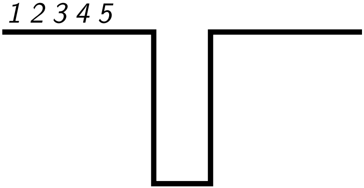
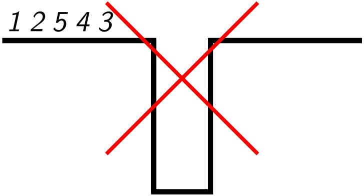


Let $\pi = 3\ 1\ 4\ 5\ 2$

- π *is NOT stack-sortable*



Stack Sorting

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Why is $\alpha = 3\ 1\ 2\ 5\ 4$ *stack-sortable, while* $\pi = 3\ 1\ 4\ 5\ 2$ *is NOT?*

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Theorem (D. Knuth 1968)

π *is NOT stack-sortable* $\Longleftrightarrow$ π *has three entries whose relative ordering is "231".*

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$\Rightarrow \pi$ **contains** *the pattern 231*

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$\Rightarrow \pi$ **contains** *the pattern 231*

$\alpha = 3\ 1\ 2\ 5\ 4$ *is stack-sortable*
$\Rightarrow \alpha$ **avoid** *the pattern 231*

Permutation Patterns

Its easier with pictures!

$\pi = 31452$

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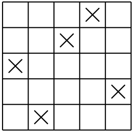
$$\pi = 31452 \quad \longmapsto$$



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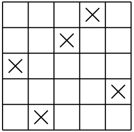

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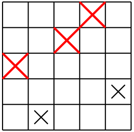

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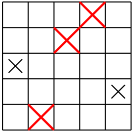

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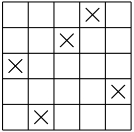

- π contains 123
- π contains 213

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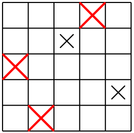

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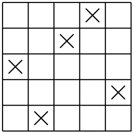

- π contains 123
- π contains 213
- π avoids 321

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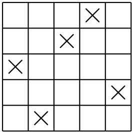

- π contains 123
- π contains 213
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Notation

Let $S_n(\tau)$ be the set of permutations of length n that avoid τ.

Permutation Patterns

Definition

We say two patterns $\tau, \sigma \in S_k$ are **Wilf-equivalent** provided

$$|S_n(\tau)| = |S_n(\sigma)|$$

for all n.

Permutation Patterns

Example (Patterns of length 2)

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What permutations avoid 21?

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$\Rightarrow S_2(21) = \{12\}.$

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What permutations avoid 21?

$$S_2 = \{12, 21\}$$

$\Rightarrow S_2(21) = \{12\}.$

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

$\Rightarrow S_3(21) = \{123\}.$

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$\Rightarrow S_3(21) = \{123\}$.

In general,

$$S_n(21) =$$

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$\Rightarrow$ 12 *is Wilf-equivalent to* 21

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Open Problem

Find a formula for $|S_n(1324)|$.

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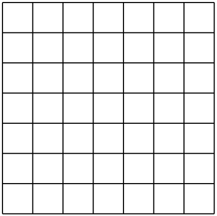

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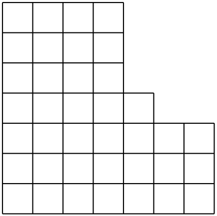
$F =$ 

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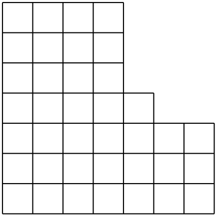

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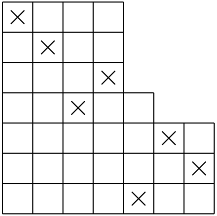

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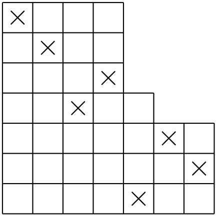
$F =$ 

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Notation

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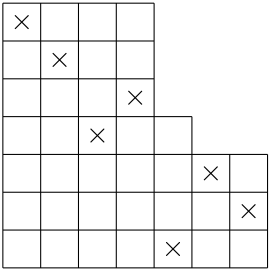
Rook Placements



