Pattern Avoidance in Ferrers Boards and Set Partitions

Jonathan Bloom
(Joint work with Sergi Elizalde & Dan Saracino)

Dartmouth College

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Ferrers Boards & Full Rook Placements

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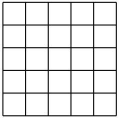

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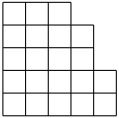

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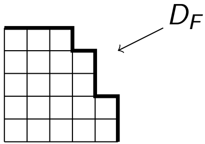

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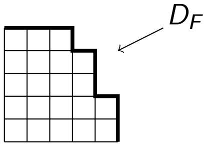

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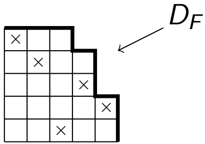

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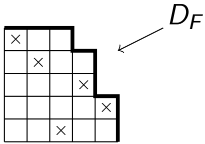

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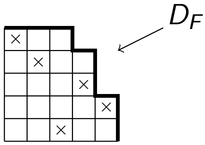

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- *$\mathcal{R}_F$ is the set of all f.r.p on $F \in \mathcal{F}_n$ and*

$$\mathcal{R}_n := \bigcup_{F \in \mathcal{F}_n} \mathcal{R}_F.$$

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We say a f.r.p. on $F \in \mathcal{F}_n$ avoids some pattern $\sigma \in S_k$ if the following happens. For any rectangle **inside** F the "permutation" in this rectangle avoids σ in the classical sense.

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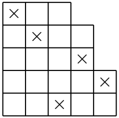

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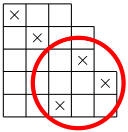

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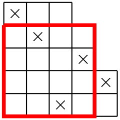

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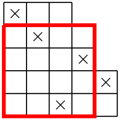

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We say two patterns $\sigma, \tau \in S_k$ are *shape-Wilf-equivalent* if for any Ferrers board $F \in \mathcal{F}_n$

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- Analyze the shape-Wilf-equivalence for pairs of patterns with length 3.

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$F \in \mathcal{F}_n$ *iff* $D_F \in \mathcal{D}_n$. *Therefore* $\mathcal{F}_n$ *and* $\mathcal{D}_n$ *are in bijection!*

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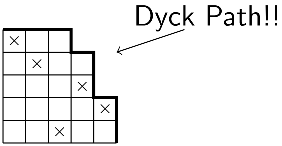
Proof by example:



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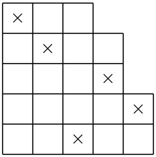

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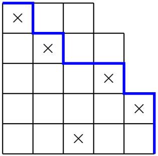

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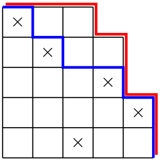

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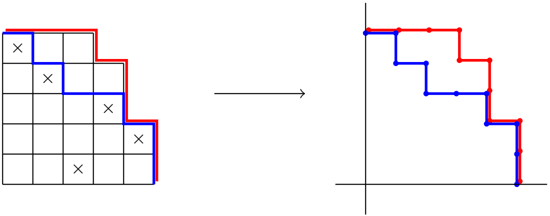

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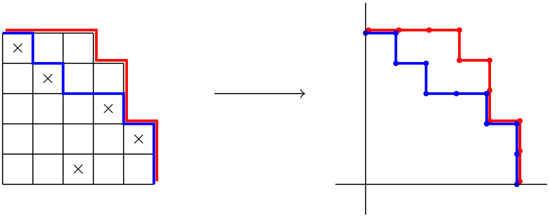

- Observe that $54132 \in S(213)$
  - Blue path results from the standard bijection $S(213) \to \mathcal{D}_n$.

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Theorem (J. Bloom, D. Saracino)

Let $F \in \mathcal{F}_n$. There exists an (simple) explicit bijection

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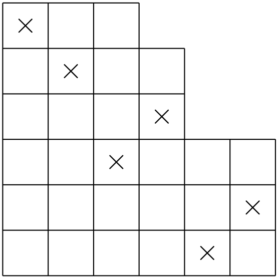

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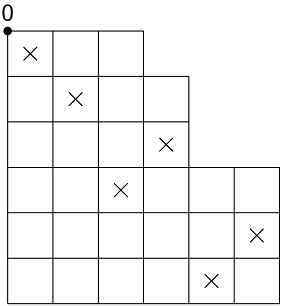

