Section 1.6

Limit Laws
So far, we have seen two different methods for determining a function’s limit:

- Determine the limit visually by graphing the function
- Determine the limit numerically by inspecting a table of function values.

Both of these methods are tedious and time-consuming; in this section we will learn a collection of rules that will allow us to evaluate limits much more rapidly.

We will begin by thinking about limits of a very simple type of function, the constant function. For example, consider the constant function \( f(x) = 4 \) below:

![Graph of f(x) = 4](image)

Using the graph, let’s evaluate the following limits:

\[
\lim_{x \to -1} 4 \quad \lim_{x \to 0} 4 \quad \lim_{x \to 5} 4.
\]

We’ll start with the limit as \( x \to -1 \). Recall that a limit is simply a prediction of the function’s value; inspecting the function’s outputs near the input \( x = -1 \), it is clear that

\[
\lim_{x \to -1} 4 = 4.
\]

The other limits are similarly easy to calculate, and return the same value:

\[
\lim_{x \to 0} 4 = 4 \quad \text{and} \quad \lim_{x \to 5} 4 = 4.
\]

In fact, for any number \( a \),

\[
\lim_{x \to a} 4 = 4.
\]

We can turn the above observation into a general rule:

For any constant \( c \), \( \lim_{x \to a} c = c \).

In other words, the limit of a constant function is the constant itself.

Let’s think now about limits of another simple type of function, the identity function \( f(x) = x \):
Using the graph, let’s evaluate the following limits:
\[
\lim_{x \to -1} x \quad \lim_{x \to 0} x \quad \lim_{x \to 5} x.
\]

Inspecting the graph, it is clear that the function’s limit as \( x \to -1 \) is just \(-1\); similarly,
\[
\lim_{x \to 0} x = 0 \quad \text{and} \quad \lim_{x \to 5} x = 5.
\]

This leads to another general rule:
\[
\lim_{x \to a} x = a.
\]

To summarize, we have just learned the following limit laws:

7. \( \lim_{x \to a} c = c \) for any constant \( c \)

8. \( \lim_{x \to a} x = a \)

We can use the identity function \( x \) and constant functions to build many new functions via addition, multiplication, division, etc: the functions
\[
f(x) = x + 1, \quad g(x) = x^4 + 3x - 2, \quad \text{and} \quad h(x) = \frac{x - 1}{x^3 + 4x}
\]
are just a few examples. We would like to be able to quickly calculate limits of such functions. Fortunately for us, the limits of these new functions are generally quite simple to determine because of the two rules above and the remaining limit laws that follow.
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Limit Laws

Let $c$ be a constant, and suppose that

$$\lim_{x \to a} f(x) = M \text{ and } \lim_{x \to a} g(x) = N,$$

where $M$ and $N$ are real numbers. Then:

1. $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = M + N$
2. $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = M - N$
3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cM$
4. $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = M \cdot N$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{M}{N}, \text{ provided that } N \neq 0$
6. $\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = M^n$

Notice that, while the limit laws above give us the power to evaluate limits of many different functions, they cannot be used for every limit. For example, law #5 does not apply to limits of fractions if the denominator of the fraction approaches 0. You must be extremely careful to use the limit laws only when they apply. Later in this section, we will learn some special techniques for handling limits to which the limit laws cannot be applied.

One other note—all of the limit laws apply to one-sided limits as well.

Example. Use the graph below and the limit laws to determine the following limits:

$$\lim_{x \to 2} (f(x) + g(x)) \quad \lim_{x \to -1} (f(x) \cdot g(x)) \quad \lim_{x \to 0} \frac{g(x)}{f(x)}$$
$$\lim_{x \to 2} (g(x))^3 \quad \lim_{x \to -1} 5g(x)$$
Limit Law #1 tells us that
\[ \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x). \]

So we merely need to determine each of these limits to be able to answer the question. By inspecting the graph, it is clear that
\[ \lim_{x \to 2} f(x) = 0 \text{ and } \lim_{x \to 2} g(x) = -3, \]
so
\[ \lim_{x \to 2} (f(x) + g(x)) = 0 - 3 = -3. \]

Limit Law #3 says that
\[ \lim_{x \to a} (f(x) \cdot g(x)) = \left( \lim_{x \to a} f(x) \right) \cdot \left( \lim_{x \to a} g(x) \right). \]

Since
\[ \lim_{x \to -1} f(x) = 3 \text{ and } \lim_{x \to -1} g(x) = 3, \]
\[ \lim_{x \to -1} (f(x) \cdot g(x)) = 3 \cdot 3 = 9. \]