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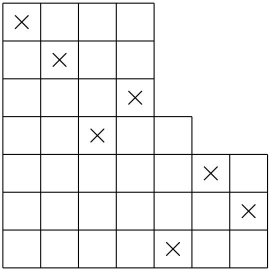
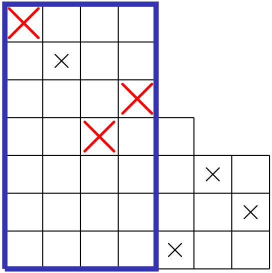


- This f.r.p. **contains** the pattern 312

Rook Placements



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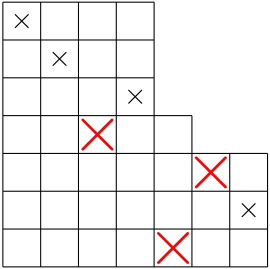

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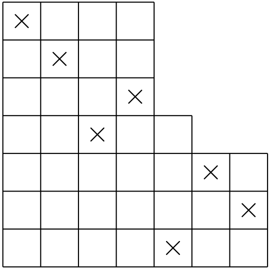

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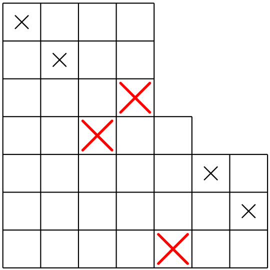

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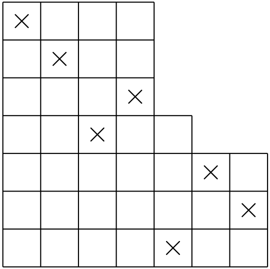

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- $\mathcal{R}_F(\tau)$ = *set of all f.r.p. on F that avoid* τ.

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Dyck Paths

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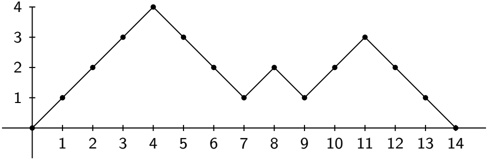

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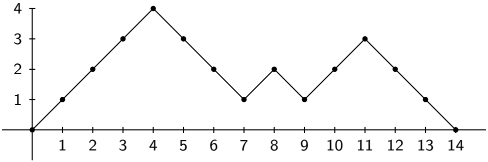
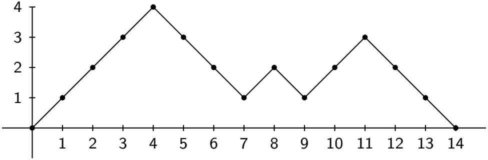


A **Dyck path** of size n is a path that:

Dyck Paths



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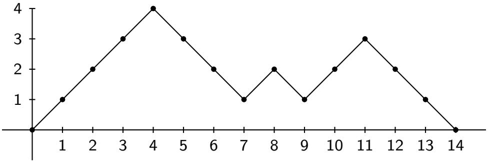

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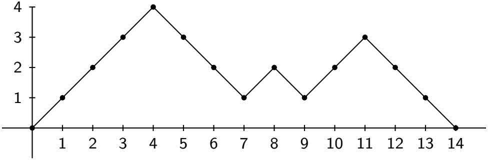

A **Dyck path** of size n is a path that:

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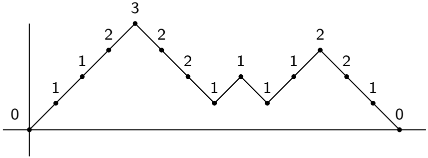
- ends at the point $(2n, 0)$
- never goes below the x-axis

Labeled Dyck paths



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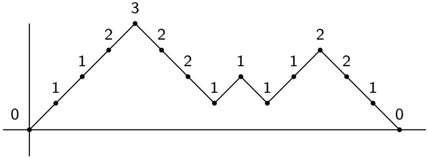

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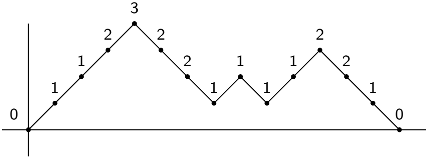

