Power Laws
in
Natural (Critical) Phenomena

Mini-talk 4
Examples of Power Laws

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Agenda

- Review of
  - Thermodynamic Functions and critical Indices
  - Scaling Hypothesis, Universality, Power laws
- Show and Tell
  - Physical
  - Other Science
- Next time:
  - Landau’s Mean Field Theory

Notable Persons of the Week

Origin of Thermodynamics

$$Z=\sum_{state} e^{-\beta \mathrm{H}} \equiv e^{\beta F} e.g.\, H(\{s_i\};J,H)=-J\sum\sum s_i s_j-\sum s_i H$$

$$Derivatives\ of\ \ F\Rightarrow$$

- Equation of state: $P=P(V,T,N)$
- Thermodynamic functions: $\mathrm{C}_H=-T\left.\frac{\partial^2 G(T,H)}{\partial T^2}\right|_H$
- Correlation: $\mathrm{g}(r,r')=<n(r)n(r')>$

- Simplicity to Complexity
- Critical Phenomena, Complex?

Parameters of Phase Transition

Table II. Parameters describing phase transition.

Physical quantity	Range of variables $\epsilon=(T-T_c)/T_c$	h	Behavior of quantity	Parameter describing quantity
$\langle p\rangle$	>0	0	$\langle p\rangle=0$	
	<0	0	$\langle p\rangle\sim\pm\lvert\epsilon\rvert^{\beta}$	β
	0	$\neq 0$	$\sim\pm\lvert h\rvert^{1/\delta}$	δ
$\chi=\partial\langle p\rangle/\partial h\rvert_{\epsilon}$	>0	0	$\sim\epsilon^{-\gamma}$	γ
	<0	0	$\sim\lvert\epsilon\rvert^{-\gamma'}$	γ'
$g(r,r')=\langle p_r p_{r'}\rangle-\langle p\rangle^2$	0	0	$\sim\lvert r-r'\rvert^{-d+2-\eta}$	η
$\xi=$ range of $g(r,r')$	>0	0	$\sim\epsilon^{-\nu}$	ν
	<0	0	$\sim\epsilon^{-\nu'}$	ν'
$C_h=$ specific heat at constant h	>0	0	$a\epsilon^{-\alpha}+b$	α
	<0	0	$a'\lvert\epsilon\rvert^{-\alpha'}+b'$	α'
or	>0	0	$A\log\epsilon^{-1}+B$	$\alpha=0$
	<0	0	$A'\log\lvert\epsilon\rvert^{-1}+B'$	$\alpha'=0$

Universality: an Illustration

TABLE XI. Correspondences between lattice gas model and Ising model.

Ising model	Lattice gas model
Number of up spins	Number of molecules
Partition function	Grand partition function
Number of spins	Volume
Free energy minus magnetic field	Pressure
Magnetization	Density minus critical density
Specific heat C_m	Specific heat C_V

Kadanoff et al, RMP 39, 395 (1967)