Limit Law #4 says that
\[ \lim_{x \to b} \frac{g(x)}{f(x)} = \frac{\lim_{x \to b} g(x)}{\lim_{x \to b} f(x)}, \]
as long as the limit in the denominator is nonzero. Since
\[ \lim_{x \to 0} f(x) = 4 \text{ and } \lim_{x \to 0} g(x) = 1, \]
\[ \lim_{x \to 0} \frac{g(x)}{f(x)} = \frac{1}{4}. \]
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Limit Law #6 states that
\[
\lim_{x \to 2} (g(x))^3 = (\lim_{x \to 2} g(x))^3,
\]
and since
\[
\lim_{x \to 2} g(x) = -3,
\]
\[
\lim_{x \to 2} (g(x))^3 = (-3)^3 = -27.
\]

Limit Law #3 tells us that
\[
\lim_{x \to 1} 5g(x) = 5(\lim_{x \to 1} g(x)),
\]
and since
\[
\lim_{x \to 1} g(x) = 3,
\]
\[
\lim_{x \to 1} 5g(x) = 5 \cdot 3 = 15.
\]

The following laws describe the limits of a few very specific functions:

9. \( \lim_{x \to a} x^n = a^n \)

10. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)} \) (if \( n \) is even, assume \( a > 0. \))

**Example.** If possible, evaluate the following limits using the limit laws:

\[
\begin{align*}
\lim_{x \to 0} x + 1 & \quad \lim_{x \to 1} x^4 + 3x - 2 & \quad \lim_{x \to 2} \frac{x - 1}{x^3 + 4x} \\
\lim_{x \to -17} \sqrt[3]{x - 10} & \quad \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}
\end{align*}
\]

Since
\[
\lim_{x \to 0} x + 1 = \lim_{x \to 0} x + \lim_{x \to 0} 1,
\]
\[
\lim_{x \to 0} x + 1 = 0 + 1 = 0.
\]

Since
\[
\lim_{x \to 1} x^4 + 3x - 2 = (\lim_{x \to 1} x)^4 + 3 \lim_{x \to 1} x - \lim_{x \to 1} 2,
\]
we know that
\[
\lim_{x \to 1} x^4 + 3x - 2 = 1^4 + 3 \cdot 1 - 2 = 2.
\]

The limit laws tell us that
\[
\lim_{x \to 2} \frac{x - 1}{x^3 + 4x} = \frac{\lim_{x \to 2} x - 1}{\lim_{x \to 2} x^3 + 4x}
\]
if the limit in the denominator is nonzero; otherwise, we will be unable to evaluate this limit. Since
\[
\lim_{x \to 2} x - 1 = 2 - 1 = 1 \quad \text{and} \quad \lim_{x \to 2} x^3 + 4x = 2^3 + 4 \cdot 2 = 8 + 8 = 16,
\]
\[
\lim_{x \to 2} \frac{x - 1}{x^3 + 4x} = \frac{1}{16}.
\]
Since
\[
\lim_{x \to 1} \sqrt[3]{x - 10} = \sqrt[3]{\lim_{x \to 1} (x - 10)},
\]
it is clear that
\[
\lim_{x \to 1} \sqrt[3]{x - 10} = \sqrt[3]{-17 - 10} = \sqrt[3]{-27} = -3.
\]

As with the earlier limit involving fractions, we will only be able to apply the limit laws to
\[
\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}
\]
if the limit in the denominator is nonzero. Unfortunately, it is clear that the limit
\[
\lim_{x \to 1} (x - 1) = 0,
\]
so the limit laws do not apply here. We will learn techniques to handle this type of limit later in this section.

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Based on the examples and laws that we have seen so far, you may have already guessed that the following statement is true:

**Direct Substitution Property** If \( f(x) \) is a polynomial function or a quotient of polynomials (i.e. a rational function), and \( a \) is in the domain of \( f(x) \), then
\[
\lim_{x \to a} f(x) = f(a).
\]

In other words, in many situations, limits match up with function values; in other words, most functions behave \((f(a))\) exactly as we expect them to behave (limit of \( f \) as \( x \to a \)).

The limit laws give us the power to determine limits for a wide variety of functions. However, as mentioned before, we cannot use them to handle every limit that we will encounter. Let’s look at some examples where the limit laws do not immediately apply.

**Example.** We saw earlier that we could not immediately evaluate
\[
\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}
\]
using the limit laws because the limit of the denominator is 0.
Example. Similarly, we cannot yet evaluate
\[ \lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2} \]
for the same reason—the denominator approaches 0 as \( x \to 2 \).

