On two recent conjectures in pattern avoidance

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Experimental Math Seminar - Rutgers University 2015

Overview

Part I

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- On a conjecture of Dokos, et al.

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- On a conjecture of Egge (2012)

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- On a conjecture of Dokos, et al.
  - REU group under Sagan
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Part II

- On a conjecture of Egge (2012)
  - A collection of pattern classes all counted by the large Schröder numbers

Part I

(A statistic-preserving bijection)

Classical Pattern Avoidance

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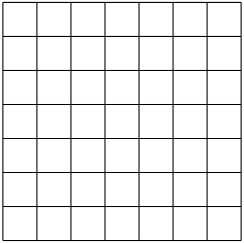

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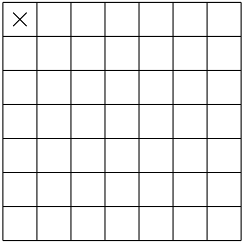

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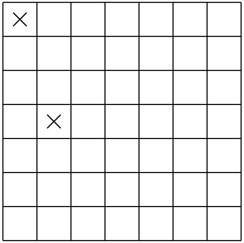

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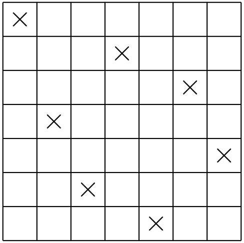

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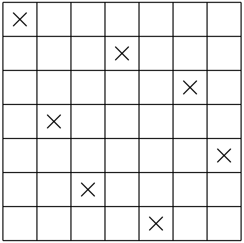

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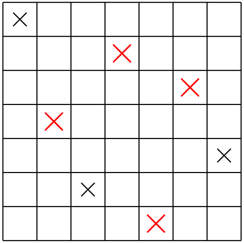

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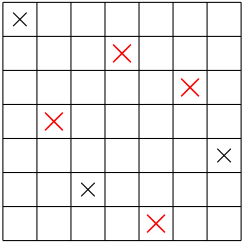

Patterns

- π contains the pattern 2 4 1 3 because...
- π avoids the pattern 1 2 3 because...

Wilf-equivalence

Notation

In general, for any $\sigma \in S_k$ *we denote by*

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the set of all permutations (length n*) that avoid* σ*.*

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All patterns τ of length 3 are Wilf-equivalent. Moreover,

$$|\mathrm{Av}_n(\tau)| = \frac{1}{1+n}\binom{2n}{n}.$$

Patterns of length 4

We have:

Class | *n*	5	6	7	8	9	...
1 4 2 3	103	512	2740	15485	91245	...
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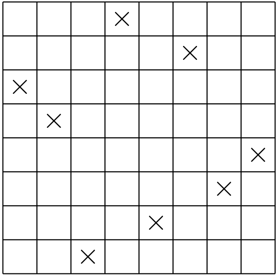
New results

- We give (first) bijective proof that 1 4 2 3 $\sim$ 2 4 1 3
- Resolve a conjecture of Dokos, et al. (2012)

Permutation Statistics

Consider the permutation $\pi = 6\ 5\ 1\ 8\ 2\ 7\ 3\ 4$



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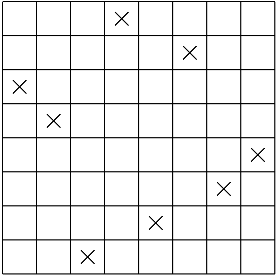

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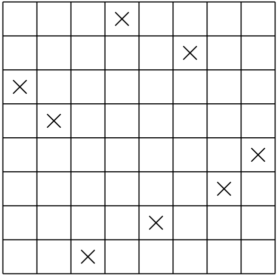

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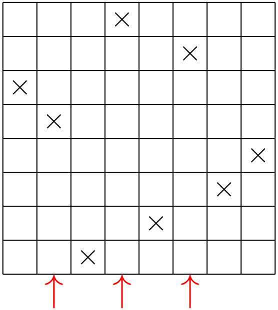

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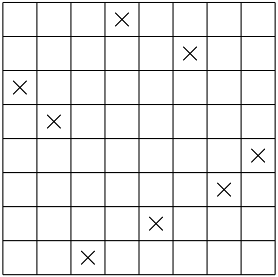

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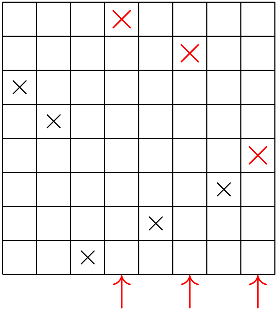

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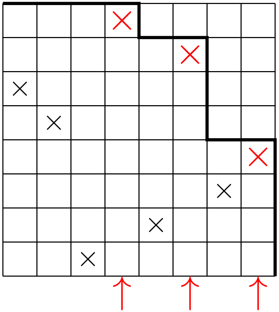

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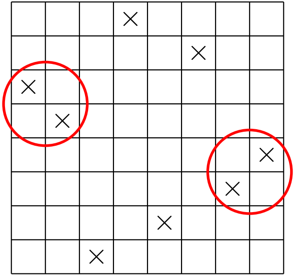

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Conjecture (Dokos, et al., 2012)

The patterns 1423 *and* 2413 *are Maj-Wilf-equivalent*

- $\mathrm{Maj}(\pi)$ *is sum of descents of* π.