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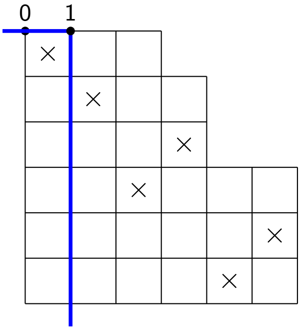

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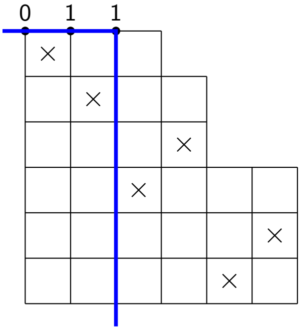

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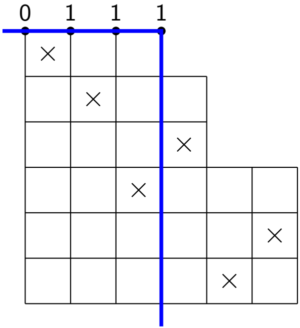

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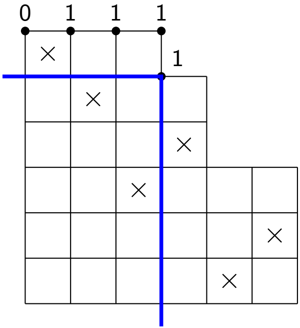

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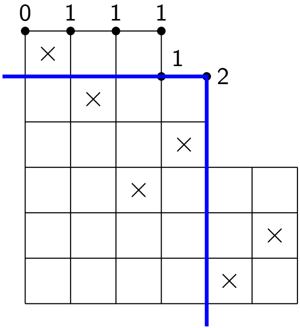

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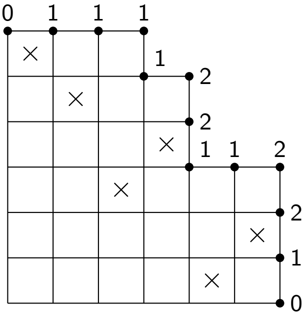

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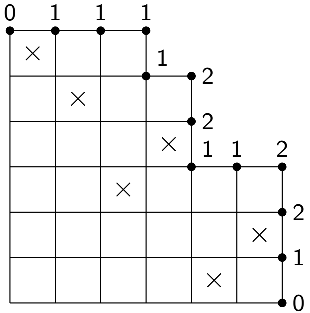

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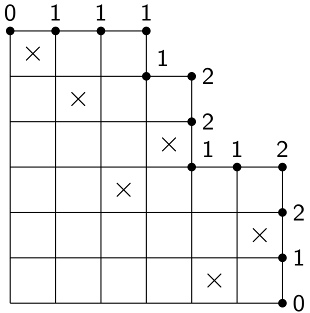

Critical Properties:

- Monotone Property
  - Increase ≤ 1 over east step.
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The Enumeratation of $\mathcal{R}_n(231)$

Theorem (J. Bloom, D. Saracino)

Let $F \in \mathcal{F}_n$. There exists an (simple) explicit bijection

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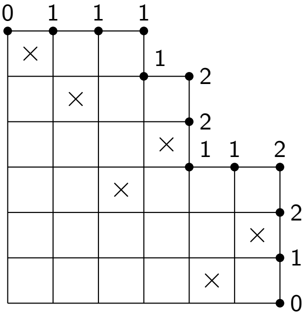

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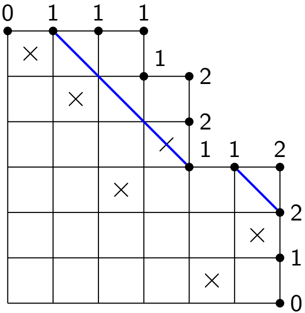

Critical Properties:

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- Zero Condition
  - Zero labels are those on the diagonal.
- Diagonal Property
  - For any diagonal: "Top" $\leq$ "Bottom".

Theorem (J. Bloom, S. Elizalde)

$$\sum_{n\geq 0} |\mathcal{R}_n(231)| z^n = \sum_{n\geq 0} |\mathcal{L}_n| z^n = \frac{54z}{1+36z-(1-12z)^{3/2}},$$

where $\mathcal{L}_n := \bigcup_{F\in\mathcal{F}_n} \mathcal{L}_F$. *Further, we obtain*

$$|\mathcal{R}_n(231)| \sim \frac{3^3}{2^5\sqrt{\pi n^5}} 12^n.$$

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⋆ Interestingly, labelings that have only the Monotone Property and the Diagonal Property are in bijection with rooted planar maps!!

