Patterns, Permutations, and Placements

Jonathan S. Bloom

Dartmouth College

Lafayette College - February 2014

Permutations

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Definition

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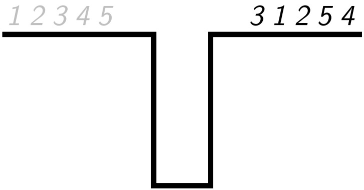

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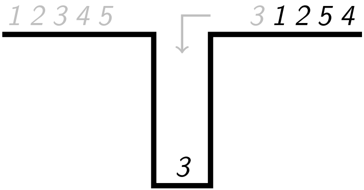

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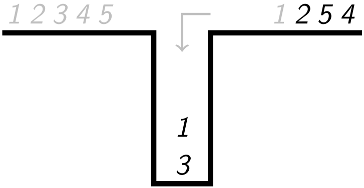

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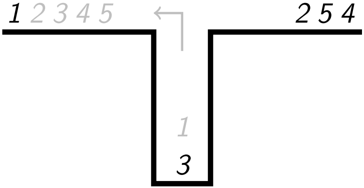

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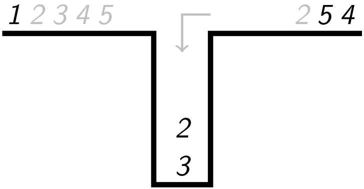

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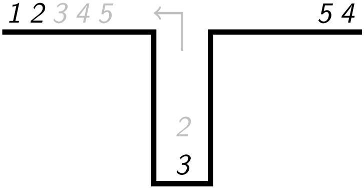

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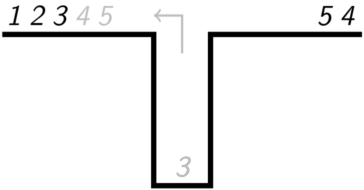

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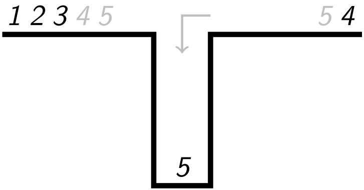

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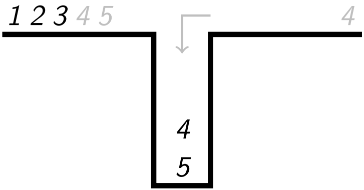

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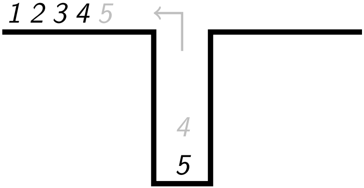

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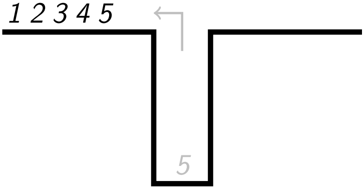

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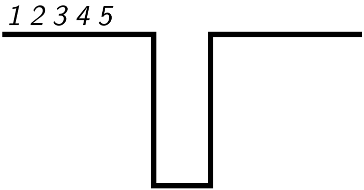

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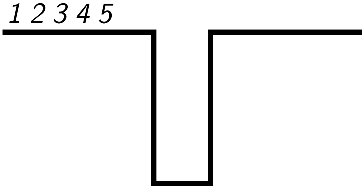

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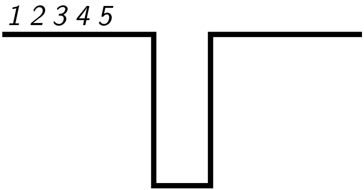

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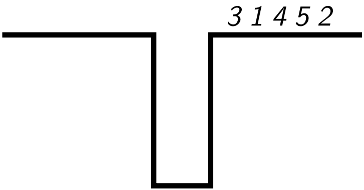

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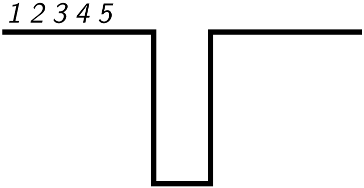

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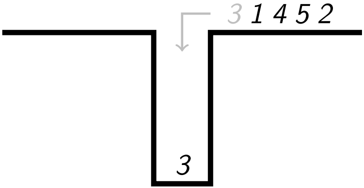

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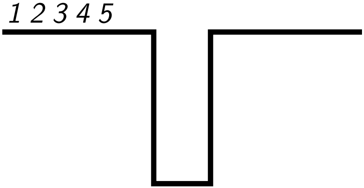