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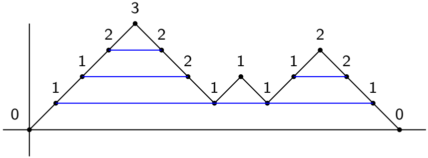

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- Zero Condition
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- Tunnel Property
  - "Left" $\leq$ "Right"

Our proof of $231 \sim 312$

An outline

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1. 231-avoiding rook placement $\mapsto$ Tunnel property

Our proof of 231 $\sim$ 312

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1. 231-avoiding rook placement $\mapsto$ Tunnel property
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Our proof of 231 $\sim$ 312

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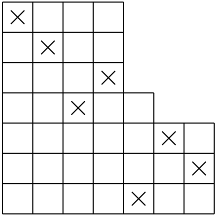

$\mathcal{R}_F(231)$

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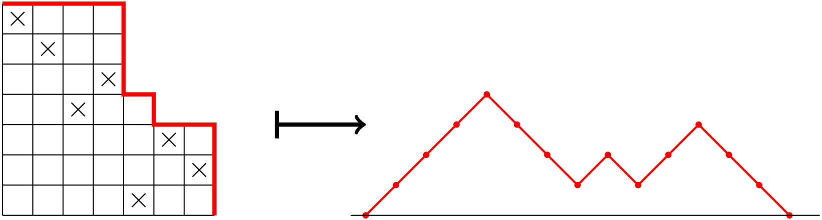

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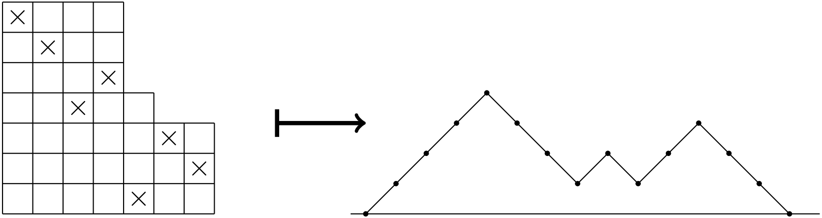

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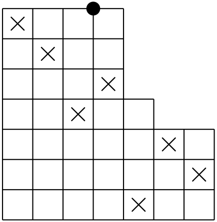

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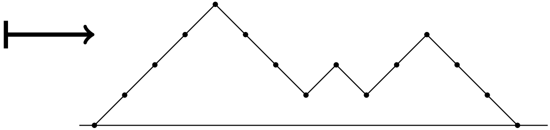

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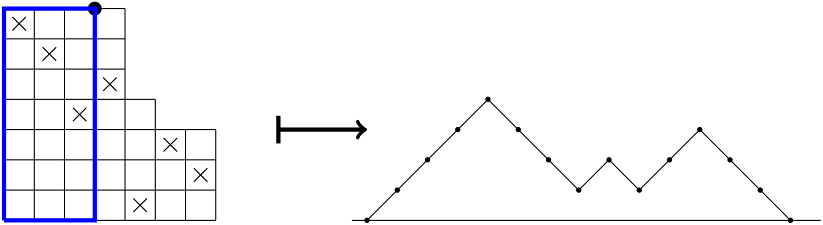

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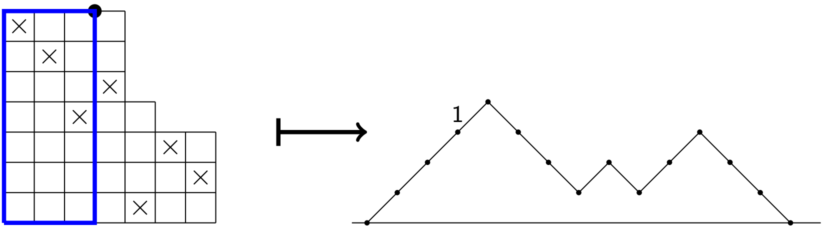

