Modified Growth Diagrams and the BWX Map ϕ^*

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Permutations and Pattern Avoidance

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For example we write

$$\sigma = 4\ 5\ 3\ 1\ 2$$

for the permutation (in 2-line notation)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$$

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but does not contain $\tau = 1\ 2\ 3$.

We write $S_n(\tau)$ for all permutations of length n which **avoid** τ.

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A classic result in the field is that

$$|S_n(\tau)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

for all $\tau \in S_3$.

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Another classic result due to Backelin, West, Xin (BWX) is:

$$|S_n(12\ldots k\rho)| = |S_n(k\ldots 1\rho)| \text{ for all n}$$

where ρ is an permutation of $\{k+1,\ldots,k+l\}$.

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An important tool in their proof is the map

$$\phi^* : S_n \to S_n(k\dots 1)$$

which is the focus of this talk.

Definition of the BWX map ϕ^*

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For example if $k = 3$ and $\sigma = 4\ 5\ 3\ 1\ 2$ then

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Key Idea: ϕ removes the smallest $k \dots 1$ pattern.

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is obtained by repeatedly applying the map ϕ until no $(k \dots 1)$-pattern remains.

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- He explicitly ask for a connection between ϕ^* and the GDA.

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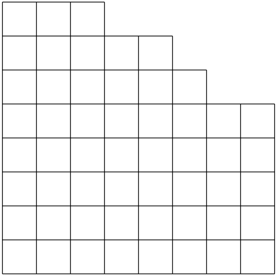
Definition (Informal): A Ferrers Board F is an array of squares obtained by removing some "northeast chunk" from the nxn array of squares leaving a staircase shape.



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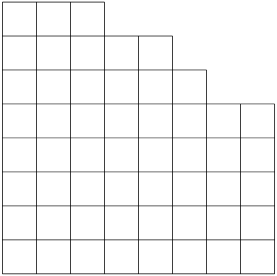

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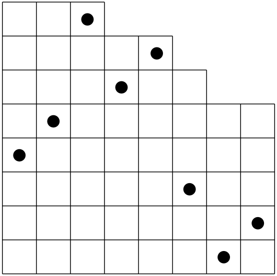

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Theorem: The length of the longest decreasing subsequence in a permutation is the number of parts in its corresponding shape.

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Key Idea

ϕ^* removes $k \ldots 1$ patterns $\longleftrightarrow$ force shape to have $<$ k parts.

Fomin's Growth Diagram Construction

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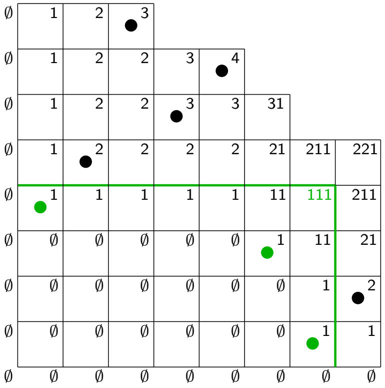

Each corner of the Ferrers board is labeled a partition which is the shape of the permutation southwest of that corner.

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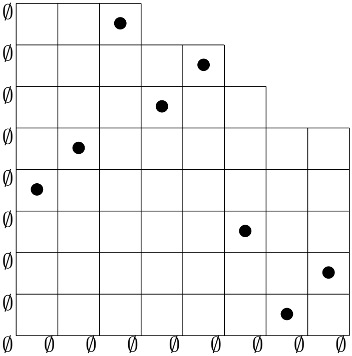

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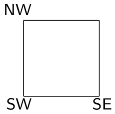
Given partitions



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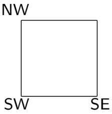

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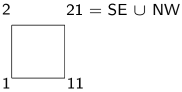

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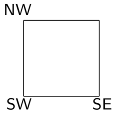

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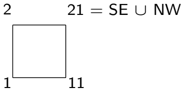

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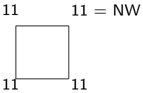

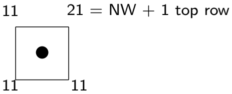

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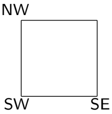

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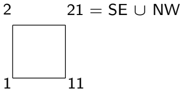

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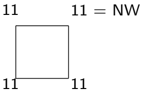

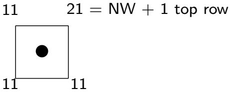

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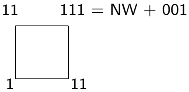

Key Idea: Only the last rule can increase the number of parts of a partition.

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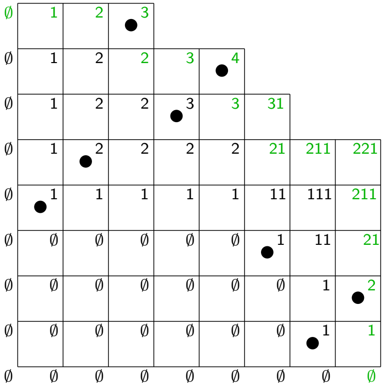

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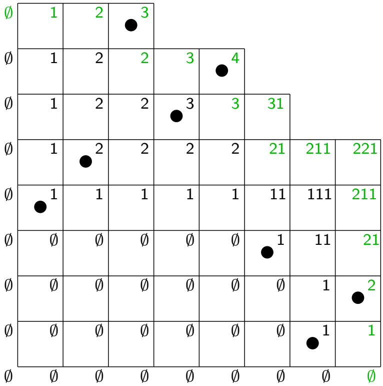