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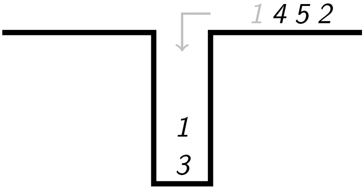

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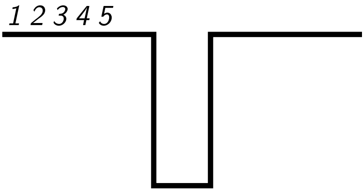

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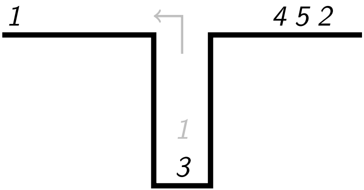

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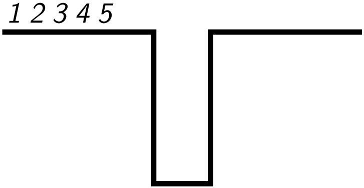

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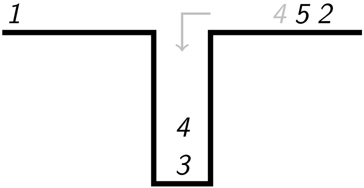

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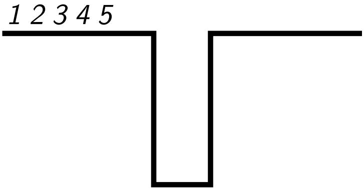

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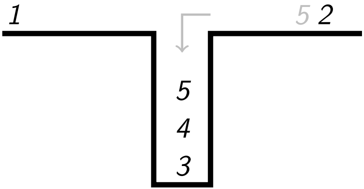

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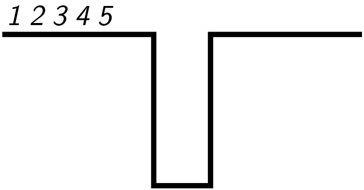

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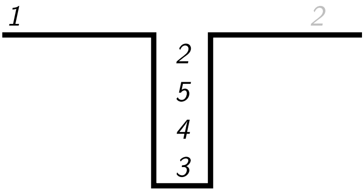

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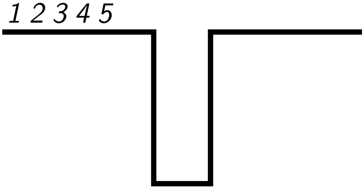

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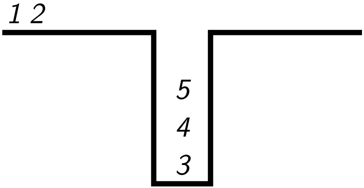

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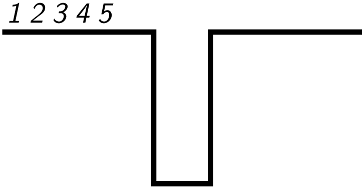

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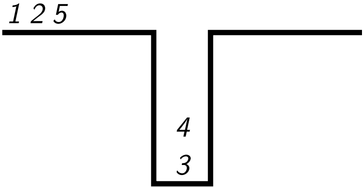

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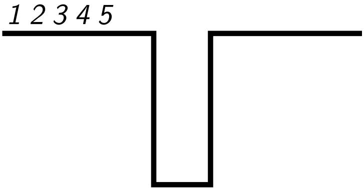

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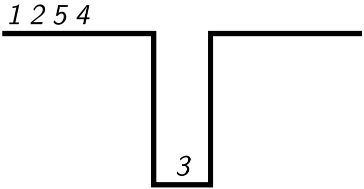

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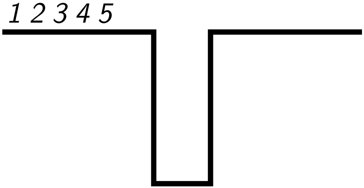

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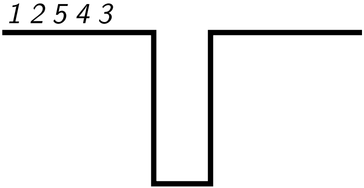

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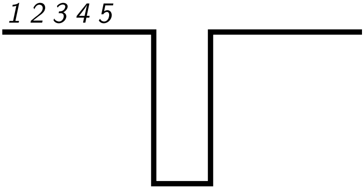
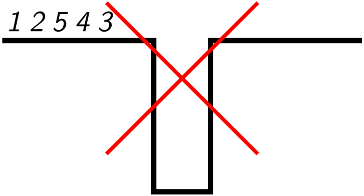


