Properties of Determinants

In the last section, we saw how determinants "interact" with the elementary row operations. There are other operations on matrices, though, such as scalar multiplication, matrix addition, and matrix multiplication. We would like to know how determinants interact with *these* operations as well.

In other words, if we know $\det A$ and $\det B$, can we use this information to find quantities such as

$$\det(kA),\ \ \det(A+B),\ \text{ and }\ \det AB?$$

It turns out that the answers to the first and third questions are quite easy to find, whereas (perhaps surprisingly), the answer to the second question is actually quite difficult, and is the topic of much current research in linear algebra (including some of my own).

Determinants and Scalar Multiplication

Let's investigate the first question raised above: If we know the determinant of matrix A, can we use this information to calculate the determinant of the matrix kA, where k is a constant?

We'll think about the question using an example from the previous section: in 2.2, we calculated that

$$\det A = \det \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2.$$

Let's try to use this information to calculate $\det(4A)$:

$$\begin{aligned} \det 4A &= \det \begin{pmatrix} 12 & 0 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 4 \end{pmatrix} \\ &= 4 \cdot \det \begin{pmatrix} 3 & 0 & 1 \\ 4 & 4 & 0 \\ 4 & 0 & 4 \end{pmatrix} \\ &= 4 \cdot 4 \cdot \det \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix} \\ &= 4 \cdot 4 \cdot 4 \cdot \det \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= 4^3 \cdot 2 \end{aligned}$$

Notice that multiplying A by 4 is the same as multiplying *each row* of A by 3. Since each of the three rows of A had an extra factor of 4, $\det 4A$ needed *three* extra factors of 4, i.e.

$$\det 4A = 4^3 \det A.$$

With this in mind, it is quite easy to understand the reasoning behind the following theorem:

Theorem. If A is an $n \times n$ matrix, and k is any constant, then

$$\det kA = k^n \det A.$$

When using the theorem, it is important to keep in mind that the constant k in the determinant formula gets multiplied by itself n times, since each of the n rows of A was multiplied by k.

Determinants and Matrix Addition

Our next question, about the relationship between $\det(A + B)$, $\det A$, and $\det B$, does not have a satisfactory answer, as indicated by the following example:

Example

Let

$$A = \begin{pmatrix} 5 & -6 \\ 0 & -12 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 0 \\ 1 & 9 \end{pmatrix}.$$

Compare $\det A$, $\det B$, and $\det(A + B)$.

The determinants of A and B are quite simple to calculate, given that they are triangular matrices: clearly $\det A = -60$, and $\det B = -27$. Now

$$A + B = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix};$$

since $A + B$ is not a triangular matrix, we'll need to revert to the formula for determinants of 2×2 matrices to calculate $\det(A + B)$:

$$\begin{aligned} \det(A + B) &= \det \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} \\ &= 2 \cdot (-3) - (-6) \cdot 1 \\ &= -6 + 6 \\ &= 0. \end{aligned}$$

Gathering our data, we see that

$$\det A = -60, \ \det B = -27, \text{ and } \det(A + B) = 0.$$

Unfortunately, it appears that there is very little connection between $\det A$, $\det B$, and $\det(A + B)$.

Key Point. In general,

$$\det A + \det B \neq \det(A + B),$$

and you should be *extremely* careful not to assume anything about the determinant of a sum.

Nerdy Sidenote

One large vein of current research in linear algebra deals with this question of how $\det A$ and $\det B$ relate to $\det(A+B)$. One way to handle the question is this: instead of trying to find the value for $\det(A+B)$, find a region on the real line that we can be certain *contains* $\det(A+B)$. It turns out that this is a rich question with many interesting and surprising answers. Some of my own research relates to this question.

Determinants and Matrix Multiplication

Perhaps surprisingly, considering the results of the previous section, determinants of products are quite easy to compute:

Theorem 2.3.4. If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B).$$

In other words, the determinant of a product of two matrices is just the product of the determinants.

Example

Compute $\det AB$, given

$$A = \begin{pmatrix} 5 & -6 \\ 0 & -12 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 0 \\ 1 & 9 \end{pmatrix}$$

from the previous example.

According to the theorem above, there are two ways to handle this problem:

1. Multiply A by B, then calculate the determinant of the product.
2. Find determinants of A and B separately, then multiply them together to get the determinant of AB.

Of course, we already know that

$$\det A = -60 \text{ and } \det B = -27,$$

so the second option is definitely easier here. By the theorem, we know that

$$\det(AB) = (\det A)(\det B) = (-60)(-27) = 1620.$$

You should verify that this is the same answer that you would get if you were to first calculate the product AB, then find its determinant.