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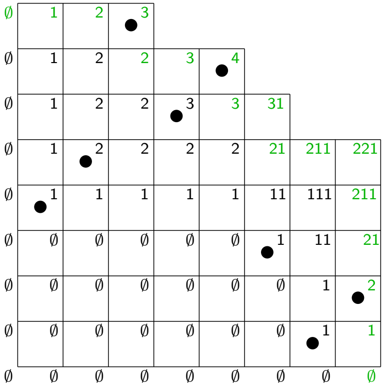

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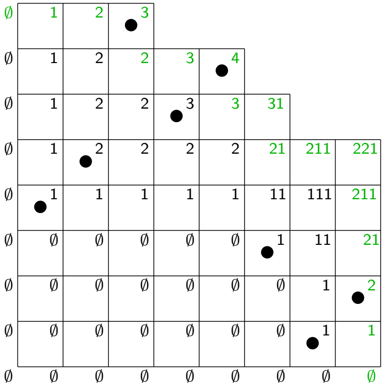

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Theorem: $seq(P, F)$ uniquely determines P.

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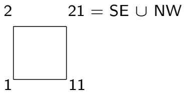

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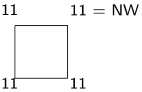

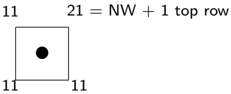

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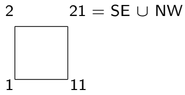

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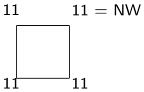

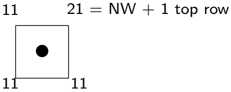

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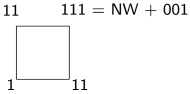

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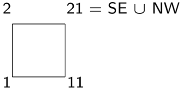

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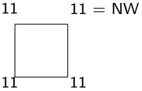

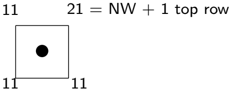

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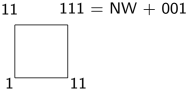
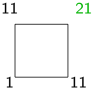


Modified Rule for GDA_k....

*if last rule rule makes $|NE| \geq k$ then



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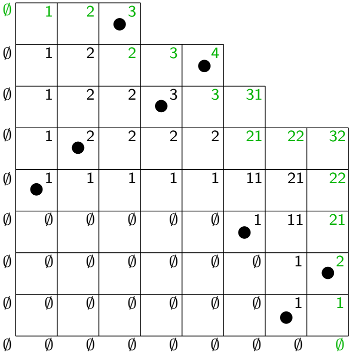
GDA_3 on (P, F)



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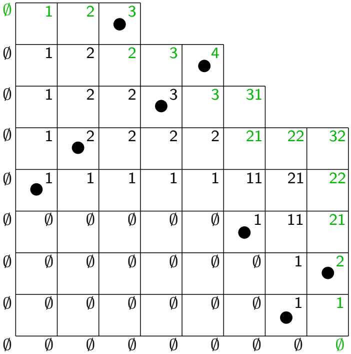
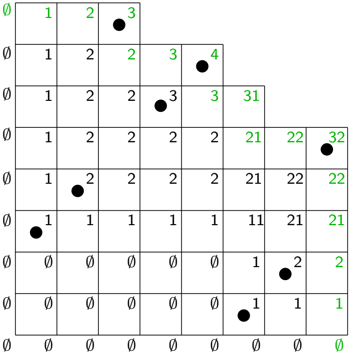


GDA on $(\phi^*(P), F)$



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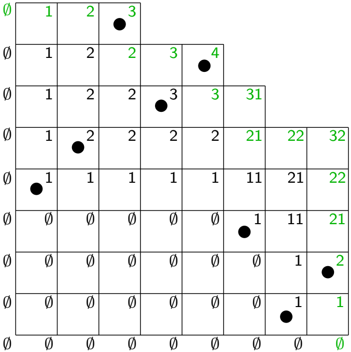

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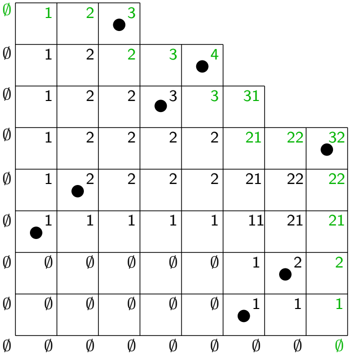

Main Theorem: For any rook placement P on a Ferrers board F,

$$seq_k(P, F) = seq(\phi^*(P), F)$$

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$$\phi^*(P') = (\phi^*(P))'$$

Proof. By the Main Theorem and the note above we have:

$$\begin{aligned} seq(\phi^*(P'), F) &= seq_k(P', F) \\ &= rev(seq_k(P, F)) \\ &= rev(seq(\phi^*(P), F) \\ &= seq((\phi^*(P))', F) \end{aligned}$$

Our Commutation Result

Definition: Let P' denote the inverse of a placement.

Note: $seq(P', F) = rev(seq(P, F))$

Corollary: For any rook placement P on a Ferrers board F,

$$\phi^*(P') = (\phi^*(P))'$$

Proof. By the Main Theorem and the note above we have:

$$\begin{aligned} seq(\phi^*(P'), F) &= seq_k(P', F) \\ &= rev(seq_k(P, F)) \\ &= rev(seq(\phi^*(P), F) \\ &= seq((\phi^*(P))', F) \end{aligned}$$

Hence we conclude that $\phi^*(P') = (\phi^*(P))'$.