Some Consequences of a New Bijective Proof of the shape-Wilf-equivalence of 231 and 312

Jonathan S. Bloom
Dartmouth College
(with Dan Saracino)

Joint Math Meetings - Baltimore 2014

Shape-Wilf-equivalence

Definition

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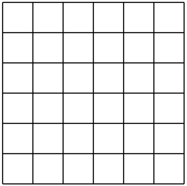

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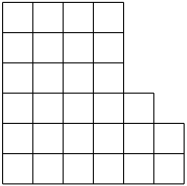
$F =$ 

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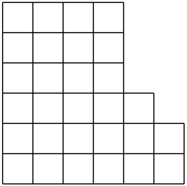
$F =$ 

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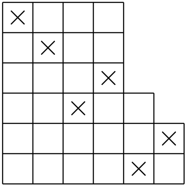
$F =$ 

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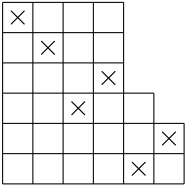

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Notation

$\mathcal{R}_F =$ set of all f.r.p. on fixed board F

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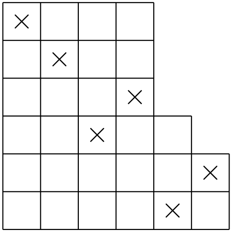
Definition (by example)

The following f.r.p. **contains** the pattern 312 because...



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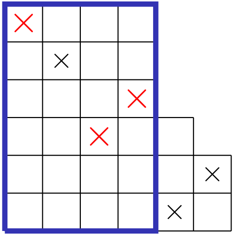

Read using cartesian coordinates!

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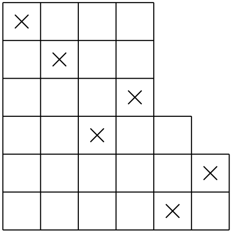

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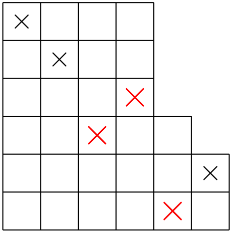

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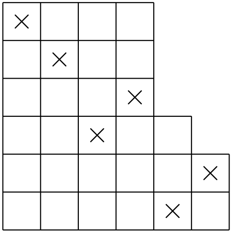

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- On the other hand this f.r.p. **avoids** the pattern 231.

Notation

- $\mathcal{R}_F(\tau)$ = *set of all f.r.p. on F that avoid* τ

Shape-Wilf-equivalence

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We say two patterns $\sigma, \tau \in S_k$ are **shape-Wilf-equivalent** and write $\sigma \sim \tau$ if

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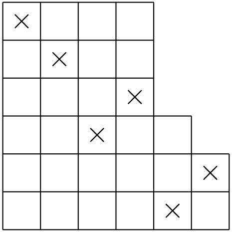
- A bijective and (we think) simple proof that $231 \sim 312$
  - Yields many nice enumerative results.

Our Bijection



$\mathcal{R}_F(231)$

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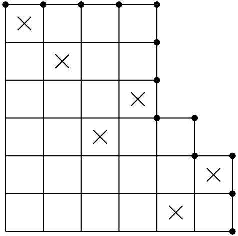

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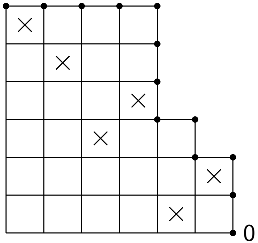

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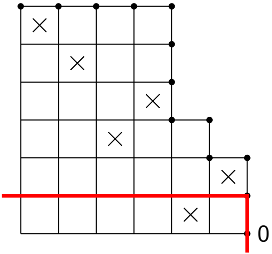

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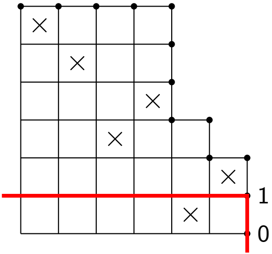

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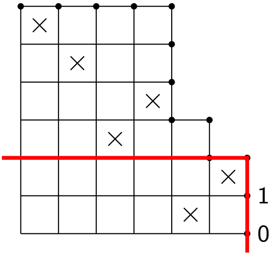

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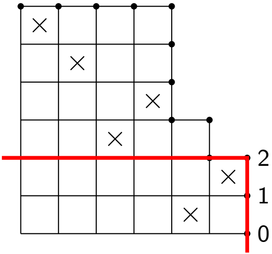

