0.1 Exponents

The expression \( x^a \) is an exponential expression with base \( x \) and exponent \( a \). If the exponent \( a \) is a positive integer, then the expression is simply notation that counts how many times the number \( x \) is being multiplied by itself. For example, the exponent 5 in the expression \( 2^5 \) indicates that 2 should be multiplied by itself 5 times, so that

\[
2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.
\]

In general, if \( a \) is a positive integer, the notation \( x^a \) indicates the number

\[
x \cdot x \cdot x \cdot \ldots \cdot x.
\]

\( a \) factors

The exponential 0 is a special case. If \( b \) is any nonzero real number, then

\[
b^0 = 1.
\]

If two expressions with the same base are multiplied together, the expression can be rewritten by adding the exponents. For example, \( 2^5 \cdot 2^3 \) is a number the first of whose factors contains five copies of 2; the second factor contains three copies of two, i.e.

\[
2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8.
\]

Overall, there are eight 2s, so

\[
2^5 \cdot 2^3 = 2^{5+3} = 2^8.
\]

In general, we write

\[
x^a \cdot x^b = x^{a+b}.
\]

Two numbers with the same exponent but different bases may be combined as well. For example, the expression \( 2^5 \cdot 3^5 \) may be rewritten as

\[
2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) = 6 \cdot 6 \cdot 6 \cdot 6 = 6^5.
\]

In general,

\[
x^a \cdot y^a = (x \cdot y)^a.
\]

An exponential expression raised to a power, such as \((7^3)^2\), is not too hard to understand. The exponent 2 indicates that there are two copies of \( 7^3 \), so

\[
(7^3)^2 = (7^3) \cdot (7^3) = (7 \cdot 7 \cdot 7) \cdot (7 \cdot 7 \cdot 7) = 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^6.
\]
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In general,

\[(x^a)^b = x^{a \cdot b}.\]

A fractional exponent indicates that a root is involved in the expression. For example, \(16^{1/4}\) may be rewritten as

\[16^{1/4} = \sqrt[4]{16}.\]

Since

\[(16^{1/4})^4 = 16^{(1/4) \cdot 4} = 16^1 = 16,
\]

we know that \(16^{1/4}\) is a number \(x\) so that \(x^4 = 16\). In this case,

\[\sqrt[4]{16} = 2\] since \(2^4 = 16\).

In general,

\[x^{a/b} = \sqrt[b]{(x^a)} = (\sqrt[b]{x})^a.\]

A negative exponent indicates that a fraction is involved in the expression. For example, \(2^{-1}\) may be rewritten as

\[2^{-1} = \frac{1}{2}.\]

On the other hand, an expression such as \(\frac{1}{2^{-1}}\) may be rewritten as

\[\frac{1}{2^{-1}} = \frac{1}{\frac{1}{2}} = \frac{1 \cdot 2}{1} = 2.\]

In general,

\[x^{-a} = \frac{1}{x^a}.\]

One more note about exponents: for all practical purposes, \((x + y)^a\) is never the same as \(x^a + y^a\), and I will ridicule you openly if you naively assume this. As an extremely simple example,

\[(1 + 1)^3 \neq 1^3 + 1^3\]

since

\[(1 + 1)^3 = 2^3 = 8\]

but

\[1^3 + 1^3 = 1 + 1 = 2.\]

To reiterate,

\[(x + y)^a \neq x^a + y^a.]\]
A \textit{function} is an expression that gives a predictable relationship between an independent and a dependent variable; each value of the independent variable corresponds to no more than one value of the dependent variable.

The expression

\[ y = x^2 \]

defines a function with independent variable \(x\) and dependent variable \(y\). Notice that any specific choice for the independent variable \(x\) corresponds to a \textit{single} value of the dependent variable. For example,

\[ x = -2 \text{ corresponds to } y = (-2)^2 = 4, \]

and this is the only value of \(y\) that corresponds to this choice of \(x\). We often write

\[ f(x) = x^2 \text{ to mean } y = x^2. \]

On the other hand, the relationship

\[ y = \pm x \]

is an expression that \textit{does not} define \(y\) as a function of \(x\), for now the independent variable \(x\) corresponds to \textit{two different} values for \(y\). For example,

\[ x = -2 \text{ corresponds to } y = 2 \text{ and } y = -2. \]

We can inspect a graph of a relationship to determine whether or not the relationship defines a function; this process is known as the vertical line test. If any vertical line passing through the curve only touches the curve in at most a single point, then the graph is the graph of a function. The relationships \(y = x^2\) and \(y = \pm x\) are graphed below; it is clear that \(y = x^2\) passes the vertical line test, while \(y = \pm x\) does not.
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Vertical line touches \( f \) in one point.

