Another Look at Pattern Avoiding Permutations

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So, $\Gamma(\sigma) = 6\ 4\ 3\ 5\ 1\ 2$

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Here we will give a pictorial description that will completely elucidate these properties.

Permutation Templates...

Definition: Given a $\sigma \in S_n$ a **permutation template** T for σ is a shading of an $n \times n$ grid such that if, for each row, we inductively place a dot in the left most unshaded square which has no dot in a square above then the squares with dots are precisely in row i and column σ_i for $i \in [n]$.

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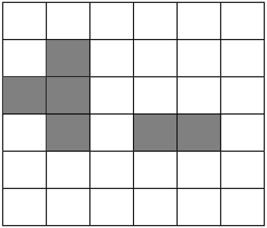

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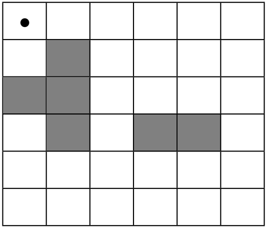

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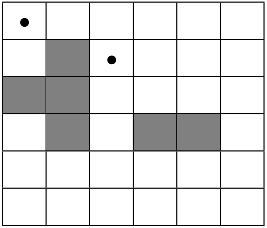

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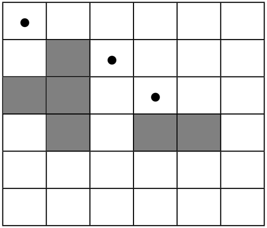

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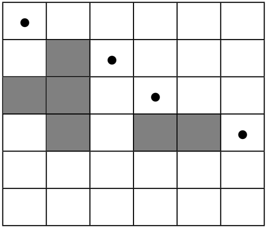

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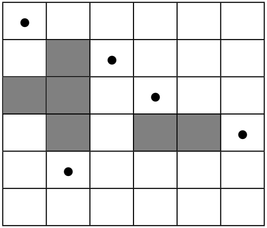

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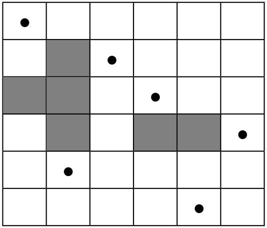

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Fact (Monotonicity): For $\sigma \in S_n(321)$ the 2-elements are in increasing order from left to right. Likewise, the 1-elements are in increasing order from left to right.

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4. Record the position of the 2-element and the value of the 1-element for each of the 21-patterns found above.

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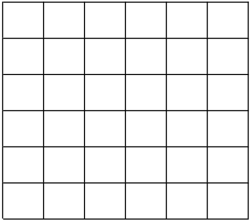

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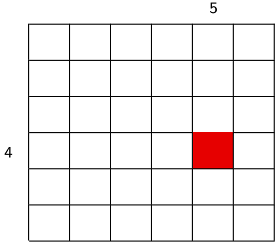

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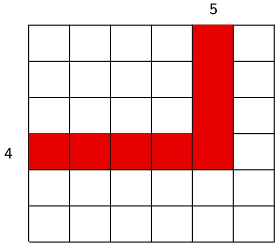

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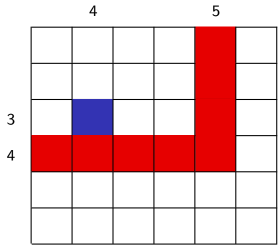

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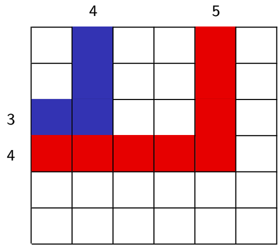

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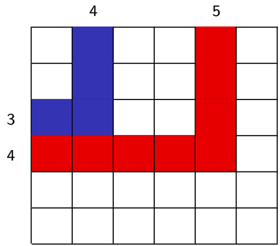

Theorem (Bloom, Saracino): If $\sigma \in S_n(321)$ then a permutation template for σ, call it T_σ, is the union of non-overlapping reversed L's.