Let $\pi = 3\ 1\ 4\ 5\ 2$

- π *is NOT stack-sortable*



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Theorem (D. Knuth 1968)

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$\pi =$ **3** 1 4 **5** **2** *is NOT stack-sortable*
$\Rightarrow \pi$ **contains** *the pattern 231*

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$\pi = \mathbf{3}\ 1\ 4\ \mathbf{5}\ \mathbf{2}$ *is NOT stack-sortable*
$\Rightarrow \pi$ **contains** *the pattern 231*

$\alpha = 3\ 1\ 2\ 5\ 4$ *is stack-sortable*
$\Rightarrow \alpha$ **avoids** *the pattern 231*

Permutation Patterns

It's easier with pictures!

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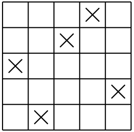

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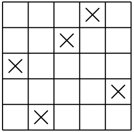

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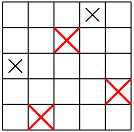

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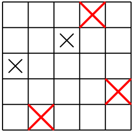

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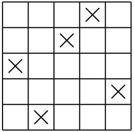

- π contains 132
- π avoids 321

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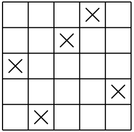

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Notation

- $S_n(\tau) =$ *the set of permutations of length n that avoid τ.*

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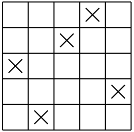

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Definition

Two patterns σ and τ are **Wilf-equivalent** if for all n,

$$|S_n(\tau)| = |S_n(\sigma)|.$$

Permutation Patterns

Example (Patterns of length 2)

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$$S_n(21) =$$

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Permutation Patterns

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$$S_n(21) = \{123\ldots n\}.$$

$$S_n(12) = \{n\ldots 321\}.$$

$\Rightarrow$ 12 *is Wilf-equivalent to* 21

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Patterns of length 3

If τ is any pattern of length 3, then

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|S_6(\tau)| &= 132 \\
&\vdots
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|S_n(\tau)| &= \frac{1}{n+1}\binom{2n}{n} = n\text{th Catalan number}
\end{aligned}
$$

Permutation Patterns

Patterns of length 3

If τ is any pattern of length 3, then

$$
\begin{aligned}
|S_3(\tau)| &= 5 \\
|S_4(\tau)| &= 14 \\
|S_5(\tau)| &= 42 \\
|S_6(\tau)| &= 132 \\
&\vdots \\
|S_n(\tau)| &= \frac{1}{n+1}\binom{2n}{n} = n\text{th Catalan number}
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- ALL length 3 patterns are Wilf-equivalent

Permutation Patterns

Patterns of length 4

Permutation Patterns

Patterns of length 4

$n =$	5	6	7	8	9
$\|S_n(2314)\|$	103	512	2740	15485	91245
$\|S_n(1234)\|$	103	513	2761	15767	94359
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What's Known?

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Open Problem

Find a formula for $|S_n(1324)|$.

Rook Placements

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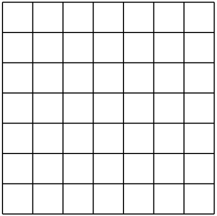

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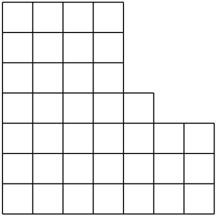

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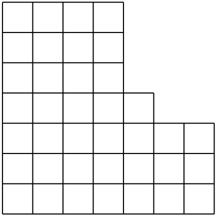
$F =$ 

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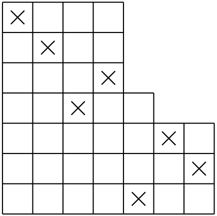
$F =$ 

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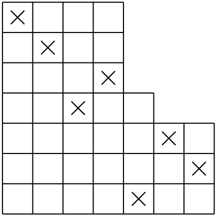
$F =$ 

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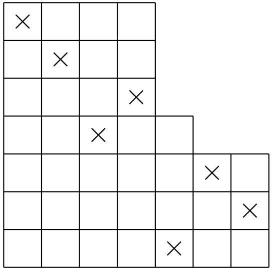
Notation

- $\mathcal{R}_F =$ *set of all f.r.p.'s on the Ferrers board* F

Rook Placements

Patterns?