$1423 \sim 2413$ revisited

Theorem (Bloom, 2014)

There is an explicit bijection

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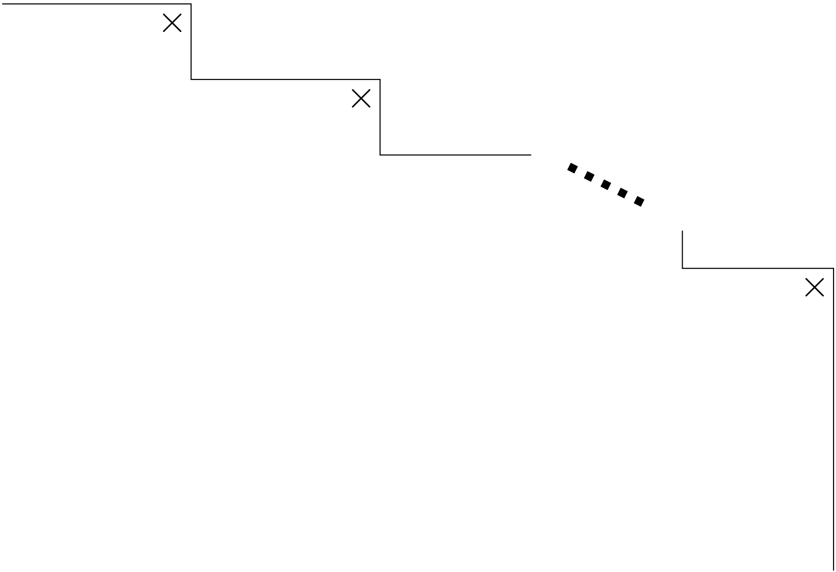
- Stankova's isomorphism does not preserve these statistics.

Anatomy of a 1423



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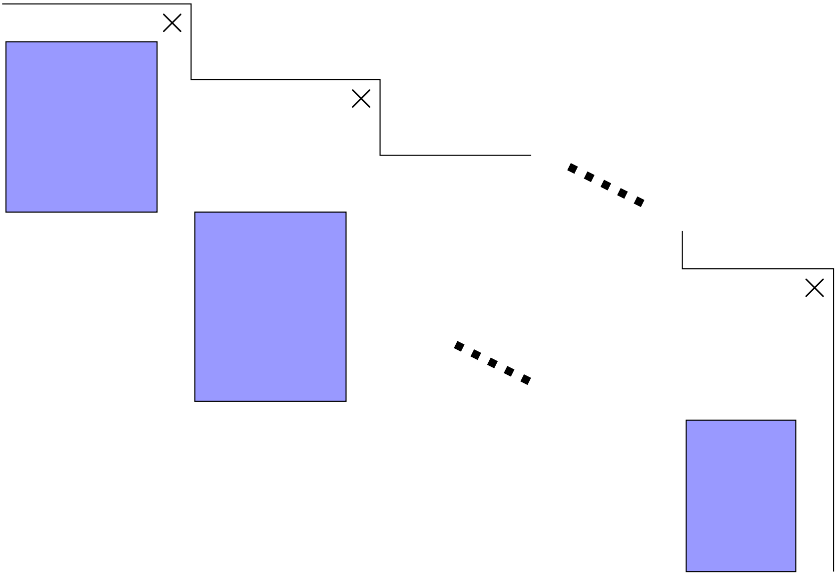

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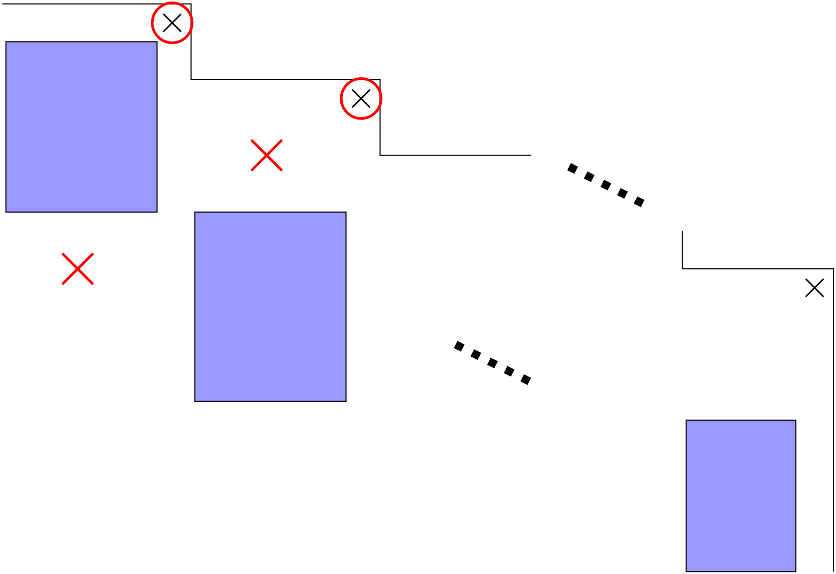

