Section 4.2

The Definite Integral

In the last section, we introduced the area problem: given a function $f(x) \geq 0$ on the interval $[a, b]$, can we find the area of the region below $f(x)$ above the $x$-axis?

We learned that we can think of the area of this region as a limit of sums of areas of approximating rectangles, given by the formula

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$ 

This idea is extremely important; in this section, we will give the aforementioned “limit of a sum” a new name and practice calculating it.

**Definition 2.** If $f(x)$ is defined for $a \leq x \leq b$, divide the interval $[a, b]$ into $n$ equal subintervals of length $\Delta x = \frac{b-a}{n}$, and let

- $x_0 = a$
- $x_1 = a + \Delta x$
- $x_2 = a + 2\Delta x$
- \vdots
- $x_k = a + k\Delta x$
- \vdots
- $x_n = a + n\Delta x = b$

be the endpoints of the subintervals. Then the *definite integral of $f$ from $x = a$ to $x = b$* is

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,$$
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if the limit exists. We say that \( f(x) \) is integrable if the limit exists.

One important point to note here is that the symbol \( \int_a^b f(x) \, dx \) is merely new notation to refer to the object we studied in Section 4.1.

The function \( f(x) \) is called the integrand; the endpoints \( a \) and \( b \) of the interval are the limits of integration.

Since the definition of the definite integral involves the same limit of a sum that we investigated in the previous section, we should make an observation here: if \( f(x) \geq 0 \) on \([a, b]\) and integrable, then the area of the region under \( f(x) \) from \( x = a \) to \( x = b \) is precisely

\[
A = \int_a^b f(x) \, dx.
\]

If, however, \( f(x) < 0 \) somewhere on the interval \([a, b]\), then while we can still calculate \( \int_a^b f(x) \, dx \), our interpretation of its meaning is slightly different: \( \int_a^b f(x) \, dx \) is actually the difference between areas enclosed by \( f \) above the \( x \) axis and areas enclosed by \( f \) below the \( x \) axis.

When attempting to evaluate a definite integral, we should be concerned about whether or not the integral actually exists, i.e. whether or not \( f(x) \) is integrable. Fortunately for us, Theorem 3 below tells us that the indicated limit does actually exist for many functions:

**Theorem 3.** If \( f(x) \) is continuous on \([a, b]\), then \( f(x) \) is integrable on \([a, b]\).

**Example.** In the last section, we investigated the area of the region under the curve \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \):

We wrote the area as a limit of sums of approximating rectangles,

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\sqrt{i}}{n^3}.
\]
Using the definition above, we can rewrite the sum as a definite integral:

\[
\int_0^1 \sqrt{x} \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{i}{n^3}}
\]

**Example.** Evaluate

\[
\int_0^2 3x \, dx.
\]

We could attempt to evaluate this integral using the definition

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,
\]

but we could also note that, since \( f(x) = 3x \) is greater than 0 from \( x = 0 \) to \( x = 3 \), we know that

\[
\int_0^2 3x \, dx
\]

is just the area of the region under \( 3x \) from \( x = 0 \) to \( x = 2 \), indicated below:

![Graph of the region under 3x from x = 0 to x = 2](image)

This region is just a triangle, whose area is given by \( A = \frac{1}{2} \times bh \), so the area of the shaded region above is \( A = \frac{1}{2} \times 2 \times 6 = 6 \). Thus we conclude that

\[
\int_0^2 3x \, dx = 6.
\]
Example. Evaluate \[ \int_{-1}^{2} x \, dx. \]

Unlike the previous example, this curve dips below the x-axis:

Thus we cannot interpret the integral above as an area; however, we can interpret it as the difference between the area of the region above the x-axis and the area of the region below the x-axis!