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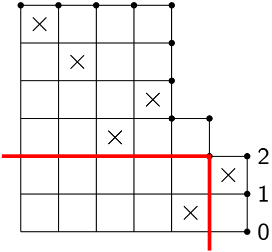

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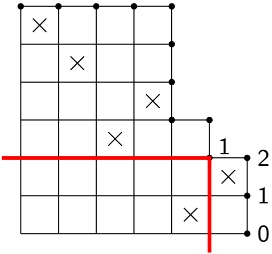

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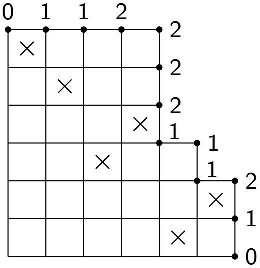

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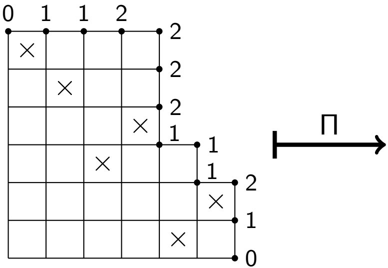

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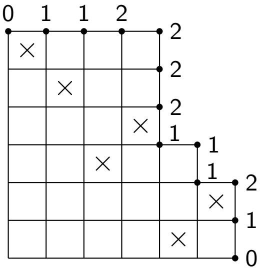

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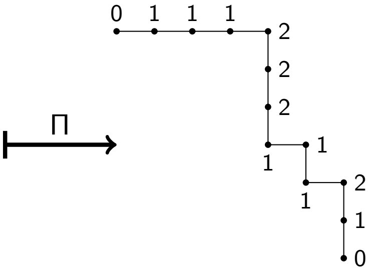

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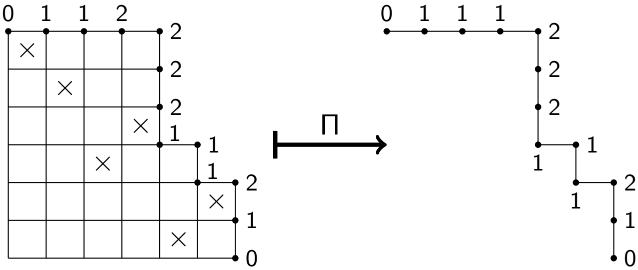

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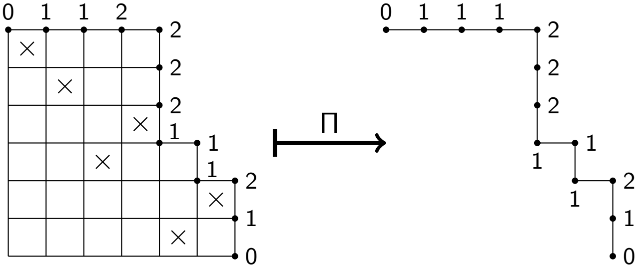

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Key Defining Property of $\mathcal{D}_F(\leq)$

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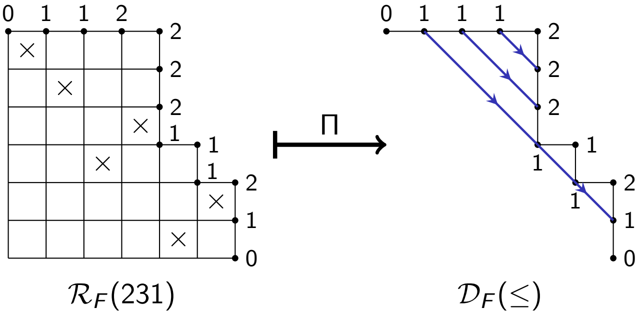

Key Defining Property of $\mathcal{D}_F(\leq)$

- Diagonal Property: “Upper” $\leq$ “Lower”

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Theorem (Bloom-Saracino '11)

The mapping

$$\Pi : \mathcal{R}_F(231) \to \mathcal{D}_F(\leq)$$

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Corollary (Bloom-Saracino '11)

There exists a bijection between $\mathcal{D}(\leq)$ *and* $\mathcal{D}(\geq)$*, thus* $231 \sim 312$.

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- Permutations that are sortable using two increasing stacks in series (M. Atkinson, M. Murphy, and N. Ruškuc 2002)
- Also led to NEW enumerative results in the study of "nestings and crossings"

Thank You!