Example. Let \( f(x) \) be the piecewise defined function given by the rule
\[
f(x) = \begin{cases} 
\sqrt{x - 1} + 2 & \text{if } x \geq 1 \\
x & \text{if } x < 1.
\end{cases}
\]
Can we find
\[ \lim_{x \to 1} f(x) \]?
Notice that the rule for \( f(x) \) changes when \( x = 1 \), which makes this problem tricky.
Notice, however, that we can evaluate
\[ \lim_{x \to -2} f(x) \text{ and } \lim_{x \to 5} f(x) : \]
since the rule for \( f(x) \) is consistent near \( x = -2 \),
\[ \lim_{x \to -2} f(x) = \lim_{x \to -2} x = -2. \]
Similarly, the rule for \( f \) is also consistent near \( x = 4 \), so
\[ \lim_{x \to 5} f(x) = \lim_{x \to 5} \sqrt{x - 1} + 2 = 4. \]
The only point at which the limit laws give us problems is the point at which the rule for \( f(x) \) changes.

So far, we have seen a few different cases where the limit laws do not help us out:
(1) Limits that produce 0 in the denominator of a fraction
(2) Limits of piecewise functions at points where the function rule changes.
We will develop ways to handle these problems below.

(1) **Limits that produce 0 in the denominator of a fraction**
There are two different possibilities here:
(a) If
\[ \lim_{x \to a} \frac{f(x)}{g(x)} \]
is a fraction so that
\[ \lim_{x \to a} g(x) = 0 \text{ but } \lim_{x \to a} f(x) \neq 0, \]
then
\[ \lim_{x \to a} \frac{f(x)}{g(x)} \text{ does not exist.} \]
(b) If
\[ \lim_{x \to a} \frac{f(x)}{g(x)} \]
is a fraction so that
\[ \lim_{x \to a} g(x) = \lim_{x \to a} f(x) = 0, \]
then we have no information. The limit may or may not exist, and we will need to do more work to determine which is the case. We can try either of the following methods, discussed in more detail below:

i. Factoring and canceling like factors.

ii. Getting rid of radicals by multiplying by conjugates.

**Example.** Let’s study
\[ \lim_{x \to 0} \frac{1}{x^2}. \]
We immediately see that we cannot apply the limit laws, because
\[ \lim_{x \to 0} x^2 = 0. \]
However,
\[ \lim_{x \to 0} 1 = 1, \]
so we may immediately conclude that the limit does not exist.

In fact, as we discussed in the previous section, we could even go a bit farther by trying to determine why the limit does not exist. Let’s analyze the right-hand and left-hand limits
\[ \lim_{x \to 0^+} \frac{1}{x^2} \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x^2}. \]

As \( x \to 0 \) from the right, say at inputs such as .1, .01, and .0001, the denominator of the fraction \( \frac{1}{x^2} \) becomes very small, so the fraction itself grows very large. So
\[ \lim_{x \to 0^+} \frac{1}{x^2} = \infty. \]

Similarly, as \( x \to 0 \) from the left, say at inputs such as −.1, −.01, and −.0001, the denominator of the fraction \( \frac{1}{x^2} \) again becomes very small (and is positive), and the fraction itself grows very large. So
\[ \lim_{x \to 0^-} \frac{1}{x^2} = \infty \]
as well.

So we can say more about the general limit:
\[ \lim_{x \to 0} \frac{1}{x^2} = \infty. \]

The graph of this function, which we studied in the last section, confirms our conclusion:
Example. Now let’s think about the limit

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}.$$ 

In this case,

$$\lim_{x \to 1} x^2 - 3x + 2 = \lim_{x \to 1} (x - 1) = 0,$$

so we can make no immediate conclusions about the limit. In a case such as this, we should try factoring both the numerator and denominator, and deleting any common factors that we find. In this case, the numerator factors as

$$x^2 - 3x + 2 = (x - 2)(x - 1),$$

so we may rewrite the original function as

$$\frac{x^2 - 3x + 2}{x - 1} = \frac{(x - 2)(x - 1)}{x - 1}.$$

Now let’s compare the functions

$$\frac{(x - 2)(x - 1)}{x - 1}$$

and $x - 2$.

the only point at which they differ is at $x = 1$. Since

- A limit at $x = 1$ is a prediction of the function’s value at $x = 1$ based on the function’s behavior near $x = 1$, and