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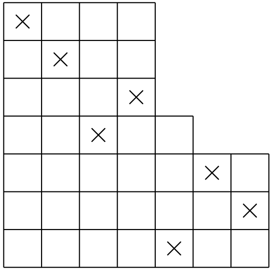

- **contains** the pattern 312

Rook Placements

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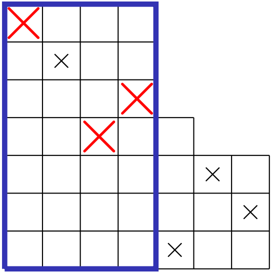

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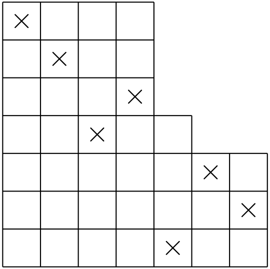

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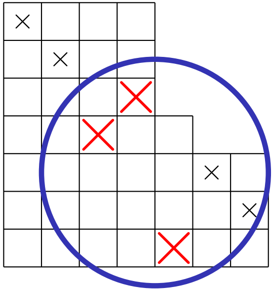

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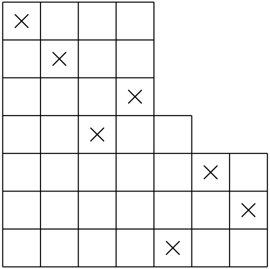

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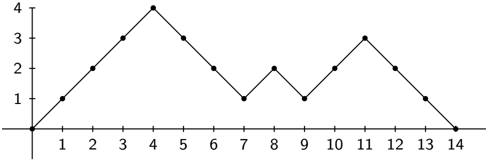
  - Can count things!

Dyck Paths

Dyck Paths



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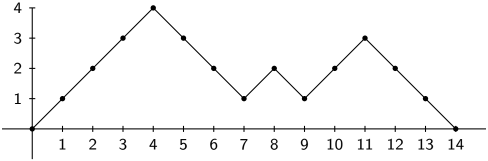
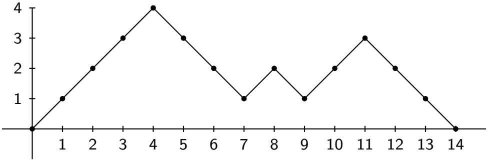


A **Dyck path** of size *n* is a path that:

Dyck Paths



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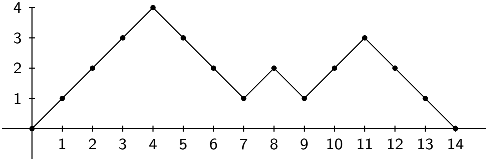

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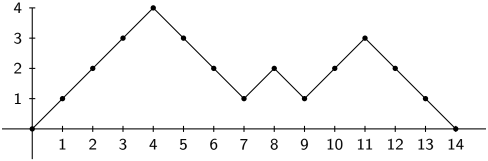
Dyck Paths



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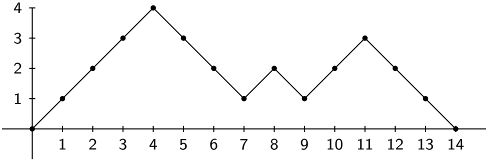
Dyck Paths



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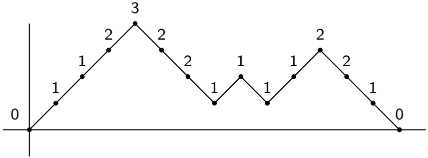
It is well known that

$$\# \text{ Dyck paths of size } n = \frac{1}{n+1}\binom{2n}{n}$$

Labeled Dyck paths



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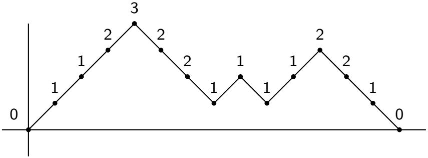

We label the Dyck path so that:

- Monotonicity
  - $+1/0$ up step and $-1/0$ down step

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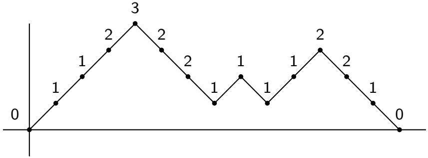