Graph of the expression \( y = \pm x \).
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Vertical line touches \( y \) in two points, thus graph fails vertical line test.

0.2.1 Domain and Range

The **domain** of a function is the set of all numbers which can be input into the function, i.e. all of the values which the independent variable can take on. For example, the function \( f(x) = x^2 \) has domain *all real numbers*: since every real number may be squared, we may replace the independent variable \( x \) with any real number.

The function \( f(x) = \sqrt{x - 4} \) has domain \([4, \infty)\), because if \( x \) is less than 4, then

\[
x - 4 < 0
\]

and thus is a negative number; even-indexed radicals cannot be evaluated at negative numbers.

In this course, we will spend most of our time studying polynomials, rational functions, trig functions, exponentials, and logarithms (all of which we will discuss in detail as we progress through the course), as well as combinations of these various types of functions. The possible domain restrictions for such functions are listed below:

1. Denominators of fractions must be nonzero
2. Even-indexed roots (\( \sqrt{\cdot}, \sqrt[4]{\cdot}, \sqrt[6]{\cdot}, \ldots \)) cannot be evaluated at negative numbers
3. Log functions cannot be evaluated at zero or at negative numbers (discussed in Chapter 6)
4. Tangent, cotangent, secant, and cosecant functions have restricted domains (discussed in the next section)

As an example of restriction 1, consider the function

\[
f(x) = \frac{x - 1}{x^2 - x - 6}.
\]

The denominator of the fraction is 0 when \( x = -2 \) or \( x = 3 \) (you should check this!), so neither of these numbers is in the domain of \( f \). However, these are the only restrictions that occur, so the domain of \( f(x) \) is \((-\infty, -2) \cup (-2, 3) \cup (3, \infty)\).
A function may have multiple restrictions. For example,

\[ h(x) = \frac{1}{\sqrt{x}} \]

has two restrictions: the denominator must be nonzero, and the input \( x \) (in \( \sqrt{x} \)) must be nonnegative. So the domain of \( h(x) \) is 

\((0, \infty)\).

The range of a function is the set of all possible outputs, i.e. all of the values that the dependent variable can take on.

For example, the function \( f(x) = x^2 \) has range \([0, \infty)\) since each number from 0 to \( \infty \) is the perfect square of some other number.

The function \( g(x) = \frac{1}{x} \) has range \((-\infty, 0) \cup (0, \infty)\) since every number other than 0 can be written as a fraction whose numerator is 1.

0.3 Polynomial Functions

A polynomial function is any function of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]

where \( n \) and each \( a_k \) are constants and \( n \) is an integer greater than or equal to 0. For example,

\[ f(x) = 4x^8 - 3x^3 - 11x^2 + 1 \]

is a polynomial with \( n = 8 \), \( a_8 = 4 \), \( a_3 = -3 \), \( a_2 = -11 \), \( a_0 = 1 \), and all other \( a_k = 0 \).

On the other hand,

\[ g(x) = x^4 \ln x \]

is not a polynomial, because it involves a factor of \( \ln x \) (which is not neither a constant nor a polynomial itself).

0.3.1 Linear Functions

A linear function is a polynomial function whose highest power of \( x \) is the first, i.e. of the form

\[ y = mx + b, \]

where \( m \) and \( b \) are constants. The graph of a linear function is always a line; the constant \( m \) is the line’s slope \( \left( \frac{\text{rise}}{\text{run}} \right) \), and \( b \) is its \( y \) intercept.