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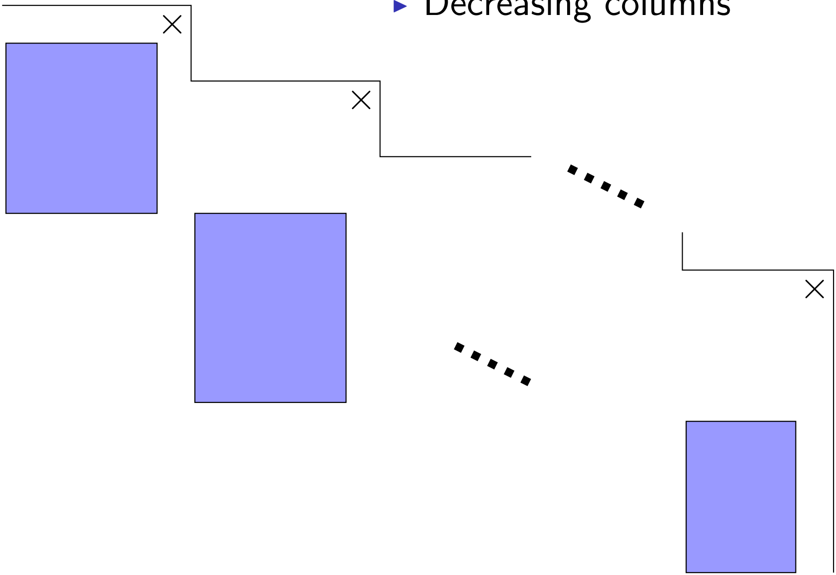
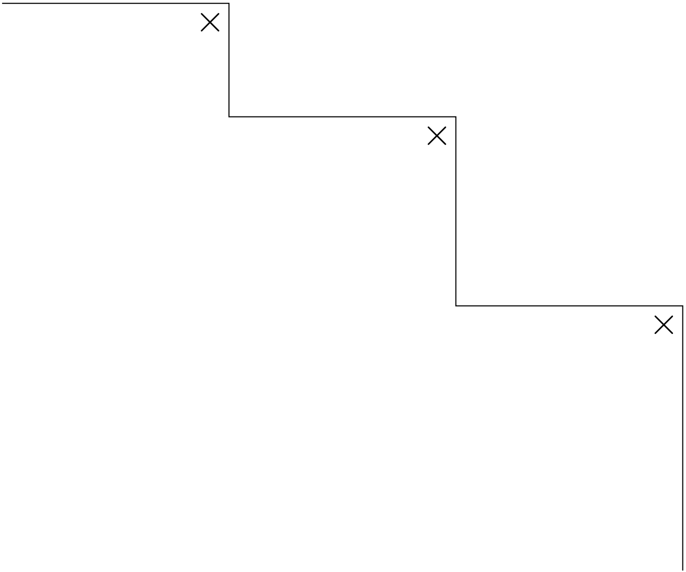


- Decreasing columns

Anatomy of a 2413



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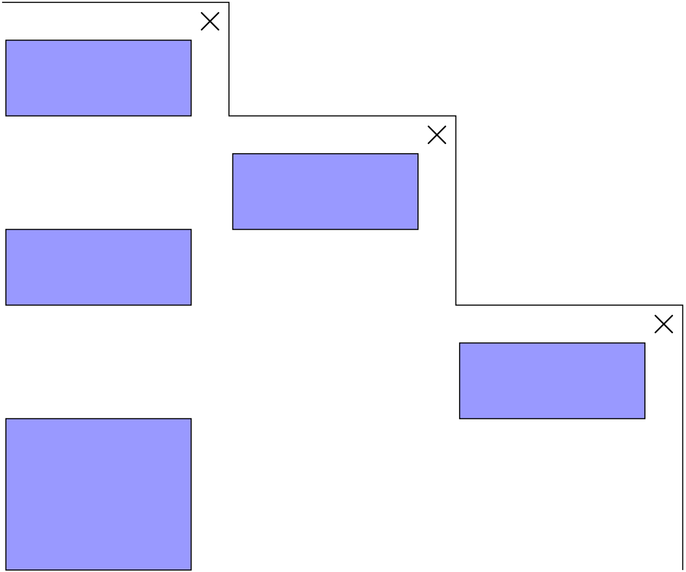