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Any $\pi \in S_n(2314)$ may be thought of as a full rook placement on a **minimal** Ferrers board.

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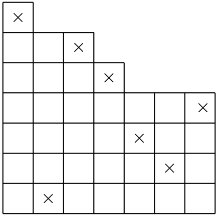

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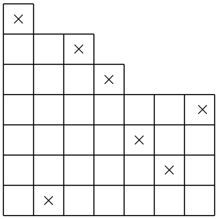

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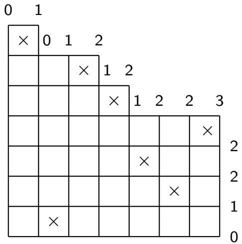

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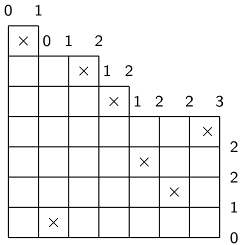

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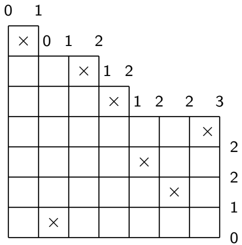

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Notation

- *Let $\mathcal{L}_n^\times \subset \mathcal{L}_n$ be all labelings with the peak property.*

The pattern 2314

Lemma (J. Bloom, S. Elizalde)

This composition of maps gives a bijection

$$S_n(2314) \to \mathcal{L}_n^{\times}(231),$$

and therefore

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Doing so we obtain

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Matchings & Set Partitions

A *perfect matching* M is graph such that every vertex is "matched" with another vertex.

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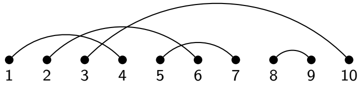

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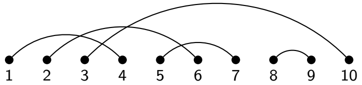

A set partition may also be represented by a graph.

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$$\{\{1\},\{2,5\},\{3,7\},\{4,12\},\{6,8,11\},\{9\},\{10\}\}$$

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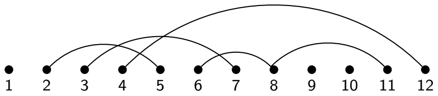

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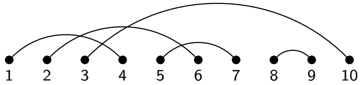

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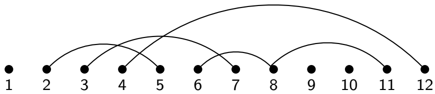


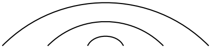
Notation

- $\mathcal{M}_n$ *is set of all matchings on* $2n$ *vertices.*
- $\mathcal{P}_n$ *is set of all set partitions on* n *vertices.*

Patterns in Matchings and Set Partitions

In this context, a pattern is a certain configuration of arcs. For example consider:

3-crossing



3-nesting

⋆ Both contain a 3-crossing and both avoid a 3-nesting.

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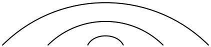

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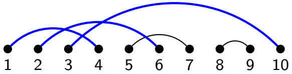

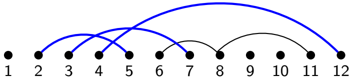

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Now consider our examples:

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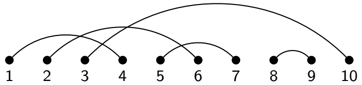
Notation

If τ is a configuration let...

- $\mathcal{M}_n(\tau)$ *be matchings on* $2n$ *vertices that avoid* τ*.*
- $\mathcal{P}_n(\tau)$ *be set partitions of* $[n]$ *that avoid* τ*.*

Matchings & Full Rook Placements



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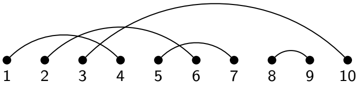

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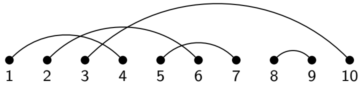

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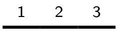

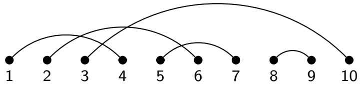

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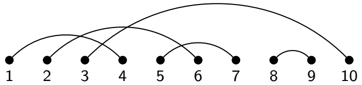

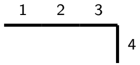

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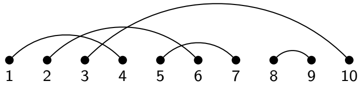