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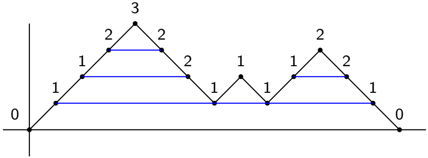

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- Zero Condition
  - All zeros lie precisely on the x-axis
- Tunnel Property
  - "Left" $\leq$ "Right"

Our proof of $231 \sim 312$

An outline

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1. 231-avoiding rook placement $\mapsto$ Tunnel property

Our proof of 231 $\sim$ 312

An outline

1. 231-avoiding rook placement $\mapsto$ Tunnel property
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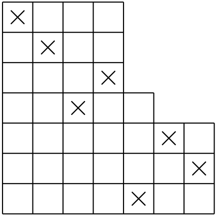

$\mathcal{R}_F(231)$

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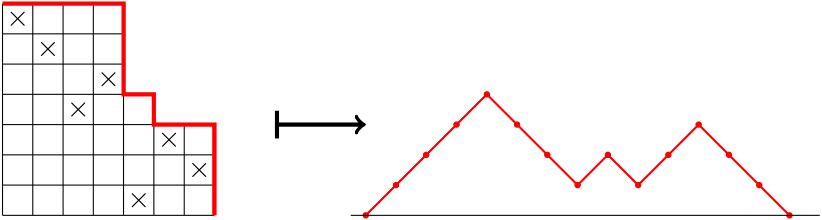

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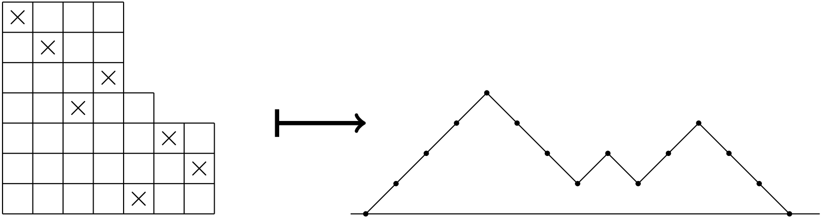

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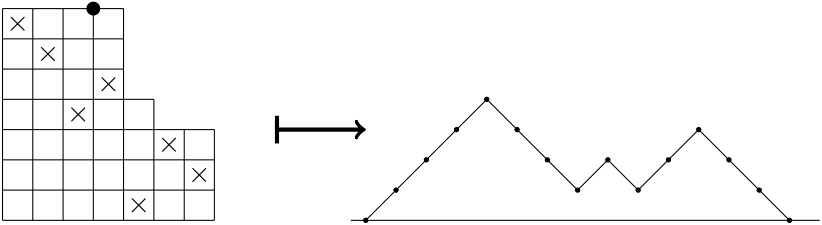

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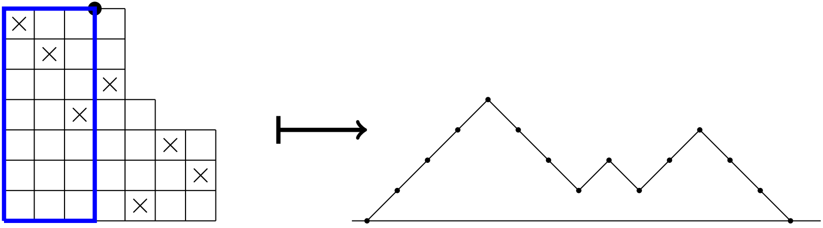

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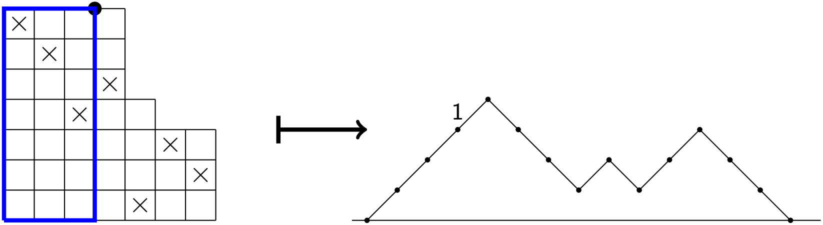

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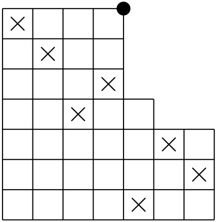

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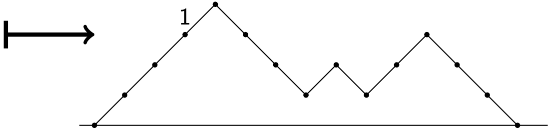