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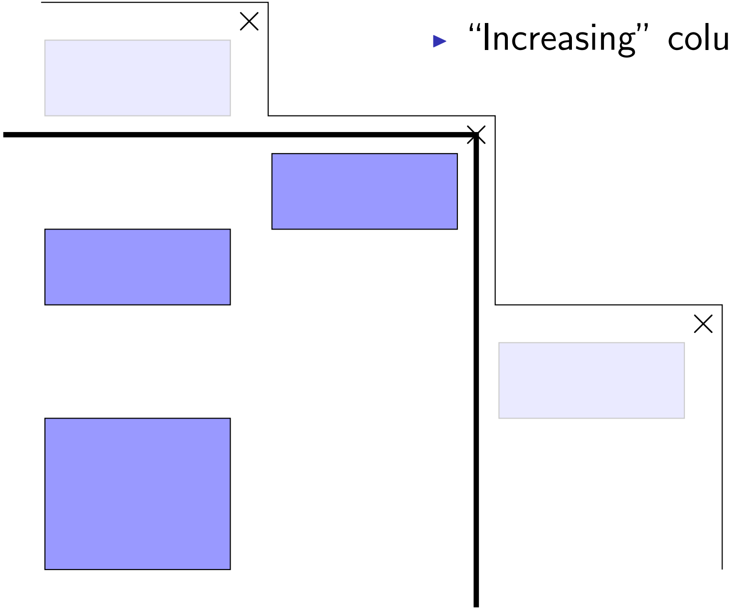

- "Increasing" columns

Anatomy of a 2413

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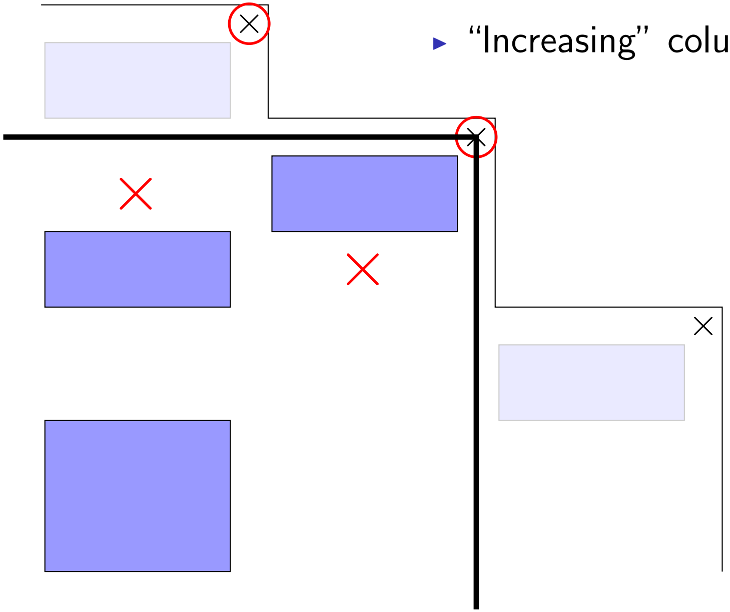

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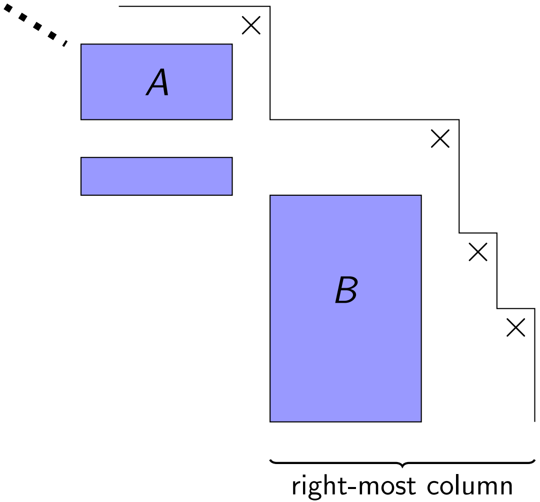