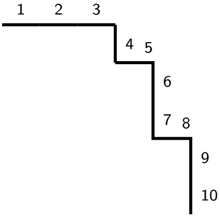

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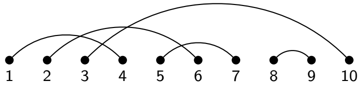

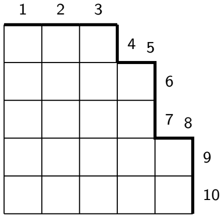

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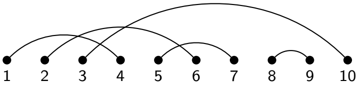

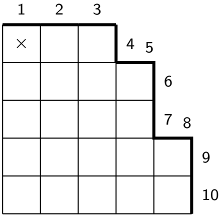

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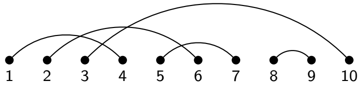

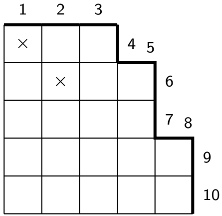

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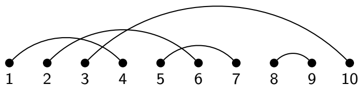

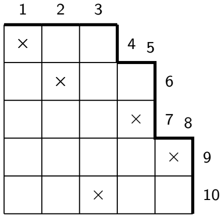

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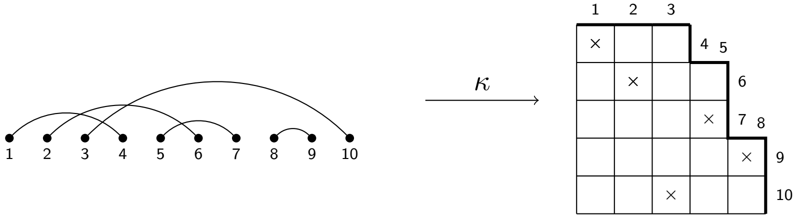

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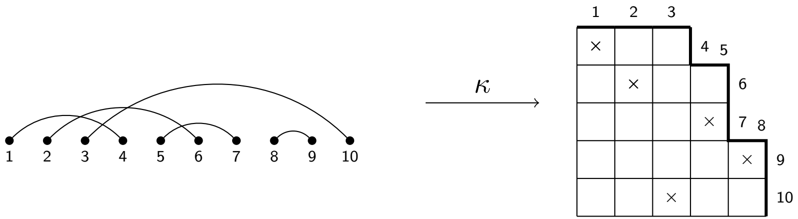

The mapping (due to C. Krattenthaler) $\kappa : \mathcal{M}_n \to \mathcal{R}_n$ is a bijection.

- ⋆ This will permit us to translate between matchings and f.r.p.

Matchings & Partitions that avoid 231

In particular, $\kappa : \mathcal{M}_n(231) \to \mathcal{M}_n(\tau)$ where τ is the configuration:

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In light of this the following notation makes sense...

Notation

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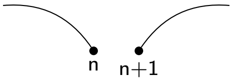
⋆ The other patterns of length 3 correspond to "nice" configuration of 3 arcs as well.
For example,

- 123 $\mapsto$ 3-nesting.
- 321 $\mapsto$ 3-crossing.

Matchings & Partitions that avoid 231

Definition

In a matching M a *valley* is the occurrence of a "closer" followed by an "opener", i.e.,



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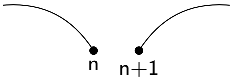

Lemma (J. Bloom, S. Elizalde)

Let τ *be* **any** *configuration. Then, given*

$$A(v,z) = \sum_{n\geq 0} \sum_{M \in \mathcal{M}_n(\tau)} v^{\mathrm{val}(M)} z^n$$

we have

$$\sum_{n\geq 0} |\mathcal{P}_n(\tau)| z^n = \frac{1}{1-z} A\left(\frac{1}{z}, \frac{z^2}{(1-z)^2}\right)$$

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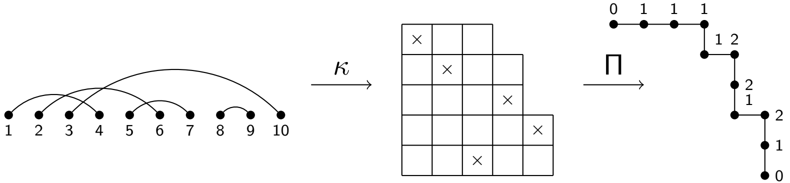