- We know that $\frac{(x-2)(x-1)}{x-1} = x - 2$ everywhere except at $x = 1$,

then the two limits below must be the same:

$$\lim_{x \to 1} \frac{(x - 2)(x - 1)}{x - 1} = \lim_{x \to 1} (x - 2) = -1.$$ 

We can see that this makes sense by comparing graphs of the two functions:
Example. Now consider the limit

$$\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2}.$$ 

Again,

$$\lim_{x \to 2} \sqrt{x^2 + 5} - 3 = \lim_{x \to 2} (x - 2) = 0,$$

so we can make no immediate conclusions about the limit. Let’s try getting rid of the radicals by multiplying by conjugates: the conjugate of $\sqrt{x^2 + 5} - 3$ is $\sqrt{x^2 + 5} + 3$ (just change the sign between the radical and the other term), so we will rewrite our original fraction as

$$\frac{\sqrt{x^2 + 5} - 3}{x - 2} = \frac{\sqrt{x^2 + 5} - 3}{x - 2} \cdot \frac{\sqrt{x^2 + 5} + 3}{\sqrt{x^2 + 5} + 3}.$$

Simplifying, we have

$$\frac{\sqrt{x^2 + 5} - 3}{x - 2} = \frac{\sqrt{x^2 + 5} - 3}{x - 2} \cdot \frac{\sqrt{x^2 + 5} + 3}{\sqrt{x^2 + 5} + 3} = \frac{(x^2 + 5) - 3\sqrt{x^2 + 5} + 3\sqrt{x^2 + 5} - 9}{(x - 2) \cdot (\sqrt{x^2 + 5} + 3)} = \frac{x^2 + 5 - 9}{(x - 2) \cdot (\sqrt{x^2 + 5} + 3)} = \frac{x^2 - 4}{(x - 2) \cdot (\sqrt{x^2 + 5} + 3)} = \frac{(x - 2)(x + 2)}{(x - 2) \cdot (\sqrt{x^2 + 5} + 3)} = \frac{x + 2}{\sqrt{x^2 + 5} + 3},$$

(except of course when $x = 2$).
Again, since
\[
\frac{\sqrt{x^2 + 5} - 3}{x - 2} = \frac{x + 2}{\sqrt{x^2 + 5} + 3}
\]
everywhere except when \(x = 2\), their limits at 2 are the same:
\[
\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2} = \lim_{x \to 2} \frac{x + 2}{\sqrt{x^2 + 5} + 3} = \frac{4}{6} = \frac{2}{3}.
\]
So
\[
\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2} = \frac{2}{3}.
\]

(2) Limits of piecewise functions at points where the function rule changes.

Example. Earlier, we looked at the function
\[
f(x) = \begin{cases} 
\sqrt{x-1} + 2 & \text{if } x \geq 1 \\
x & \text{if } x < 1
\end{cases}
\]
and saw that we could not evaluate its limit at 1 because the function rule changes here. However, think back to a rule we learned in the previous section:
\[
\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.
\]
If the right-hand and left-hand limits do not match, then the general limit does not exist. On the other hand, if they do match up, then the general limit exists. So let’s evaluate left-hand and right-hand limits for this function.

We’ll start with the left-hand limit. According to the function rule, as \(x \to 1\) from the left (i.e., when we plug in numbers slightly smaller than 1), \(f(x)\) behaves like the function \(x\). So
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1.
\]
On the other hand, as \(x \to 1\) from the right (i.e., when we plug in numbers slightly larger than 1), \(f(x)\) behaves like the function \(\sqrt{x-1} + 2\). So
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} + 2 = 2.
\]
In this case, the left-hand and right-hand limits do not match up, so we can say that
\[
\lim_{x \to 0} f(x) \text{ does not exist.}
\]
We can verify this by inspecting a graph of the function:
Example. Let’s calculate \( \lim_{x \to 0} |x| \). The function \( f(x) = |x| \) is actually a piecewise defined function, with

\[
f(x) = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0.
\end{cases}
\]

Since the function rule changes at \( x = 0 \), we can evaluate the limit at \( 0 \) by checking the left-hand and right-hand limits here.

Let’s begin with the left-hand limit. According to the function rule, as \( x \to 0 \) from the left (i.e., when we plug in numbers slightly smaller than \( 0 \)), \( f(x) \) behaves like the function \(-x\). So

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0.
\]

On the other hand, as \( x \to 0 \) from the right (i.e., when we plug in numbers slightly larger than \( 0 \)), \( f(x) \) behaves like the function \( x \). So

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.
\]

In this case, since the left-hand and right-hand limits match up, we can say that

\[
\lim_{x \to 0} |x| = 0.
\]

The function’s graph is shown below:
In general, to evaluate the limit of piecewise functions at a point where the function rule changes, evaluate the left-hand and right-hand limits separately, and compare them as we did here.