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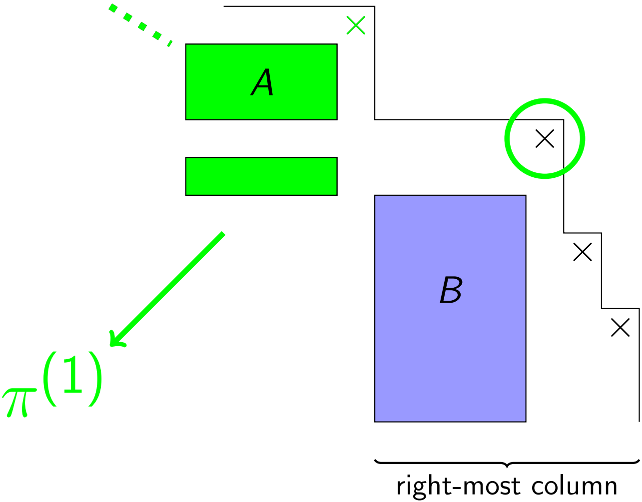

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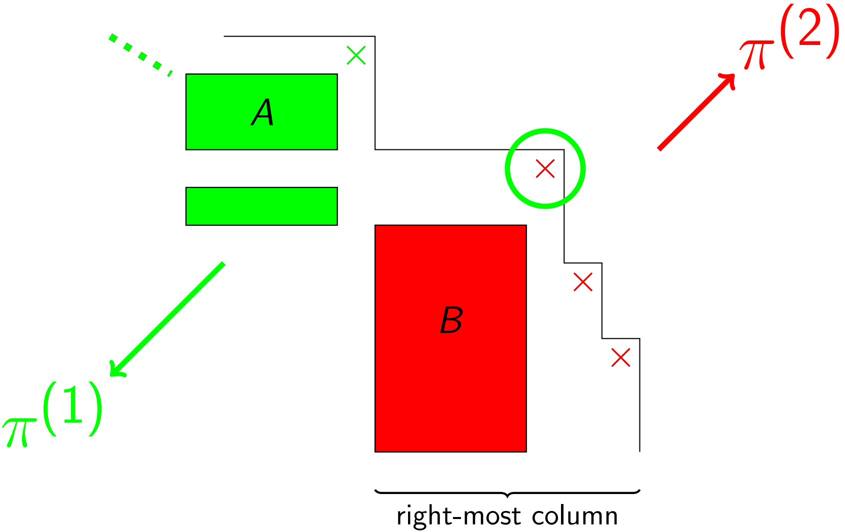

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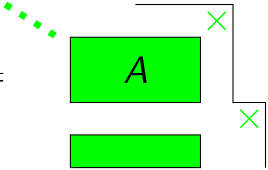

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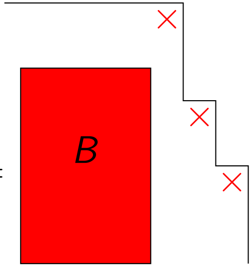
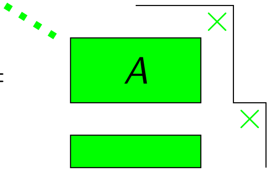
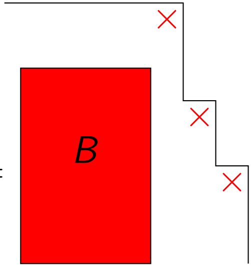


$\pi^{(1)} =$ 

$\pi^{(2)} =$ 

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exists and preserves statistics

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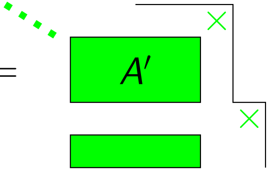
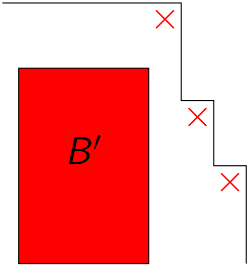
$\Theta(\pi^{(1)}) =$ 

$\Theta(\pi^{(2)}) =$ 

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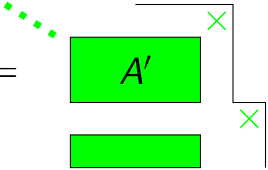

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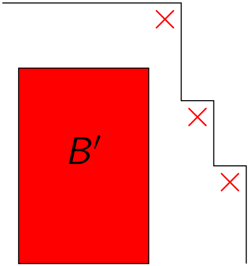

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★ Applying Θ to each part maintains structure!

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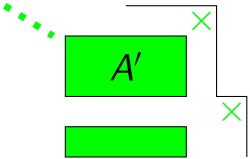

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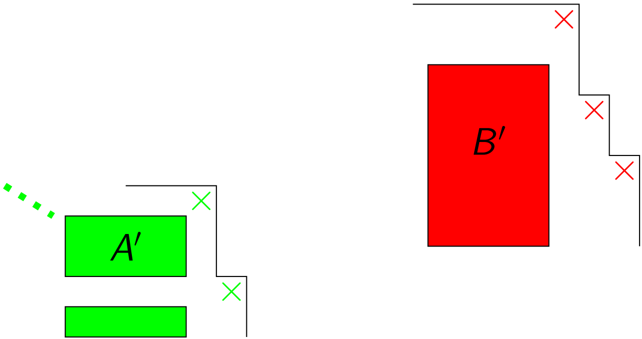

