Logarithmic Functions

A class of functions that are closely related to exponential functions are logarithmic functions. If $a > 1$, $x > 0$, then the function

\[ \log_a x \]

is called the logarithmic function with base $a$; the notation for the function is equivalent to the exponential notation indicated below:

\[ \log_a x = y \iff a^y = x. \]

In a sense, logarithmic functions offer us an alternative way to talk about exponential functions. In mathematically precise language, the function $\log_a x$ is the inverse of the exponential function $a^x$; the two functions reverse each other. In particular,

\[ \log_a a^x = x \text{ and } a^{\log_a x} = x. \]

Logarithmic functions with several different bases are graphed below:

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Example. Let $f(x) = \log_3 x$.

1. Find the inverse of $f(x)$.
2. Evaluate $f(1)$.
3. Evaluate $f(3)$.
4. Evaluate $f(27)$.

1. The inverse of $f(x)$ is the function $g(x) = 3^x$.
2. Evaluating $f(1) = \log_3 1$ is equivalent to finding the number $y$ so that $3^y = 1$; clearly $y = 0$, so $f(1) = 0$. 
Finding $f(3) = \log_3 3$ is the same as finding the number $y$ so that $3^y = 3$; since $3^1 = 3$, $f(3) = 1$.

To find $f(27) = \log_3 27$, we need to find the number $y$ so that $3^y = 27$. Since $3^3 = 27$, $y = 3$ and $f(27) = 3$.

**Properties of Logarithmic Functions**

The following list details the domain, range, and rules for combining logarithmic functions:

Let $a > 1$.

1. $\log_a x$ is continuous
2. $\log_a x$ has domain $(0, \infty)$
3. $\log_a x$ has range $(-\infty, \infty)$
4. $\log_a x$ is an increasing function.
5. $\log_a a = 1$
6. $\log_a(xy) = \log_a x + \log_a y$
7. $\log_a \frac{x}{y} = \log_a x - \log_a y$
8. $\log_a x^y = y \log_a x$

**Limits of Logarithmic Functions**

Recall that $a > 1$.

Since logarithmic functions are continuous, finite limits agree with function values:

$$\lim_{x \to c} \log_a x = \log_a c,$$ for any real number $c > 0$.

We would like to understand limits at infinity as well. You may have already guessed the following fact based on the graphs above:

If $a > 1$, then

1. $\lim_{x \to \infty} \log_a x = \infty$
2. $\lim_{x \to 0^+} \log_a x = -\infty$.  

Example. Find 

\[ \lim_{x \to -\infty} \log_2(3^x). \]

We know that we can rewrite the limit as

\[ \lim_{x \to -\infty} \log_2(3^x) = \log_2( \lim_{x \to -\infty} 3^x); \]

as we saw in the previous section,

\[ \lim_{x \to -\infty} 3^x = 0 \]

since \(3 > 0\). Since

\[ \log_2 x \to -\infty \text{ as } x \to 0^+, \]

we know that

\[ \lim_{x \to -\infty} \log_2(3^x) = -\infty. \]

The Natural Logarithmic Function

We can choose any number \(a > 0\) to be the base for the logarithmic function

\[ \log_a x; \]

but if we choose \(a = e\), we end up with the *natural logarithmic function*

\[ \log_e x = \ln x. \]

The natural logarithmic function has several important properties:

1. \(\ln e = 1\)
2. \(\ln e^x = x\)
3. \(e^{\ln x} = x, \ x > 0.\)

In addition, the natural log function allows us to easily change bases of other log functions, as indicated by the following rule:

Change of Base Formula. For any \(a > 0, \ a \neq 1,\)

\[ \log_a x = \frac{\ln x}{\ln a}. \]