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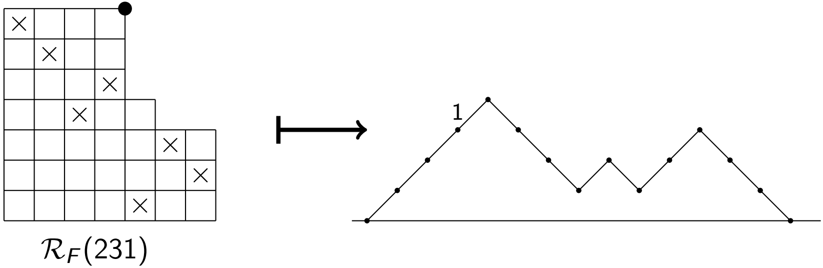

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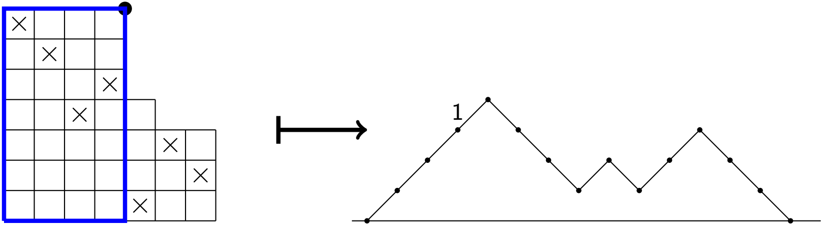

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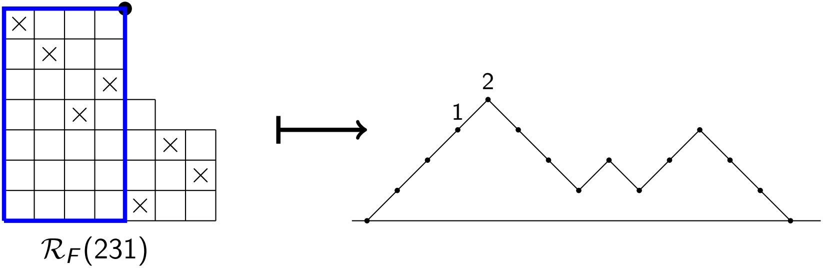

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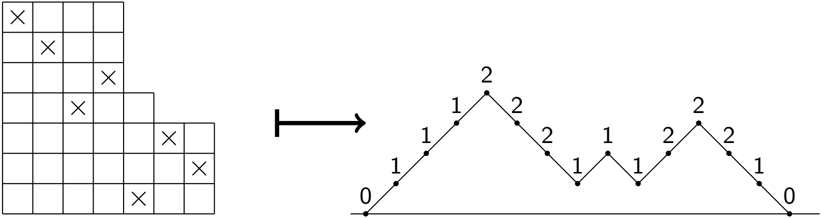

$\mathcal{R}_F(231)$

$\leq$

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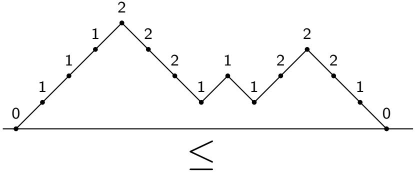
2. Tunnel property $\Rightarrow$ Reverse tunnel property



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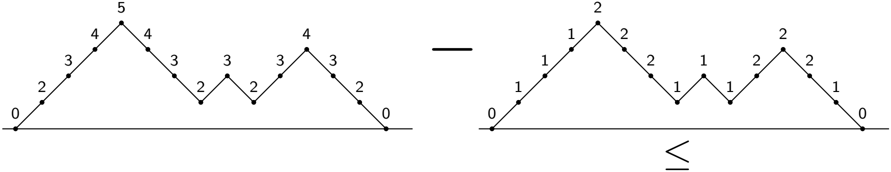

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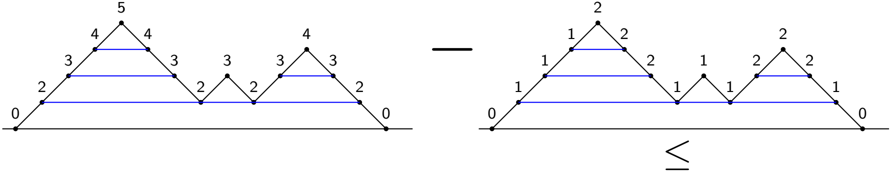

