Implicit Differentiation

In the beginning of this chapter, we have focused on finding the derivative \( f'(x) \) of the function \( f(x) \); we have used the data we collected about the derivative to understand (1) slopes of tangent lines to \( f(x) \), and (2) the instantaneous rate of change of \( f(x) \).

However, there will be times when we wish to calculate slopes of lines tangent to curves that are not functions: as an example, we might wish to find an equation of the line tangent to the ellipse with equation

\[
\frac{x^2}{25} + \frac{y^2}{4} = 1
\]

(shown below) at the point \((3, 8/5)\):

Since the ellipse is not a function of \(x\) (it fails the vertical line test), we can't use our normal calculus techniques to answer this question about its tangent line.

However, the relationship given by the equation for the curve is only satisfied by a select few pairs of points (the ones that fall on the ellipse!); once we choose a value for \(x\), we have (in this case) only two choices for the value for \(y\). For example, if we choose \(x = 3\), then \(y\) must either be \(8/5\) or \(-8/5\).

In other words, the equation gives us an implicit relationship between \(x\) and \(y\); we say that \(y\) is defined implicitly as a function of \(x\) (actually, \(y\) must be broken up into two pieces to get two different functions of \(x\)). We think of \(y\) as an unknown function of \(x\), and since \(y\) is indeed a function of \(x\), \(y' = \frac{dy}{dx}\) is its derivative.

Let's go back to the original problem: we want to find the slope of the line tangent to the ellipse with equation

\[
\frac{x^2}{25} + \frac{y^2}{4} = 1,
\]

i.e. we want to find \(y' = \frac{dy}{dx}\). If the equation said \(y = \text{(function of } x\text{)}\), then we would just take the derivative and be done. However, this is not the case for our equation—in particular, the equation has the term \(\frac{y^2}{4}\) in it—how do we differentiate this?

Keep in mind that \(y\) is just an unknown function of \(x\), and that \(y' = \frac{dy}{dx}\) is the unknown function that is the derivative of \(y\). Perhaps \(y = \sqrt{x}\) (in which case \(y' = 1/(2\sqrt{x})\)), or even \(y = \tan x\) (so that \(y' = \sec^2 x\))—we just don’t know! Regardless of what the actual value for \(y\) is, we can still
find the derivative
\[ \frac{d}{dx} \frac{y^2}{4} \]
just as if we knew that \( y \) was a specific function like \( \tan x \).

Pretend for a moment that we did know that \( y = \tan x \) (it’s not, so use your imagination). Then
\[ \frac{y^2}{4} = \frac{(\tan x)^2}{4}, \]
and we would find
\[ \frac{d (\tan x)^2}{dx} \]
using the chain rule.

So we’ll do exactly the same thing to find \( \frac{d}{dx} \frac{y^2}{4} \):

Differentiate using the chain rule. Since \( y \) is a function of \( x \), the “inside function” is \( g(x) = y \), and the outside function is \( f(x) = \frac{x^2}{4} \). So the chart for the chain rule looks like this:

\[
\begin{align*}
  f(x) &= \frac{x^2}{4} \\
g(x) &= y \\
f'(x) &= \frac{x}{2} \\
g'(x) &= y' = \frac{dy}{dx}
\end{align*}
\]

Using the chain rule, we see that the derivative of the composition function \( f(g(x)) \) is
\[
f'(g(x)) \cdot g'(x) = f'(y) \cdot \frac{dy}{dx} = \frac{y}{2} \cdot \frac{dy}{dx}.
\]

The method of implicit differentiation uses the technique above to find \( y' \) when \( y \) is defined implicitly as a function of \( x \).

**Technique of Implicit Differentiation**

1. Differentiate both sides of the given equation with respect to \( x \), treating \( y \) as an unknown function of \( x \) and using the chain rule to find its derivative.