**Squeeze Theorem**

While we have learned many methods for determining limits, we can still only evaluate a few of the many limits that could come up. For example, consider the limit

\[ \lim_{x \to 0} x^2 \cos\left( \frac{1}{x} \right). \]

We have not yet learned how to calculate limits involving trigonometric functions, so we are already stuck. Regardless, it is clear that 0 is not in the domain of the function because of the fraction \( 1/x \). The function is graphed below:

It appears that the limit is 0, but it's hard to tell. This example is a case in which we will need a new tool: the Squeeze Theorem.

To begin to understand what the theorem says, consider the function \( g(x) \) below:
We would like to evaluate the limit of \( g(x) \) as \( x \to 3 \); inspecting the graph visually, it is clear that the limit is 10. However, if we were to try to determine the limit algebraically using the limit laws, we would be unable to do so (the open circle indicates that 3 is not in the domain of the function).

However, suppose that I can find two new functions, \( f(x) \) and \( h(x) \), so that:

1. \( f(x) \leq g(x) \leq h(x) \) near \( x = 3 \), and
2. We can calculate \( \lim_{x \to 3} f(x) \) and \( \lim_{x \to 3} h(x) \), and
3. \( \lim_{x \to 3} f(x) = \lim_{x \to 3} h(x) \).

The graph below illustrates these conditions:

![Graph illustrating conditions](image-url)

Notice that the function \( f(x) \) is always smaller than \( g(x) \), and that \( h(x) \) is larger than \( g(x) \). In addition, \( f \) and \( h \) have no "holes" that would make it difficult to calculate their limits at \( x = 3 \); in fact, it is clear that

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} h(x) = 10.
\]

Then it is reasonable to assume that the unknown limit behaves the same way, i.e. that

\[
\lim_{x \to 3} g(x) = 10.
\]

This is precisely the idea behind the statement of the Squeeze Theorem:

**Squeeze Theorem.** If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \)), and if

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,
\]

then

\[
\lim_{x \to a} g(x) = L.
\]

The only difficulty in using the theorem is finding the functions \( f(x) \) and \( h(x) \); unfortunately, this is often an ad hoc process. We illustrate some possible techniques below:
**Example.** Use the Squeeze Theorem to evaluate

\[
\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right).
\]

We need to find the two functions \(f(x)\) and \(h(x)\); we’ll do this by picking apart the original function. The range of the function \(\cos x\) is \([-1, 1]\), which means that

\[-1 \leq \cos\left(\frac{1}{x}\right) \leq 1.\]

Compare this to the original function,

\[x^2 \cos\left(\frac{1}{x}\right):\]

the only difference between the middle term in the inequality and our function is the factor \(x^2\), and since \(x^2 \geq 0\), we may multiply the inequality through by this factor without changing anything,

\[-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \implies -x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.\]

The graph below illustrates this point:

![Graph illustrating the Squeeze Theorem example](image)

Fortunately for us, we know how to evaluate the limits of the functions on the left-hand and right-hand side of the inequalities, even though we can’t evaluate the limit of the function in the middle. We know that

\[
\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2,
\]

so by the Squeeze Theorem, we can conclude that

\[
\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0
\]

as well.

**Example.** Use the Squeeze Theorem to evaluate

\[
\lim_{x \to 0} x^2 e^{\sin\left(\frac{1}{x}\right)}.
\]
As with the previous example, it is helpful to recall that the range of the sine function is \([-1, 1]\), so that
\[-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.\]

We can create three new functions
\[e^{-1}, \ e^{\sin\left(\frac{1}{x}\right)}, \text{ and } e^1\]
by using the terms from the inequality above as the powers for \(e\). Fortunately for us, the function \(e^x\) increases as its inputs increase, which means that the inequality is preserved, i.e.
\[e^{-1} \leq e^{\sin\left(\frac{1}{x}\right)} \leq e.\]

Finally, multiplying this last inequality through by \(x^2\), we end up with
\[e^{-1}x^2 \leq x^2e^{\sin\left(\frac{1}{x}\right)} \leq ex^2.\]

Since \(e\) and \(e^{-1}\) are just constants, limits of the functions on the left-hand and right-hand side of the inequality are easy to compute:
\[\lim_{x \to 0} e^{-1}x^2 = 0 = \lim_{x \to 0} ex^2,\]
so by the Squeeze Theorem, we know that
\[\lim_{x \to 0} x^2e^{\sin\left(\frac{1}{x}\right)} = 0.\]