For example, consider the line graphed below:
The line has $y$ intercept 8, and we can calculate its slope using the formula

$$m = \frac{\text{rise}}{\text{run}} :$$

Thus its slope is

$$m = \frac{-3}{1} = -3,$$

so the linear function associated to this graph is

$$y = 8 - 3x.$$

There are other forms for the equation of a line; above we used the slope-intercept form. Another common form is the point-slope form: given a point $(x_0, y_0)$ on a line and its slope $m$, the line has equation

$$y - y_0 = m(x - x_0).$$

We can demonstrate this form using the graph above: note that the graph passes through the point $(1, 5)$ and has slope $m = -3$. Thus we can write an equation for the associated function in the form

$$y - 5 = -3(x - 1).$$
If desired, we can simplify this equation as

\[y - 5 = -3x + 3 \quad \text{or} \quad y = 8 - 3x,\]

the same function we wrote earlier.

0.3.2 Quadratic Functions

A quadratic function is a polynomial function whose highest powered term is \(x^2\), i.e. of the form

\[y = ax^2 + bx + c,\]

where \(a\), \(b\), and \(c\) are constants. The graph of a quadratic function is always a parabola. The direction in which the parabola opens is determined by the sign of the squared term: if \(x^2\) has a positive coefficient, the parabola will open up, and if it has a negative coefficient, the parabola will open down. For example, the graph of \(f(x) = 4x^2 - 1\) below opens up:

\[
\text{Graph of } f(x) = 4x^2 - 1.
\]

On the other hand, the graph of \(g(x) = -2x^2 - 1\) opens down:

\[
\text{Graph of } g(x) = -2x^2 - 1.
\]
It is important to be able to find the roots of a quadratic function, i.e. the $x$ coordinates of the points at which the graph of the function crosses the $x$ axis. The roots of $f(x) = 4x^2 - 1$ are marked on the graph:

We will discuss factoring later, but for now it suffices to recall the quadratic equation. The quadratic equation is an equation which outputs the real roots of a quadratic function:

$$f(x) = ax^2 + bx + c \text{ has roots } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

if and only if $b^2 - 4ac \geq 0$. If $b^2 - 4ac < 0$, then $f(x)$ has no real roots.

For example, the function $f(x) = 4x^2 - 1$ has $a = 4$, $b = 0$, $c = -1$, so the quadratic equation gives its roots as

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{-4(4 \cdot (-1))}}{2 \cdot 4} = \frac{\pm\sqrt{16}}{8} = \pm 1/2,$$

which are exactly the $x$ values indicated in the previous graph:

On the other hand, $g(x) = -2x^2 - 1$ has $a = -2$, $b = 0$, and $c = -1$; filling in the quadratic equation

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm\sqrt{-4((-2) \cdot (-1))}}{2 \cdot (-2)} = \frac{\pm\sqrt{-8}}{-1},$$

9
we see that the numerator of the equation is imaginary; thus $g(x)$ has no real roots, as is confirmed by its graph, which does not cross the $x$ axis:

\[ g(x) = -2x^2 - 1 \text{ has no real roots.} \]

### 0.4 Factoring Polynomials

It will be extremely helpful to be able to factor polynomial expressions quickly. Factoring is simply the process of collecting like factors from the terms of a sum and rewriting the sum as a product. For example, the expression

\[ x^3 - x \]

can be factored as $x(x^2 - 1)$ since there is a copy of $x$ in each term of the original expression.

#### 0.4.1 Special Forms

Certain polynomials have a special form that allows us to factor them quickly. The main ones to know are the following:

\[
\begin{align*}
x^2 + 2xy + y^2 &= (x + y)^2 \\
x^2 - 2xy + y^2 &= (x - y)^2 \\
x^2 - y^2 &= (x - y)(x + y) \\
x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\
x^3 - y^3 &= (x - y)(x^2 + xy + y^2)
\end{align*}
\]

For example, we can factor the polynomial $x^2 - 3$ by thinking of it as a difference of squares:

\[ x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3}). \]

We can factor $2x^3 + 54$ quickly if we first factor out the common 2:

\[
\begin{align*}
2x^3 + 54 &= 2(x^3 + 27) \\
&= 2(x + 3)(x^2 - 3x + 9).
\end{align*}
\]
0.4.2 Factoring Quadratic Polynomials

I always prefer to factor quadratics like $ax^2 + bx + c$ using the quadratic equation, because the quadratic equation will always yield the factors with no guess-and-check. However, if you prefer to try to factor quadratics using ad-hoc methods, there are a few things to keep in mind. If $b$ and $c$ are positive real numbers, then

$$x^2 + bx + c \text{ must factor as } (x + \_ ) (x + \_ )$$

$$x^2 - bx + c \text{ must factor as } (x - \_ ) (x - \_ )$$

$$x^2 + bx - c \text{ must factor as } (x + \_ ) (x - \_ )$$

$$x^2 - bx - c \text{ must factor as } (x + \_ ) (x - \_ )$$

Filling in the blanks is largely a matter of guess-and-check.