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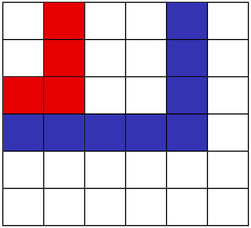

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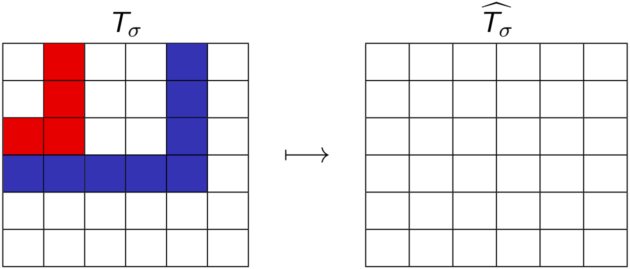

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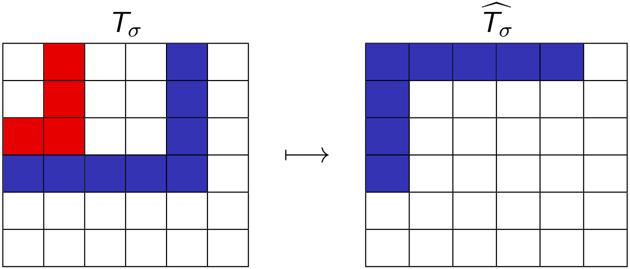

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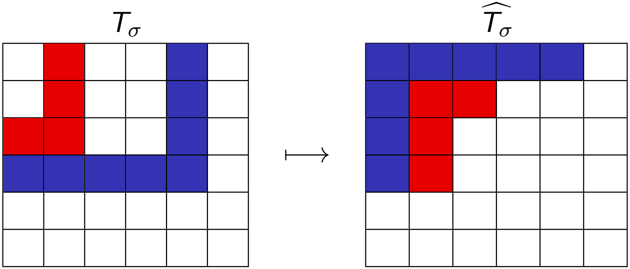

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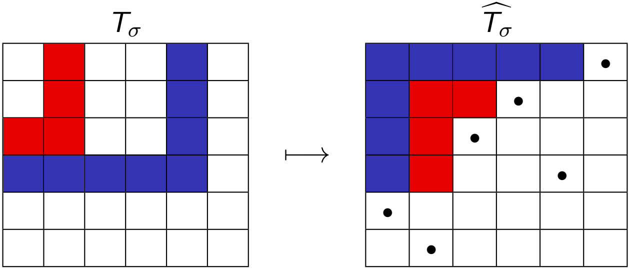

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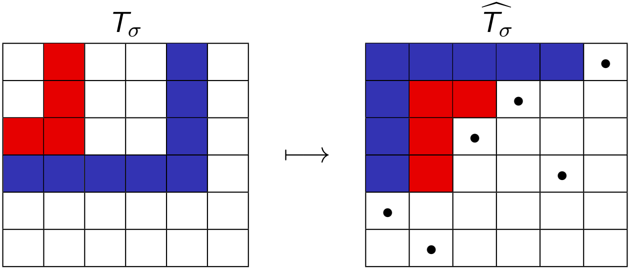

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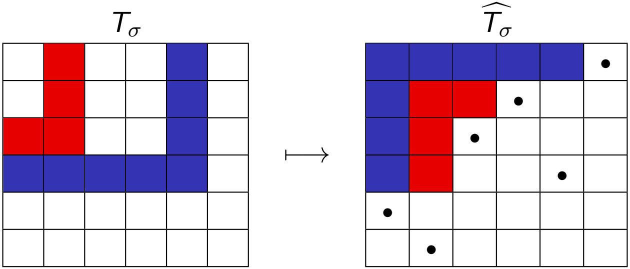

