Section 3.3

Derivatives and the Shapes of Graphs

In this section, we will specifically discuss the information that $f'(x)$ and $f''(x)$ give us about the graph of $f(x)$; it turns out understanding the first and second derivatives can give us a great deal of concrete information about the shape of the graph of $f(x)$.

Increasing/Decreasing Behavior

Example. Consider the graph of the function $f(x) = -x^2 + 4x - 1$ below:

Notice that (from left to right), the $y$ values for $f(x)$ increase when $x < 2$. On the other hand, the $y$ values for $f(x)$ decrease when $x > 2$. Let’s break the graph of $f(x)$ into two sections, based on where $f$ is increasing and where it is decreasing:
We say that $f(x)$ is increasing on $(-\infty, 2)$ and $f(x)$ is decreasing on $(2, \infty)$.

We would like to be able to use calculus to determine the intervals on which $f$ increases and on which it decreases, without having to inspect the graph. To understand how this might work, think about tangent lines to $f(x)$ on the interval where $f(x)$ is increasing, $(-\infty, 2)$:

Notice that each tangent line has positive slope.

Similarly, let’s think about tangent lines to $f(x)$ on the interval where $f(x)$ is decreasing, $(2, \infty)$; several such tangent lines are drawn below:
Notice that each tangent line has *negative slope*.

There is one more interesting feature of the graph to point out; $f(x)$ *changes* from increasing to decreasing at $x = 2$. Let’s draw the tangent line to $f(x)$ at $x = 2$:

It is clear that

- $f(x)$ has a local maximum value at $x = 2$
- $f'(2) = 0$.

In the previous example, we observed the following behavior:
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\[
\begin{array}{c|c|c}
\text{Interval} & \text{Behavior of } f(x) & \text{Behavior of } f'(x) \\
\hline
(-\infty, 2) & \text{increases} & > 0 \\
(2, \infty) & \text{decreases} & < 0 \\
(2, 1) & \text{changes from increasing to decreasing} & = 0 \text{ at } x = 2
\end{array}
\]

All of the relationships between the graph of \( f(x) \) and the behavior of the function \( f'(x) \) are actually true in general:

**Increasing/Decreasing Test.** Let \( f(x) \) be differentiable.

1. If \( f'(x) > 0 \) on \((a, b)\), then \( f(x) \) is increasing on \((a, b)\).
2. If \( f'(x) < 0 \) on \((a, b)\), then \( f(x) \) is decreasing on \((a, b)\).

In essence, the test says that if we wish to determine where \( f(x) \) is increasing, we simply need to determine the intervals on which \( f'(x) > 0 \); to determine where \( f(x) \) is decreasing, find the intervals on which \( f'(x) < 0 \).

**Example.** The function \( f(x) \) has:

\[
\begin{array}{c|c}
\text{Interval} & \text{Behavior} \\
\hline
(-2) & 0 \\
(-1) & 2 \\
f'(x) > 0 & \text{when } -2 < x < -1 \\
f'(x) < 0 & \text{when } x < -2 \text{ and } x > -1
\end{array}
\]

Use the information above and the ideas from the Increasing/Decreasing Test to sketch a graph of \( f(x) \).

From the information in the chart, we see that:

- \( f(x) \) is increasing on the interval \((-2, -1)\)
- \( f(x) \) is decreasing on the intervals \((-\infty, -2)\) and \((-1, \infty)\).

Using this data, we get the following sketch:
Locating Local Extremes on an Open Interval

In Section 3.1, we saw that the local extremes of a function $f(x)$ can only occur at critical numbers of the function, i.e. numbers $c$ in the domain of $f(x)$ so that

1. $f'(c) = 0$, or
2. $f'(c)$ DNE.

However, as we saw above, not all critical numbers correspond to local extremes. For example, the function $f(x) = x^3 + 1$ graphed below has a critical number at $x = 0$, since

$$f'(x) = 3x^2 \text{ and } f'(0) = 0 :$$
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However, inspecting the graph, we realize that \( f(x) \) does not actually have a local extreme at \( x = 0 \).