0.4.3 Completing Squares

We will sometimes factor a quadratic expression by completing the squares—i.e., making use of the identity $x^2 + 2xy + y^2 = (x + y)^2$. For example, we might wish to factor the left hand side of the equation

$$x^2 + 3x = 1.$$ 

We can do this by adjusting the equality so that the left side is a sum of squares. In other words, we wish to find a number $q$ so that $x^2 + 3x + q$ can be factored easily; then we will rewrite

$$x^2 + 3x = 1 \text{ as } x^2 + 3x + q = 1 + q.$$ 

It is quite easy to find $q$: divide the coefficient of the linear (i.e. $x^1$) term by 2, then square the result. In this case,

$$q = \left( \frac{3}{2} \right)^2 = \frac{9}{4}.$$ 

We rewrite

$$x^2 + 3x = 1 \text{ as } x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4}$$

and note that the left hand side is now a perfect square:

$$x^2 + 3x + \frac{9}{4} = \left( x + \frac{3}{2} \right)^2,$$

so

$$x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4}$$

may be rewritten as

$$\left( x + \frac{3}{2} \right)^2 = \frac{13}{4}.$$ 

In general, to complete the squares in the expression

$$x^2 + bx = c,$$
rewrite it as
\[ x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2, \]
whose left hand side is factorable:
\[ (x + \frac{b}{2})^2 = c + \left(\frac{b}{2}\right)^2. \]

### 0.4.4 Factoring Higher Order Polynomials

It is usually not a simple process to factor higher order polynomials, but occasionally it can be done quickly if we recognize patterns such as the one in the following example.

The polynomial
\[ x^3 - x^2 + 3x - 3 \]
can be factored by grouping: we notice that there is a common 3 in the last two terms, so we factor 3x − 3 as 3(x − 1). There is a pattern here—the first two terms can also be rewritten with a factor of x − 1 since
\[ x^3 - x^2 = x^2(x - 1). \]
So we have
\[
\begin{align*}
  x^3 - x^2 + 3x - 3 &= x^2(x - 1) + 3(x - 1) \\
                     &= (x - 1)(x^2 + 3).
\end{align*}
\]

We will rarely need to factor higher order polynomials that do not display a pattern similar to the one above.

### 0.5 Pythagorean Identity

The Pythagorean identity describes the relationship between the lengths of the three sides of any right triangle. If a right triangle has side lengths a, b, and c, with c the length of the longest side (hypotenuse), then a, b, and c satisfy the relationship
\[ a^2 + b^2 = c^2. \]
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For example, suppose that a right triangle has hypotenuse with length 4, and remaining sides of equal length. Then $a = b$ in the graph above, and the Pythagorean identity says that

$$a^2 + a^2 = 4^2,$$

i.e.

$$2a^2 = 16.$$

This means that $a = \sqrt{8}$, so that both remaining sides have length $\sqrt{8}$.

The Pythagorean identity can only be applied to right triangles. Side lengths of non-right triangles are not governed by the identity.

0.6 Basic Graphs

It is important that you know the shapes of graphs of basic functions. Below are graphs of some of the most important basic functions that we will encounter. I will not go into it in this review, but you should be comfortable with the way that the shapes of basic graphs change when the associated functions are altered (for example, how does the graph of $(x + 1)^2$ compare to the graph of $x^2$?).
0.6.1 Log functions

Graph of $y = \ln x$

Graph of $y = \log_{10} x$

0.6.2 Exponential functions

Graph of $y = e^x$
0.6.3 Constant functions

Graph of $y = 3$

0.6.4 Linear functions

Graph of $y = x$
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0.6.5 Quadratic functions

Graph of \( y = x^2 \)

0.6.6 Cubic functions

Graph of \( y = x^3 \)
0.6.7 Basic rational functions

Graph of $y = \frac{1}{x}$

Graph of $y = \frac{1}{x^2}$

0.6.8 Roots of basic functions

Graph of $y = \sqrt{x}$