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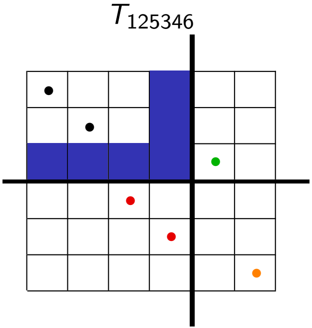

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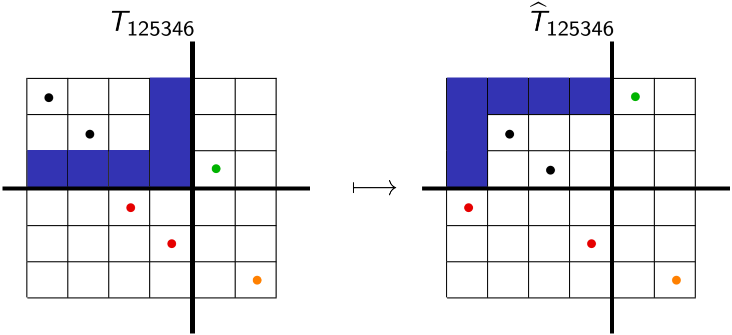

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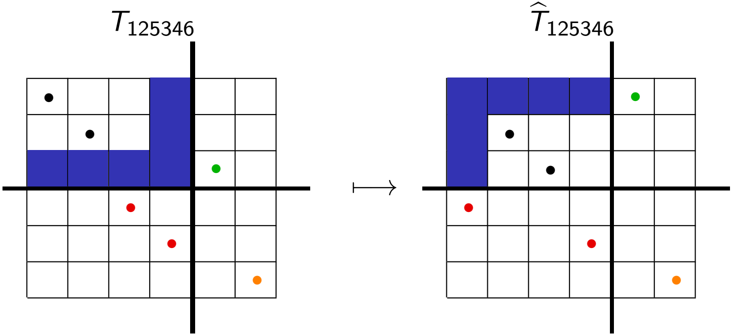

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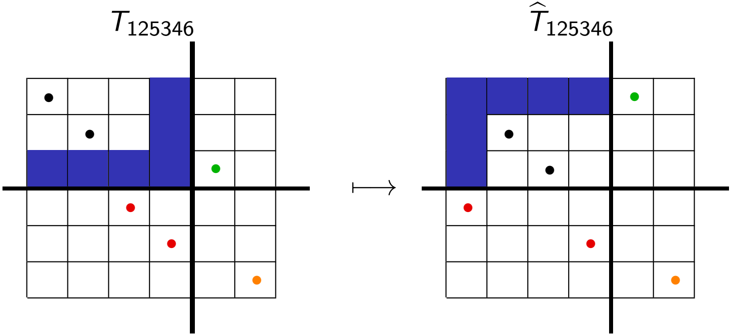

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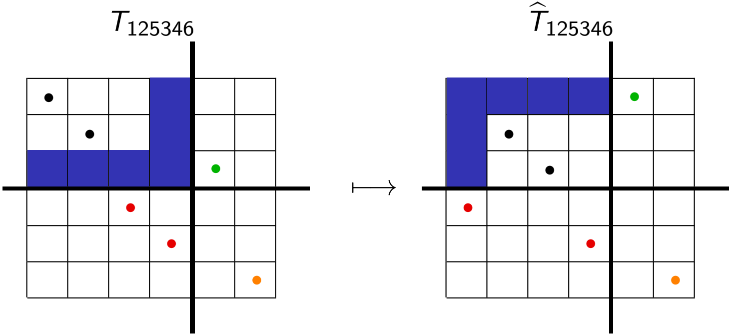

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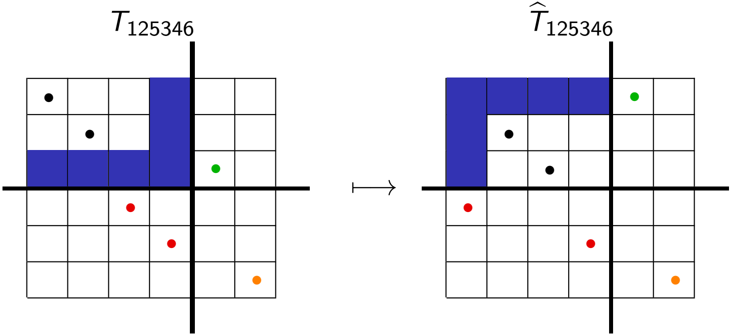
Theorem: Γ preserves the number of fixed points and excedances.