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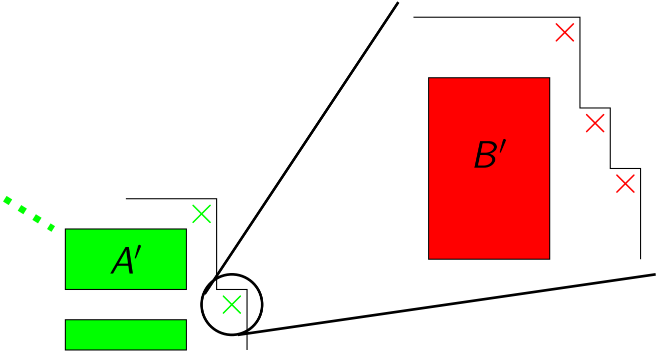
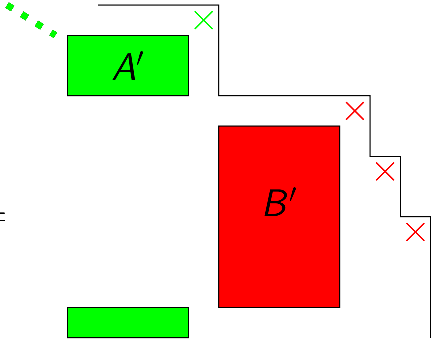


Doing this we obtain our final result:

$$\Theta(\pi) =$$



Part II

(Pattern classes & large Schröder numbers)

Egge's motivation

Consider the following table

$n =$	2	3	4	5	6	7	...
$\mathrm{Av}_n(2143, 3142)$	2	6	22	90	**395**	1823	...
nth large Schröder #	2	6	22	90	**394**	1806	...

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Question:

Are there any patterns $\tau \in S_6$ such that the sets

$$|\,\mathrm{Av}_n(2143, 3142, \tau)|$$

are counted by the large Schröder numbers?

Egge triples & unbalanced Wilf-equivalences

Conjecture (Egge, AMS Fall Eastern Meeting in 2012)

Fix $\tau \in \{246135, 254613, 524361, 546132, 263514\}$. *Then*

$$\sum_{n\geq 0} |\operatorname{Av}_n(2143, 3142, \tau)|x^n = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2},$$

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- Bloom and Burstein proved the remaining 4 cases
  - 263514: simple permutations
  - 254613, 524361, 546132: decomposition using LR-maxima

Unbalanced Wilf-equivalence

It is well known that the separable permutations, i.e., $\mathrm{Av}(2413, 3142)$ are also counted by large Schröder numbers, so

$$|\mathrm{Av}_n(2413, 3142)| = |\mathrm{Av}_n(2143, 3142, \tau)|,$$

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- Examples of unbalanced Wilf-equivalence abound!

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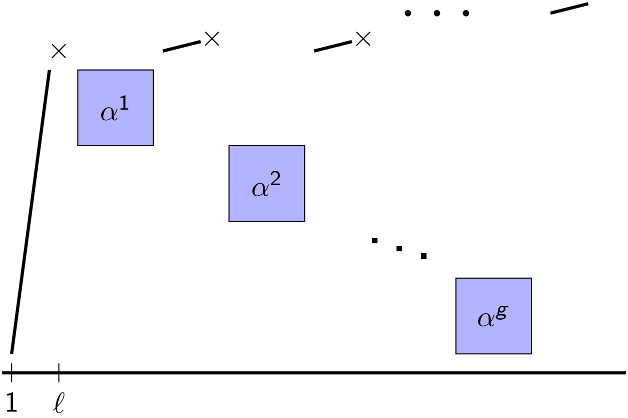

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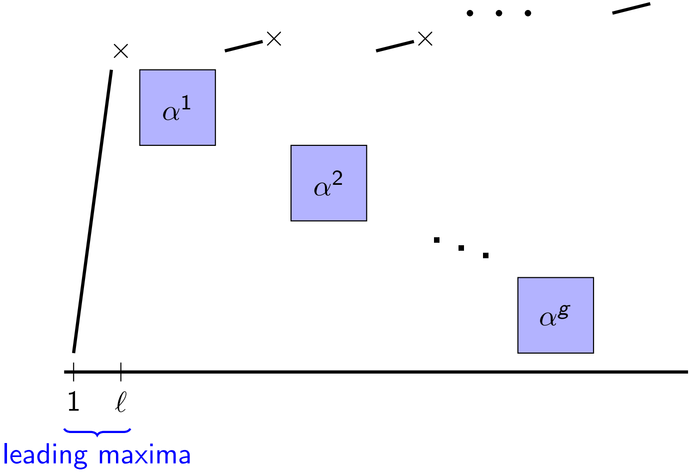