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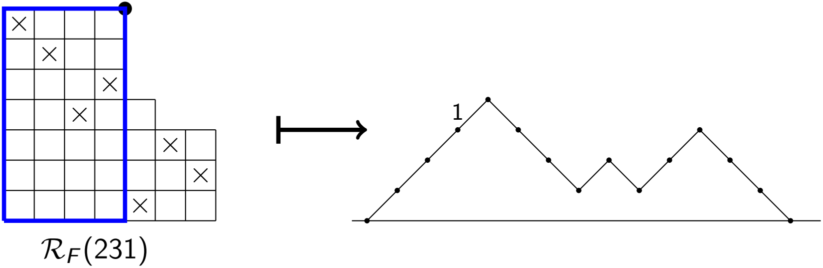

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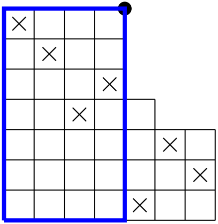

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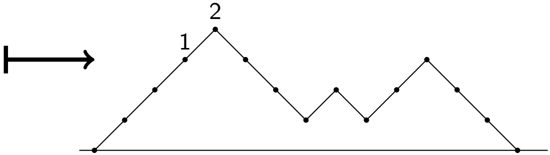

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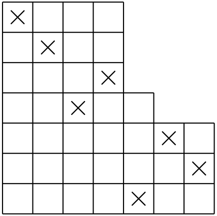

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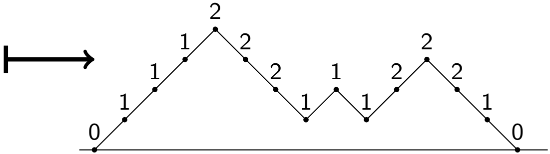
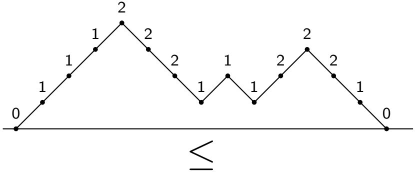


$\leq$

Our proof of 231 $\sim$ 312

2. Tunnel property $\mapsto$ Reverse tunnel property



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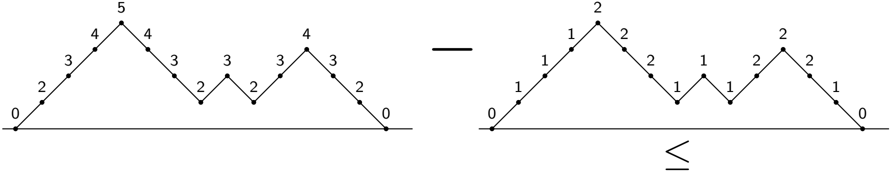

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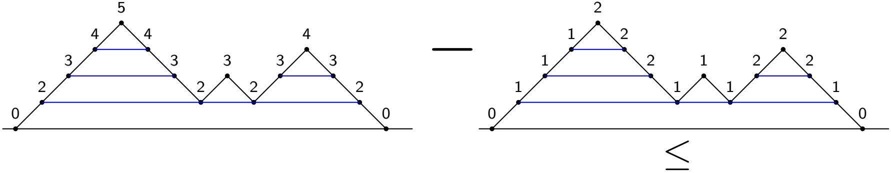

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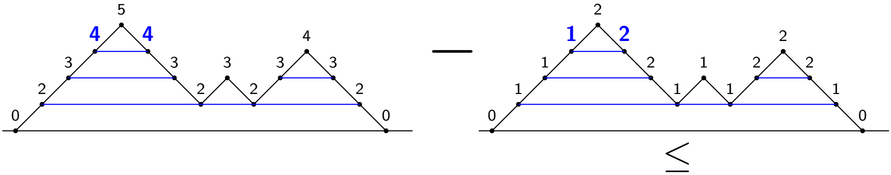