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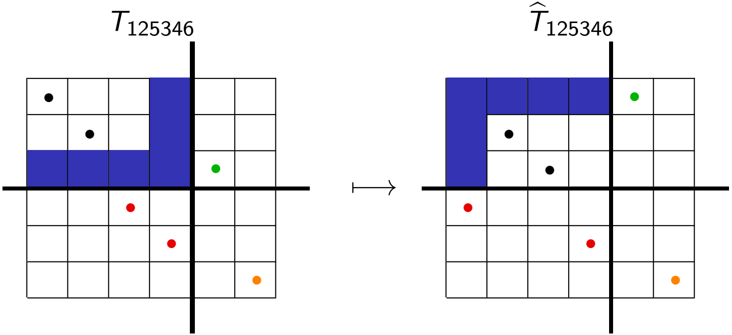

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Γ Commutes with Inverses - The Easy Way!

Let $\sigma = 1\ 3\ 4\ 6\ 2\ 5 \in S_6(321)$

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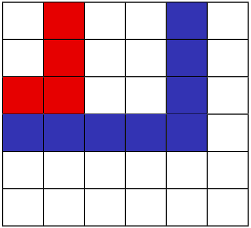

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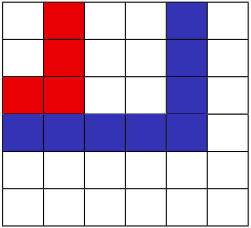

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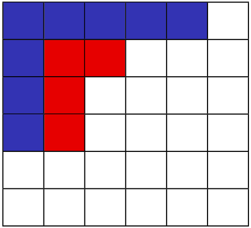
$\widehat{T_\sigma}$ = Template for $\Gamma(\sigma)$



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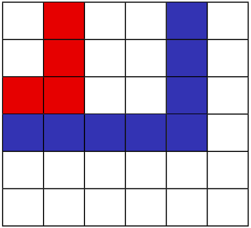

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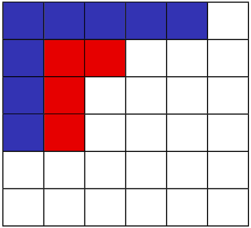
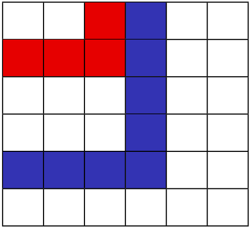


$T_{\sigma^{-1}}$



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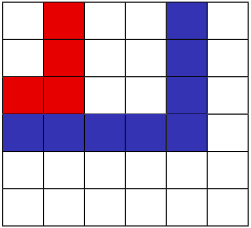

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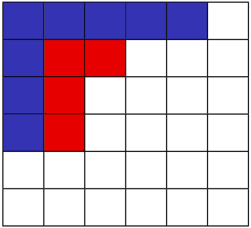

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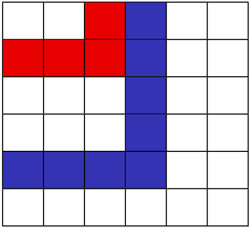

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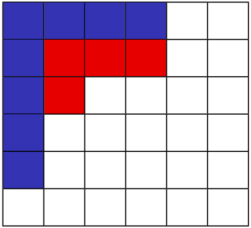

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Theorem: If $\sigma \in S_n(321)$ then $\Gamma(\sigma^{-1}) = \Gamma(\sigma)^{-1}$.

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- As a template uniquely determines a permutation we must have $\Gamma(\sigma^{-1}) = \Gamma(\sigma)^{-1}$