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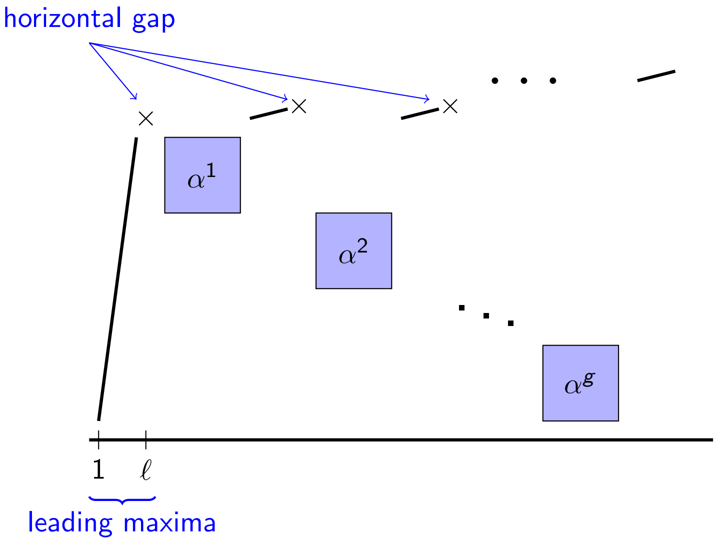

Counting $\tau = 254613$

Idea We consider three cases:

- No horizontal gaps
- Exactly 1 horizontal gap
- At least 2 horizontal gaps

Set

$$A(t,x) = \sum_{\pi \in \mathsf{Av}(2143,3142,\tau)} x^{|\pi|} t^{\ell(\pi)},$$

where $\ell(\pi)$ is the number of leading maxima in π.

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Case 1: No Horizontal gap

$\pi \in \text{Av}_n(2143, 3142, \tau)$ has no horizontal gap iff

$$\pi = 1\ 2\ \ldots\ n.$$

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Counted by

$$\frac{1}{1 - tx}.$$

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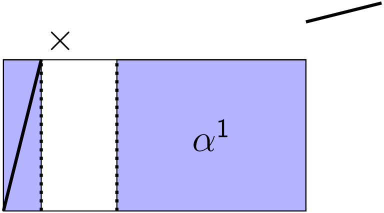
Case 2: Exactly 1 horizontal gap



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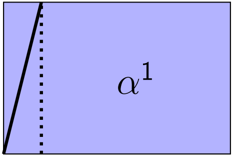

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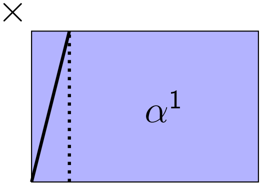

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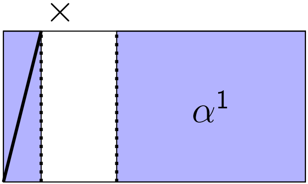

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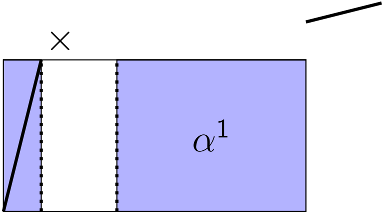

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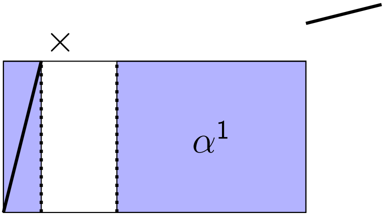


This translates to

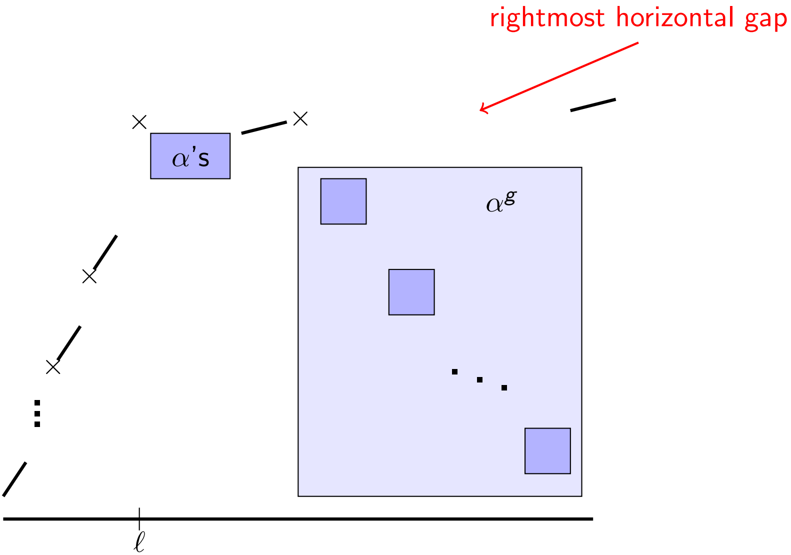
$$\frac{txE}{1-x}$$

where

$$E(t,x) = \frac{B - tA}{1-t} - \frac{1}{1-tx} \qquad \text{and} \qquad B = A(1,x).$$