Determinants and Invertibility

Several sections ago, we introduced the concept of *invertibility*. Recall that a matrix A is *invertible* if there is another matrix, which we denote by A^{-1}, so that

$$AA^{-1} = I.$$

For example, it is easy to see that the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

has inverse

$$A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

In a sense, matrix inverses are the matrix analogue of *real number* multiplicative inverses. Of course, it is quite easy to determine whether or not a real number has an inverse:

$$a \text{ has inverse } \frac{1}{a} \text{ if and only if } a \neq 0.$$

In other words, *every* real number other than 0 has an inverse.

Unfortunately, the question of whether or not a given *square matrix* has an inverse is not quite so simple. Certainly the 0 matrix

$$0_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

has no inverse; however, *many* nonzero matrices also fail to have inverses, such as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So how can we determine whether or not a given square matrix does actually have an inverse? We saw a list of conditions in section 1.5 that can help us out:

Theorem. Let A be an $n \times n$ matrix. Then the following are equivalent:

- A is invertible.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is a product of elementary matrices.

However, it turns out that there is a much cleaner way to make the determination, as indicated by the following theorem:

Theorem 2.3.3. A square matrix A is invertible if and only if $\det A \neq 0$.

In a sense, the theorem says that matrices with determinant 0 act like the *number* 0–they don't have inverses. On the other hand, matrices with nonzero determinants act like all of the other real numbers–they *do* have inverses.

With this theorem in mind, we now expand the list of equivalent conditions for a matrix A to be invertible:

Theorem 2.3.8. Let A be an $n \times n$ matrix. Then the following are equivalent:

- A is invertible.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is a product of elementary matrices.
- $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
- $\det A \neq 0$.

Example

Determine if the following matrices are invertible:

1.
$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

2.
$$C = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$$

1. In the previous section, we saw that

$$\det A = \det \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2.$$

   By the theorem, we know that A is invertible.

2. We calculated the determinant of

$$C = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$$

   earlier in this section:

$$\det C = \det \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} = 0.$$

   Again using the theorem, we know that C is *not* invertible.

Key Point. Note that, even though the theorem can tell us that the matrix A above has an inverse, we have *no* information as to what matrix A^{-1} actually is. The theorem only tells us that A^{-1} exists.

Determinants of Inverses

Now that we have an easy way to determine whether or not A^{-1} exists by using determinants, we should demand an easy way to calculate $\det(A^{-1})$, when A^{-1} exists. Fortunately, there is an easy way to make the calculation:

Theorem 2.3.5. If A^{-1} exists, then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Cramer's Rule

We spent a lot of time in chapter 1 on a discussion of solving systems of linear equations using the form

$$A\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is the vector of unknowns.

It turns out that one among the many uses of the determinant is in a formulaic solution to many such systems, as indicated by the following theorem:

Theorem 2.3.7. *Cramer's Rule* If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns so that the determinant of the coefficient matrix A is nonzero, i.e. $\det A \neq 0$, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$x_1 = \frac{\det A_1}{\det A},\ x_2 = \frac{\det A_2}{\det A}, \ldots, \text{ and } x_n = \frac{\det A_n}{\det A}.$$

Each of the matrices A_i is obtained from A by replacing the ith column of A by

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Example

Use Cramer's rule to find the solution $\mathbf{x}$ to the system of equations whose matrix equation is given by

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Since the coefficient matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

is square and invertible (we've already calculated that $\det A = 2$), we can apply Cramer's rule to the problem. The rule says that the solution to this system is given by

$$x_1 = \frac{\det A_1}{\det A},\ x_2 = \frac{\det A_2}{\det A}, \text{ and } x_3 = \frac{\det A_3}{\det A}.$$

As mentioned above, we already know that $\det A = 2$, so we can solve the system by simply calculating the determinants of three more matrices, A_1, A_2, and A_3. The rule says that we get the matrix A_1 by replacing the first column of A with

$$\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$$

so we have

$$A_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$A_2 = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Fortunately, the determinants of each of these matrices are quite easy to calculate: by cofactor expansion along the 2nd row of A_1, we see that

$$\det A_1 = \det \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1.$$

By cofactor expansion along the 2nd row of A_2, we have

$$\det A_2 = \det \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = -1.$$

Finally, cofactor expansion along the 2nd column of A_3 gives us

$$\det A_3 = \det \begin{pmatrix} 3 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1.$$

Thus Cramer's rule tells us that the solution to the system

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

is given by

$$x_1 = \frac{1}{2}, \; x_2 = -\frac{1}{2}, \; \text{and } x_3 = \frac{1}{2}.$$