Our proof of 231 ∼ 312

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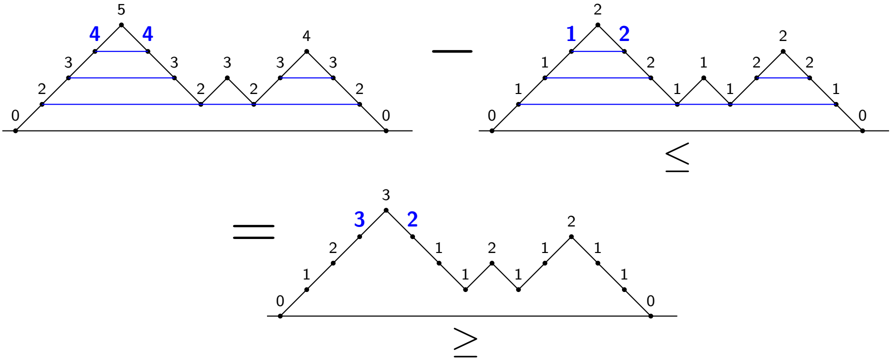
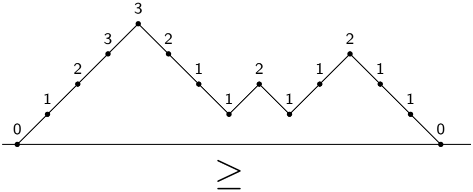


Our proof of $231 \sim 312$

3. Reverse tunnel property $\mapsto$ 312-avoiding f.r.p.



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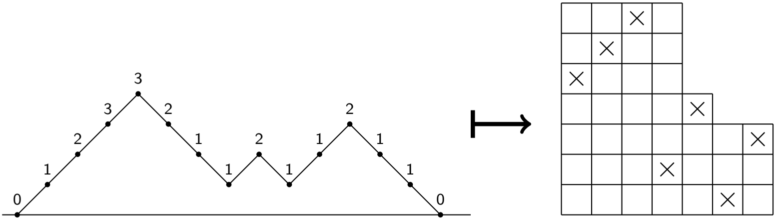

$\geq$

$\mathcal{R}_F(312)$

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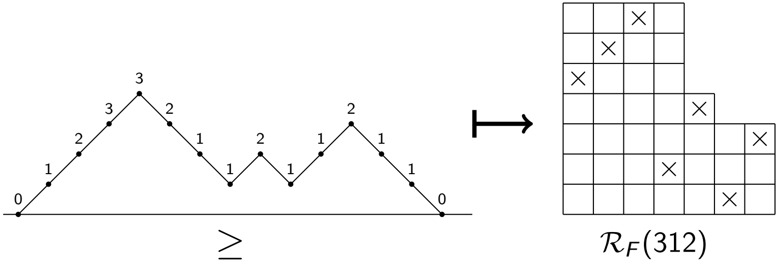

Theorem (Bloom–Saracino '11)

This mapping is a bijection between $\mathcal{R}_F(231)$ and $\mathcal{R}_F(312)$.
$\Rightarrow$ *231 and 312 are shape-Wilf-equivalent.*

Enumerative Results: 2314-Avoiding Permutations

In 1997, Bóna proved the celebrated result:

$$|S_n(2314)| = (-1)^n \left[\frac{-7n^2 + 3n + 2}{2} + 6 \sum_{i=2}^{n} (-2)^i \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2} \right]$$

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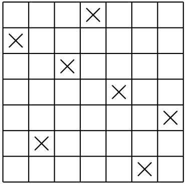

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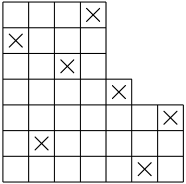

$S_n(2314)$ $\qquad$ $R_F(231)$

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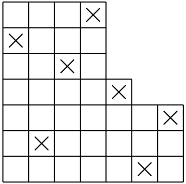

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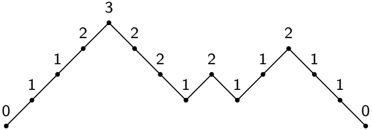

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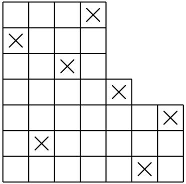
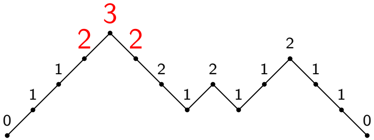
6257413 $\longmapsto$  $\longmapsto$ 

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Thank you!

Appendix: New Enumerative Results

- In 2012, D. Callan and V. Kotesovec conjectured that

$$\sum_{n=0}^{\infty} |S_n(2314, 1234)| z^n = \frac{1}{1 - C(zC(z))}$$
$$= 1 + z + 2z + 6z^2 + 22z^3 + \cdots$$

where $C(z)$ is the generating function for the Catalan numbers.

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- New enumerative results in the theory of perfect matchings and set partitions.