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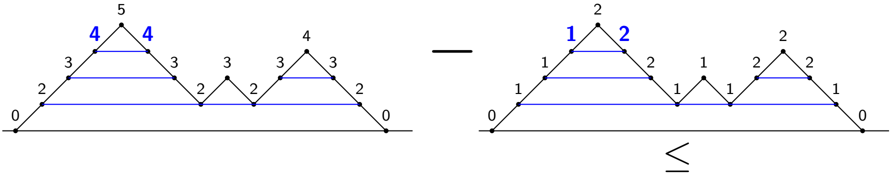

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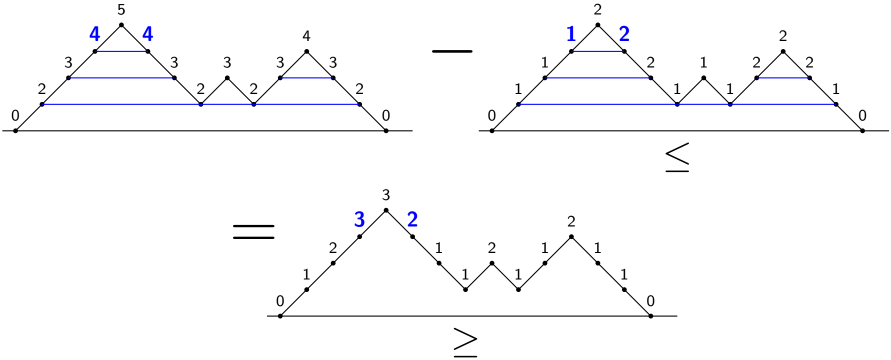

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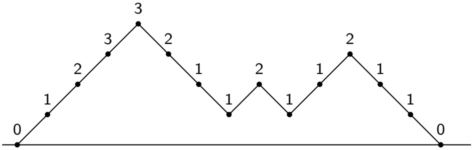

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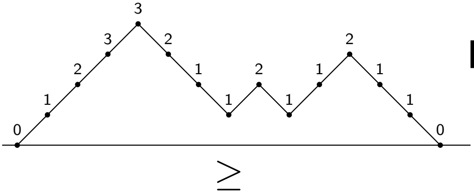

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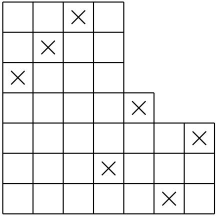

$\mathcal{R}_F(312)$

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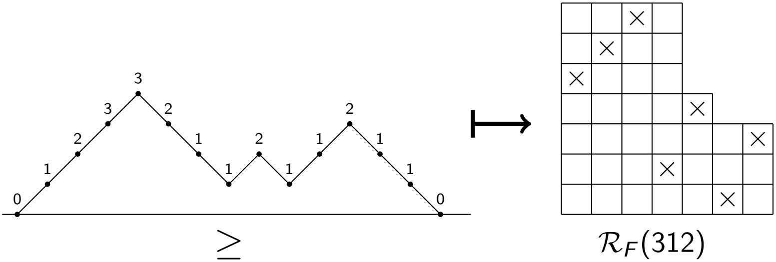

Theorem (Bloom–Saracino '11)

This mapping is a bijection between $\mathcal{R}_F(231)$ and $\mathcal{R}_F(312)$.
$\Rightarrow$ *231 and 312 are shape-Wilf-equivalent.*

Generating Functions

The **generating function** for a sequence of integers

$$a_0, a_1, a_2, a_3, \ldots$$

is the "formal" series

$$\sum_{n=0}^{\infty} a_n z^n.$$

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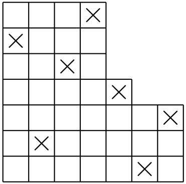

$S_n(2314)$ $\qquad$ $R_F(231)$

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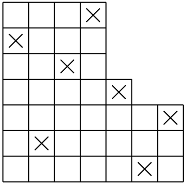

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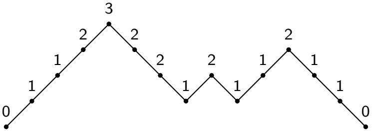

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New Enumerative Results

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- New enumerative results in the theory of perfect matchings and set partitions.

Thank you!