The key point to note here is that finding critical numbers is \textit{not enough information} to be able to determine if \( f(x) \) has a local extreme. Indeed, we will need more information to determine \textit{whether or not a critical number yields a local extreme}.

We can get a feel for the type of information we will need by inspecting the graph again:

Notice that, even though \( f \) has a critical number at \( x = 0 \), \( f(x) \) \textit{does not} change from increasing to decreasing at \( x = 0 \); this is actually the reason that there is no local extreme at \( x = 0 \).
One of our main goals for this section is to be able to locate local extremes of a function on an open interval, which we know must occur at critical numbers. However, once we have found a critical number $c$, we need more data to determine if $c$ does actually correspond to a local extreme. Earlier, we saw several examples of functions $f(x)$ that had local extremes at numbers at which $f(x)$ changed from increasing to decreasing, or vice-versa. The test below incorporates this idea into our search for local extremes; in particular, the First Derivative Test gives us a way to differentiate between critical numbers which yield a local extreme, and critical numbers which do not:

**First Derivative Test.** Let $c$ be a critical number of a continuous function $f(x)$.

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(x)$ has a local maximum at $c$.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(x)$ has a local minimum at $c$.
3. If $f'(x)$ does not change signs at $x = c$, then $f(x)$ does not have a local extreme at $c$.

The three cases are illustrated below:
There is one more important point to note here: if $f(x)$ is continuous, then $f'(x)$ can only change signs at critical numbers of $f$. In other words, once we find critical numbers of $f$, we can be certain that $f'$ is either always positive or always negative between the critical numbers.
To find all of the local extremes of $f(x)$, follow the procedure below:

1. Find all critical numbers $c$ of $f(x)$
2. Test a number close to $c$ on either side in $f'$
3. Use the First Derivative Test to determine if each critical number $c$ is a local max, min, or neither.

**Example.** Let $f(x) = 3x^{1/3}(x + 2)$.

1. Find all critical numbers of $f(x)$.
2. Determine the intervals on which $f(x)$ is increasing.
3. Determine the intervals on which $f(x)$ is decreasing.
4. For each critical number in the previous step, determine if the number corresponds to a local maximum, a local minimum, or neither.

1. To determine the critical numbers of $f(x)$, we note that $f(x)$ has domain $(-\infty, \infty)$; we need to know when $f'(x) = 0$ or when $f'(x)$ DNE. So we need to calculate $f'(x)$:

   
   \[
   f(x) = 3x^{1/3}(x + 2) \\
   f'(x) = \frac{1}{x^{2/3}}(x + 2) + 3x^{1/3}
   \]

   Recall that critical numbers can occur when

   (a) $f'(x) = 0$, or
   (b) $f'(x)$ DNE.

   Let’s determine when $f'(x) = 0$: since

   
   \[
   f'(x) = \frac{1}{x^{2/3}}(x + 2) + 3x^{1/3},
   \]

   we want to know when

   \[
   \frac{1}{x^{2/3}}(x + 2) + 3x^{1/3} = 0,
   \]

   or

   \[
   \frac{1}{x^{2/3}}(x + 2) = -3x^{1/3}.
   \]
We can simplify the equation significantly by multiplying both sides by \(x^{2/3}\), which will rid it of fractions:

\[
\frac{x + 2}{x^{2/3}} = -3x^{1/3}
\]
\[
\frac{x + 2}{x^{2/3}} x^{2/3} = -3x^{1/3} x^{2/3}
\]
\[
x + 2 = -3x.
\]

Now we may rewrite the equation \(x + 2 = -3x\) as

\[4x + 2 = 0.\]

Now we can solve for \(x\):

\[4x = -2 \text{ so that } x = -\frac{1}{2}.\]

Since \(x = -1/2\) is in the domain of \(f(x)\), it is indeed a type 1 critical number.

To find type 2 critical numbers, we need to know when \(f'(x)\) DNE. Since

\[f'(x) = \frac{1}{x^{2/3}}(x + 2) + 3x^{1/3},\]

the only number at which \(f'(x)\) has domain issues is at \(x = 0\), when the denominator of the fraction in the first term is 0; 0 is in the domain of \(f(x)\), thus is a type 2 critical number.