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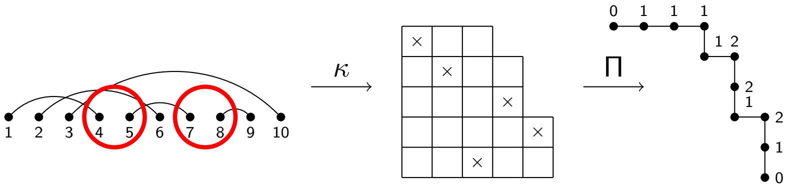

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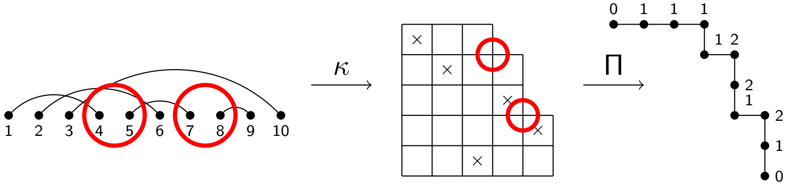

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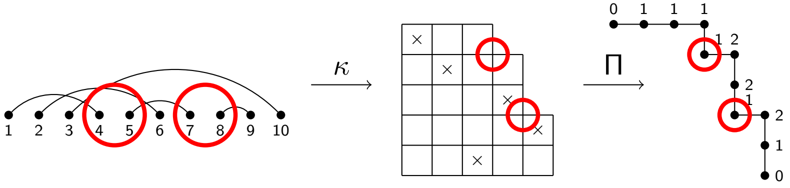

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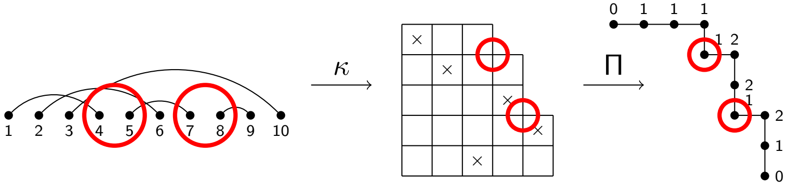

Translating to generating functions:

$$\sum_{\mathcal{M}_n(231)} v^{\mathrm{val}(M)} z^n = \sum_{\mathcal{R}_n(231)} v^{\mathrm{val}(F)} z^n = \sum_{\mathcal{L}_n} v^{\mathrm{val}(D)} z^n$$

Theorem (J. Bloom, S. Elizalde)

The generating function $\sum_{n\geq 0} |\mathcal{P}_n(231)| z^n$ *is a root of the cubic polynomial*

$$(z-1)(5z^2-2z+1)^2X^3$$
$$+(-9z^5+54z^4-85z^3+59z^2-14z+3)X^2$$
$$+(-9z^4+60z^3-64z^2+13z-3)X+(-9z^3+23z^2-4z+1).$$

The asymptotic behavior of its coefficients is given by

$$|\mathcal{P}_n(312)| \sim \delta n^{-5/2}\,\rho^n,$$

where $\delta \approx 0.061518$ *and*

$$\rho = \frac{3(9+6\sqrt{3})^{1/3}}{2+2(9+6\sqrt{3})^{1/3}-(9+6\sqrt{3})^{2/3}} \approx 6.97685.$$

Shape-Wilf-Equivalent Pairs

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Class	Shape-Wilf Equivalent Pairs
I	$\{123, 213\} \sim \{132, 213\} \sim \{132, 231\} \sim \{132, 312\} \sim \{213, 231\}$ $\sim \{213, 312\} \sim \{231, 312\} \sim \{231, 321\} \sim \{312, 321\}$
II	$\{123, 231\}$
III	$\{123, 312\}$
IV	$\{123, 321\}$
V	$\{213, 321\}$
VI	$\{123, 132\}$
VII	$\{132, 321\}$

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IV	$\{123,321\}$
V	$\{213,321\}$
VI	$\{123,132\}$
VII	$\{132,321\}$

Class	Matchings	Set partitions
I	$\dfrac{4}{3+\sqrt{1-8z}}$	$\dfrac{2-3z+z^2-z\sqrt{1-6z+z^2}}{2(1-3z+3z^2)}$
II & III	Solution of a cubic	Solution of a cubic
IV	$\dfrac{1-5z+2z^2}{1-6z+5z^2}$	$\dfrac{1-10z+32z^2-37z^3+12z^4}{(1-z)(1-10z+31z^2-30z^3+z^4)}$
V	Solution of a functional equation	Unknown
VI & VII	Unknown	Unknown