Counting $\tau = 254613$

Case 3: At least 2 horizontal gap



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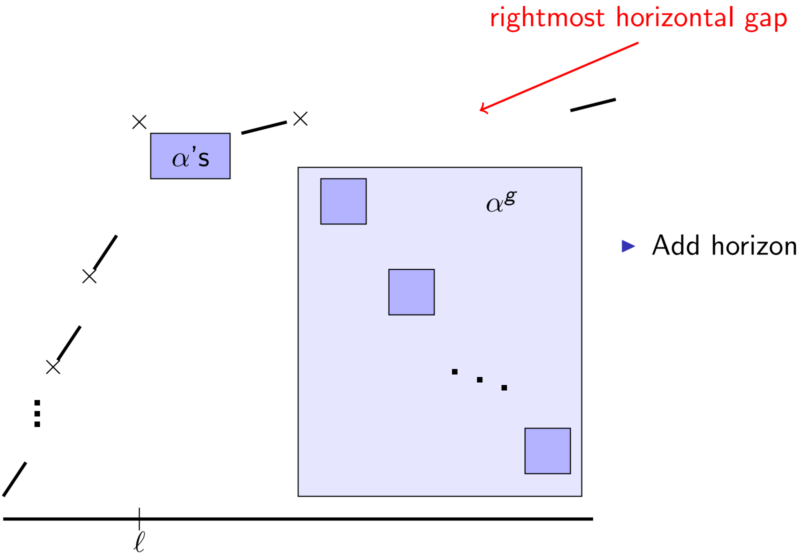

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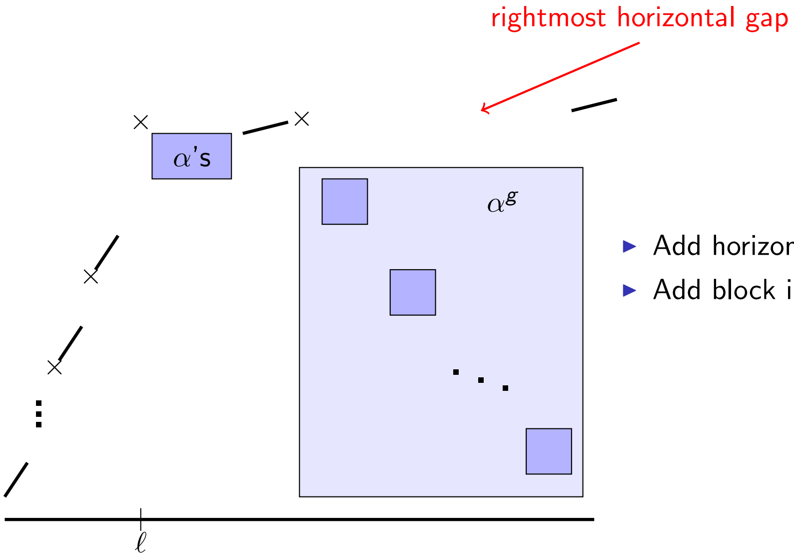

- Add horizontal gap
- Add block into gap

Counting $\tau = 254613$

All Together...

$$A(t,x) = \frac{1}{1-tx} + \frac{txE}{1-x} + \left(A - \frac{1}{1-tx}\right)\left(\frac{x(B-1)}{(1-x)(1-tx)}\right)\left(\frac{1}{1-\frac{tx(B-1)}{1-tx}}\right),$$

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$$\left(\frac{Bt^3x^2 + Bt^2x^2 - Bt^2x - Btx^2 + Bx - t^2x + t - 1}{(1-t)(1-x)(1-Btx)}\right) A_*$$
$$= \frac{xt}{1-x}\left(\frac{Btx - B + 1}{(t-1)(tx-1)}\right)$$

where $A_* = A - \dfrac{1}{1-xt}$.

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$$0 = Bt^3x^2 + Bt^2x^2 - Bt^2x - Btx^2 + Bx - t^2x + t - 1.$$

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  - The RHS yields: $Bxt(x) = B - 1$

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Solving (now) yields

$$\begin{aligned} A(1,x) = B &= \frac{3-x-\sqrt{1-6x+x^2}}{2} \\ &= 1+x+2x^2+6x^3+22x^4+90x^5+\cdots \end{aligned}$$

Thank You!