2. To determine the intervals on which \(f(x)\) is increasing, we must determine the intervals on which \(f'(x) > 0\). Recall that \(f'(x)\) can only change from positive to negative at critical numbers, so we only need to test a few points–one from each interval on the number line below:

Let’s determine the signs of \(f'(-1)\), \(f'(-1/8)\), and \(f'(1)\):
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- $f'(-1) = (-1 + 2) - 3 < 0$
- $f'(1/4) = 4(-1/8 + 2) + 3(-1/8)^{1/3} = 60/8 - 3/2 > 0$
- $f'(1) = (1 + 2) + 3 > 0$

Let’s record our data on the chart below:

From the chart, we see that $f'(x) > 0$ on $(-1/2, 0)$ and $(0, \infty)$, so these are the intervals on which $f(x)$ is increasing.

3. To determine the intervals on which $f(x)$ is decreasing, we must determine when $f'(x) < 0$. Using the chart above, we know that $f'(x) < 0$ on $(-\infty, -1/2)$, so this is the interval on which $f(x)$ is decreasing.

4. The first critical number $x = -1/2$ yields a local minimum of the function since $f(x)$ changed from decreasing to increasing. The second critical number $x = 0$ does not correspond to a local extreme, since $f$ did not change from increasing to decreasing or vice-versa.

Concavity

One more feature of a graph that we would like to be able to describe mathematically is its concavity. The function $f(x) = (x - 2)^3 + 2$ is graphed below:
Notice that, on the interval \((-\infty, 2)\), the function is curving downwards:

We say that \(f(x)\) is concave down on \((-\infty, 2)\).
On the other hand, \(f(x)\) is curving upwards on the interval \((2, \infty)\):
We say that $f(x)$ is concave up on $(2, \infty)$.

As with increasing and decreasing behavior, we can describe the concavity of $f(x)$ by inspecting derivatives of $f$. Let’s draw tangent lines to $f(x)$ on the interval on which it is concave down:

Notice that the slopes of the tangent lines are getting smaller, i.e. $f'(x)$ is decreasing.

Let’s consider $f'(x)$ when $f(x)$ itself is concave up:
In this case, slopes of tangent lines are getting bigger, which means that $f'(x)$ is increasing. Let’s assemble all of this data:

<table>
<thead>
<tr>
<th>$f(x)$ concave down on $(-\infty, 2)$</th>
<th>$f'(x)$ decreasing on $(-\infty, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$ concave up on $(2, \infty)$</td>
<td>$f'(x)$ increasing on $(2, \infty)$</td>
</tr>
<tr>
<td>$f(x)$ changes concavity at $x = 2$</td>
<td>$f'(x)$ changes from decreasing to increasing at $x = 2$.</td>
</tr>
</tbody>
</table>

It seems that if $f'$ is increasing, then $f$ is concave up, and if $f'$ is decreasing, then $f$ is concave down. Of course, we know how to determine when $f'(x)$ is increasing or decreasing—check $f''(x)$!

<table>
<thead>
<tr>
<th>$f'(x)$ decreases on $(-\infty, 2)$</th>
<th>$f''(x) &lt; 0$ on $(-\infty, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$ increases on $(2, \infty)$</td>
<td>$f''(x) &gt; 0$ on $(2, \infty)$</td>
</tr>
<tr>
<td>$f'(x)$ changes from decreasing to increasing at $x = 2$</td>
<td>$f''(x) = 0$ at $x = 2$.</td>
</tr>
</tbody>
</table>

So we can actually use data about $f''(x)$ to determine the concavity of $f(x)$, as indicated by the following test:

**Concavity Test.** Let $f(x)$ be twice differentiable.

1. If $f''(x) > 0$ on $(a, b)$, then $f(x)$ is concave up on $(a, b)$.
2. If $f''(x) < 0$ on $(a, b)$, then $f(x)$ is concave down on $(a, b)$.  