Both regions are triangles; using the graph, it is easy to see that the region above the x-axis has area

\[ A_1 = \frac{1}{2} \cdot 2 \cdot 2 = 2, \]

and that the region below the x-axis has area

\[ A_2 = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}. \]

Thus the integral is the difference between these two areas, given by

\[ \int_{-1}^{2} x \, dx = 2 - \frac{1}{2} = \frac{3}{2}. \]

Unfortunately, most definite integrals are not as easy to evaluate as the previous ones; in such cases, we will have to find a way to actually evaluate the limit in

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

by hand.
To do so, it will be extremely helpful to rewrite the sum in a “closed form”; the following identities will help us do so:

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2}, \]
\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}, \]  
(1)
\[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n + 1)}{2} \right)^2, \]
\[ \sum_{i=1}^{n} c = nc, \]
\[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i, \]
(2)
\[ \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i. \]

**Example.** Evaluate

\[ \int_{0}^{1} x^2 \, dx. \]

Using the formula

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

with

\[ \Delta x = \frac{1 - 0}{n} = \frac{1}{n} \]

and

\[ x_0 = 0, \ x_1 = \frac{1}{n}, \ldots , \ x_i = \frac{i}{n}, \]
we have

\[ \int_0^1 x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{f(i/n)}{n} \frac{1}{n} \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n} \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \]

\[ = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2, \]

where the last step above is due to the fact that \( n \) is a constant with respect to the summation variable \( i \), thus may be factored out as in rule (2).

Next, we use rule (1):

\[ \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \]

\[ = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \]

\[ = \lim_{n \to \infty} \frac{(n^2 + n)(2n+1)}{6n^3} \]

\[ = \lim_{n \to \infty} \frac{2n^3 + 2n^2 + n^2 + n}{6n^3} \]

\[ = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3}. \]

Notice that there is no summation left—indeed, the equation has reduced to a limit that we know how to evaluate! Let’s finish the problem:

\[ \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \lim_{n \to \infty} \frac{\frac{2n^3}{n^3} + \frac{3n^2}{n^3} + \frac{n}{n^3}}{\frac{6n^3}{n^3}} \]

\[ = \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \]

\[ = \frac{2}{6} \]

\[ = \frac{1}{3}. \]

Thus we see that

\[ \int_0^1 x^2 \, dx = \frac{1}{3}. \]
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and since $x^2 \geq 0$ on $[0, 1]$, we can conclude that the area of the region below $f(x) = x^2$ from $x = 0$ to $x = 1$ is precisely $1/3$.

Properties of the Definite Integral

The definite integral has many properties that will be quite helpful for us to understand, and which can significantly simplify the types of computations we saw above.

The first property says that

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.$$  

In words, the property says that if reversing the order of integration simply changes the sign of the integral.

The second property says that

$$\int_a^a f(x) \, dx = 0;$$

thinking of the integral as an area computation, this property just says that the “area” under a point is $0$.

The next property tells us the value of

$$\int_a^b c \, dx,$$

where $c$ is any constant. Again thinking of the definite integral of a function as the area of the region below the curve, we can easily calculate the integral above using a graph:

The curve $c$ is just the constant function, so the region shaded above is a rectangle, whose length is $b - a$ and whose height is $c$. Thus the area of the rectangle is $c(b - a)$, so that

$$\int_a^b c \, dx = c(b - a).$$
The next property tells us that the integral of a sum or difference is simply the sum or difference of the integrals:

\[ \int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x). \]

As you might have suspected, constants can be pulled out of integrals:

\[ \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx. \]

To understand the next property, consider the graph below:

Suppose that we wish to know

\[ \int_a^b f(x) \, dx, \]

which we again think of as the area of the region below \( f(x) \) from \( x = a \) to \( x = b \). Of course, we could think of this region as the “sum” of the region below \( f(x) \) from \( x = a \) to \( x = c \) and the region below \( f(x) \) from \( x = c \) to \( x = b \), as illustrated below:
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In particular, adding up the areas of these two regions will give us the area of the larger region. Thus

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

if \( a \leq c \leq b \).