2. From the resulting equation, solve for \( y' \) to find the desired derivative.

**Example.** Use implicit differentiation to find the slope \( y' \) of the line tangent to the ellipse with equation
\[
\frac{x^2}{25} + \frac{y^2}{4} = 1
\]
at the point \((3,8/5)\).
To find $y'$, we’ll start by differentiating both sides of the equation with respect to $x$. Since

$$\frac{x^2}{25} + \frac{y^2}{4} = 1,$$

we know that

$$\frac{d}{dx} \left( \frac{x^2}{25} + \frac{y^2}{4} \right) = \frac{d}{dx} 1.$$

Let’s handle the two sides separately. On the left, we have already seen that

$$\frac{d}{dx} \left( \frac{x^2}{25} + \frac{y^2}{4} \right) = \frac{2x}{25} + \frac{y}{2} \frac{dy}{dx}.$$

On the right hand side, we know that

$$\frac{d}{dx} 1 = 0.$$

Returning to the original equation,

$$\frac{d}{dx} \left( \frac{x^2}{25} + \frac{y^2}{4} \right) = \frac{d}{dx} 1$$

becomes

$$\frac{2x}{25} + \frac{y}{2} \frac{dy}{dx} = 0.$$

Finally, we’ll solve for $\frac{dy}{dx}$:

$$\frac{y}{2} \frac{dy}{dx} = -\frac{2x}{25},$$

so that

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

Recall that $\frac{dy}{dx}$ tells us the slope of a line tangent to the given curve; so to find the slope of the line tangent to the ellipse

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

at $(3, 8/5)$, we’ll just plug the point $(3, 8/5)$ into the equation for $\frac{dy}{dx}$; at $(3, 8/5)$,

$$\frac{dy}{dx} = -\frac{4 \cdot 3}{25 \cdot \frac{8}{5}} = -\frac{3}{10}.$$

The line tangent to the ellipse at $(3, 8/5)$ is graphed on the curve below—notice that $-3/10$ does appear to be the correct slope.
Example. Find \( \frac{dy}{dx} \) if \( \frac{x}{y} = \sin y \).

We'll start by differentiating both sides of the relationship: we know that

\[
\frac{d}{dx} \frac{x}{y} = \frac{d}{dx} \sin y.
\]

Again, let's differentiate each side separately. On the left hand side, we need to differentiate the fraction \( \frac{x}{y} \), so we must use the quotient rule. With \( f(x) = x \) and \( g(x) = y \), we have

\[
f(x) = x \quad f'(x) = 1 \\
g(x) = y \quad g'(x) = y' = \frac{dy}{dx}
\]

Using the quotient rule,

\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2},
\]
we have

\[
\frac{d}{dx} \frac{x}{y} = \frac{1 \cdot y - x \cdot y'}{y^2}.
\]

On the right hand side, we need to find \( \frac{d}{dx} \sin y \). Since \( \sin y \) is a composition of two functions, we'll need to use the chain rule. The inside function is \( g(x) = y \), and the outside function is \( f(x) = \sin x \), so we have

\[
f(x) = \sin x \quad f'(x) = \cos x \\
g(x) = y \quad g'(x) = y' = \frac{dy}{dx}
\]

Using the chain rule,

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x),
\]
we have

\[
f'(g(x)) \cdot g'(x) = f'(y) \cdot \frac{dy}{dx} = \cos y \cdot \frac{dy}{dx}.
\]
Putting the two sides of the original equation

\[
\frac{dx}{dy} \frac{dy}{dx} y = \frac{d}{dx} \sin y
\]

back together, we have

\[
\frac{y - x \frac{dy}{dx}}{y^2} = \cos y \cdot \frac{dy}{dx},
\]

Now we can solve for \( \frac{dy}{dx} \):

\[
\frac{d}{dx} x = \frac{d}{dx} \sin y
\]

\[
\frac{y - x \frac{dy}{dx}}{y^2} = \cos y \cdot \frac{dy}{dx}
\]

\[
y - x \frac{dy}{dx} = \frac{y^2 \cos y}{dx} \frac{dy}{dx}
\]

\[
y = -y^2 \cos y - x \frac{dy}{dx} = -y
\]

\[
y = y^2 \cos y + x \frac{dy}{dx} = y
\]

\[
(y^2 \cos y + x) \frac{dy}{dx} = y
\]

\[
\frac{dy}{dx} = \frac{y}{y^2 \cos y + x}.
\]