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All of this data is summarized in the following table:

<table>
<thead>
<tr>
<th>$f$ concave up</th>
<th>$f'$ increasing</th>
<th>$f'' &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ concave down</td>
<td>$f'$ decreasing</td>
<td>$f'' &lt; 0$</td>
</tr>
</tbody>
</table>

There is one more important thing to notice about the graph of $f(x)$—at $x = 2$, the graph changes concavity:

Points at which curves change concavity have a special name:

**Definition.** A point $P$ at which a curve changes concavity is called an *inflection point*.

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**Finding Local Extremes Redux**

Using the ideas above, we have an alternate method for determining if critical numbers correspond to local extremes:

**Second Derivative Test.** Let $c$ be a critical number of a continuous function $f(x)$ so that $f'(c) = 0$ (the theorem won’t handle type 2 critical numbers).

1. If $f''(c) > 0$ ($f(x)$ is concave up), then $f(x)$ has a local minimum at $c$. 

---

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2. If \( f''(c) < 0 \) \((f(x) \text{ is concave down})\), then \( f(x) \) has a local maximum at \( c \).

3. If \( f''(c) = 0 \), the test is inconclusive.

The following graph illustrates the ideas behind parts 1 and 2 of the test:

\[\text{critical number = local max.}\]
\[\text{f concave down}\]
\[f' = 0\]

\[\text{critical number = local min}\]

\[\text{f concave up}\]

---

**Example.** Given \( f(x) = x^4 - 2x^2 + 8 \):

1. use the Second Derivative Test to determine if critical numbers correspond to local extremes,

2. find all intervals on which \( f(x) \) is concave up,

3. find all intervals on which \( f(x) \) is concave down, and

4. find the inflection points of \( f(x) \).

1. To find the critical numbers, we’ll need to calculate \( f'(x) \):

\[
f'(x) = 4x^3 - 4x.
\]

Since \( f(x) \) itself has domain \((-\infty, \infty)\), every number \( c \) that we find so that \( f'(c) = 0 \) or \( f'(c) \) DNE will be a critical number.

To find the type 1 critical numbers, we need to know when

\[4x^3 - 4x = 0;\]

factoring, we may rewrite as

\[4x(x^2 - 1) = 0\]

or

\[4x(x - 1)(x + 1) = 0,\]

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so $x = 0$, $x = -1$, and $x = 1$ are all type 1 critical numbers.

Since $f'(x)$ has domain $(-\infty, \infty)$, there are no type 2 critical numbers.

To use the second derivative test, we’ll need to calculate $f''(x)$:

$$f''(x) = 12x^2 - 4.$$ 

We will test each critical number in $f''$ in order to determine if they are local extremes:

- $f''(0) = -4 < 0$, so $f$ has a local maximum at $x = 0$.
- $f''(1) = 8 > 0$, so $f$ has a local minimum at $x = 1$.
- $f''(-1) = 8 > 0$, so $f$ has a local minimum at $x = -1$.

2. To determine the intervals on which $f(x)$ is concave up, we’ll need to know when $f''(x) = 0$: so we should solve

$$12x^2 - 4 = 0 \text{ or } x^2 - \frac{1}{3} = 0.$$ 

The roots of this equation are

$$x = \pm \frac{1}{\sqrt{3}}.$$ 

Thus we draw the number line below to help us out:

We need to test a number from each interval in $f''$. In $(-\infty, -\frac{1}{\sqrt{3}})$, let’s choose $-1$: 

$$f''(-1) > 0.$$ 

In $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, let’s test 0: 

$$f''(0) < 0.$$ 

Finally, in $(\frac{1}{\sqrt{3}}, \infty)$, let’s test 1: 

$$f''(1) > 0.$$
Since $f'' > 0$ on $(-\infty, -\frac{1}{\sqrt{3}})$ and on $(\frac{1}{\sqrt{3}}, \infty)$, we conclude that $f(x)$ is concave up on these intervals.

3. Since $f'' < 0$ on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, we conclude that $f(x)$ is concave down on this interval.

4. Since $f(x)$ changes concavity at $x = -\frac{1}{\sqrt{3}}$ and at $x = \frac{1}{\sqrt{3}}$, $f(x)$ has inflection points at both of these numbers.