Critical Phenomena: Divergence of some thermodynamic Observables

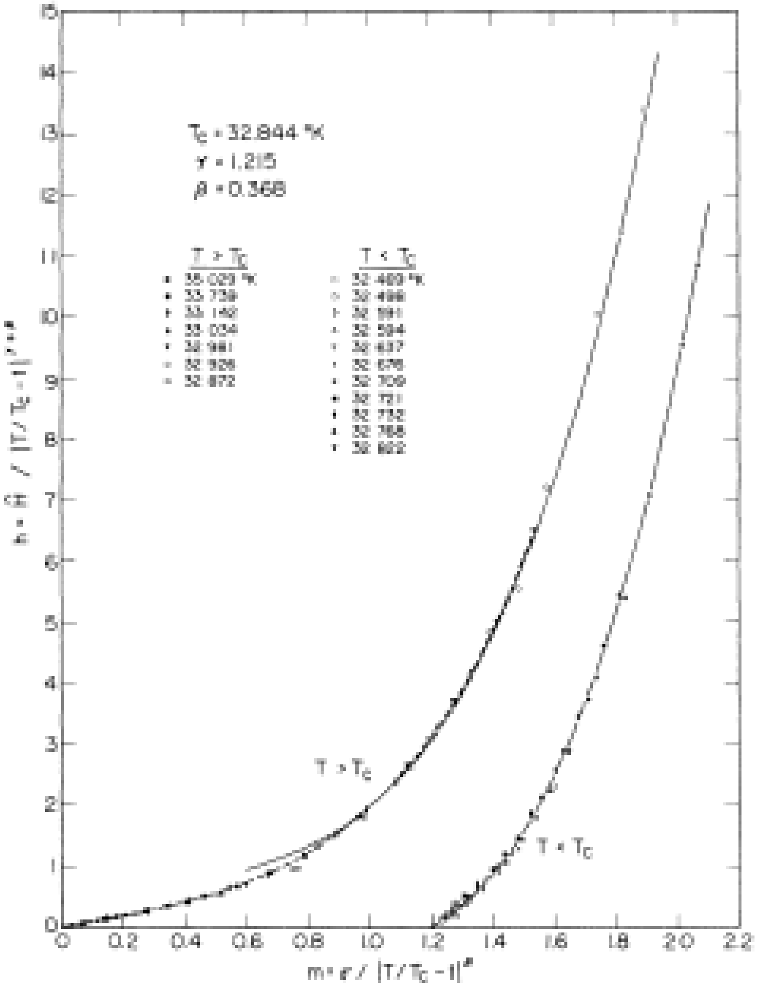


FIG. 1. A linear plot of scaled magnetic field $h(m)$ as a function of scaled magnetization m. The dashed line represents Eq. (1), and fits experimental points from $0 < m < 1.8$ when $T > T_c$. The solid lines are Eq. (2), and the dotted lines show Eq. (4) when it departs significantly from the experimental points. On this plot the origin is the critical isochore, the critical isotherm is at $h = m = \infty$, and the coexistence curve is represented by the point $h = 0$, $m \simeq 1.2$.

$$\frac{M}{\epsilon^{\beta}} = f\left(\frac{H}{\epsilon^{\beta\delta}}\right) ???$$

J.T. Ho and J.D. Litster, PRL, 22, 603 (1969)

Origin of Critical Indices

- Widom's Scaling Hypothesis
- All thermodynamic functions (Complexity)
- Homogeneous functions of 2 variables
- Simplicity $f(\lambda x, \lambda y) = \lambda^p f(x, y)$
- Relations among all indices

Figure 7: Michael Fisher, foreground, and Benjamin Widom, background, at a 65th birthday party for Michael.

Critical Indices

TABLE XII. Definitions of critical indices for liquid–gas transition. If this transition is described correctly by the lattice gas model, α, α', β, γ, γ', and δ all have the same values as in the Ising model.

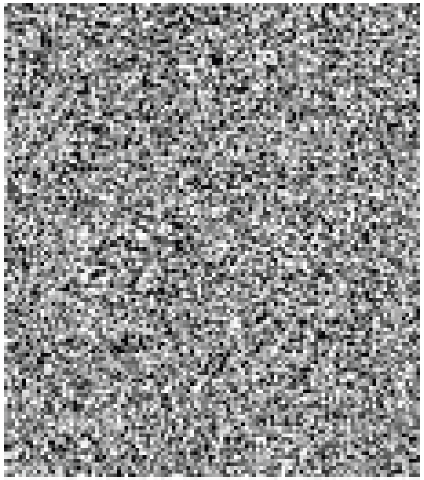
ϵ	$\epsilon=(T-T_c)/T_c$	
α'	$C_v \sim (-\epsilon)^{-\alpha'}$	$\rho=\rho_c,\ \epsilon<0$
α	$C_v \sim \epsilon^{-\alpha}$	$\rho=\rho_c,\ \epsilon>0$
β	$(\rho_L-\rho_G)\sim(-\epsilon)^{\beta}$	$\epsilon<0$, coexistence curve.
γ'	$K_T \sim (-\epsilon)^{-\gamma'}$	$\epsilon<0$, coexistence curve.
γ	$K_T \sim \epsilon^{-\gamma}$	$\rho=\rho_c,\ \epsilon>0$
δ	$\lvert P-P_c\rvert \sim \lvert \rho-\rho_c\rvert^{\delta}$	$T=T_c$
μ	Surface tension $\sim(-\epsilon)^{\mu}$	$\rho=\rho_c,\ \epsilon<0$

Kadanoff et al, RMP 39, 395 (1967)

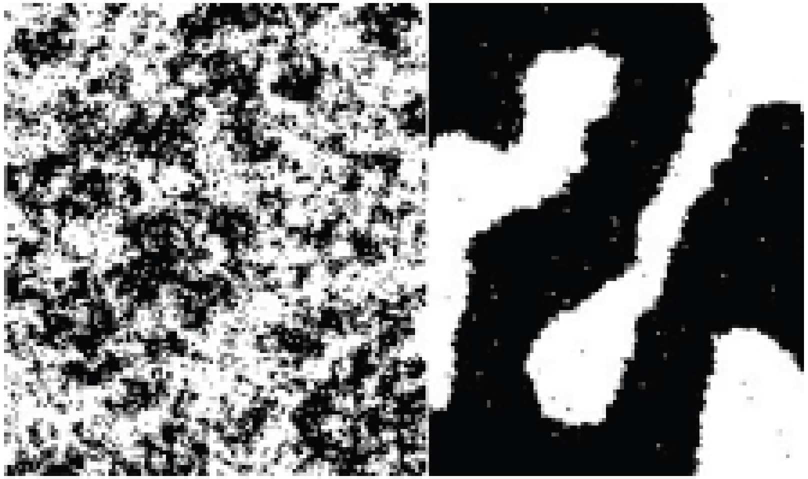
Show and Tell

- More Pictures
- Less Equations (away from G….)

2D Ising model

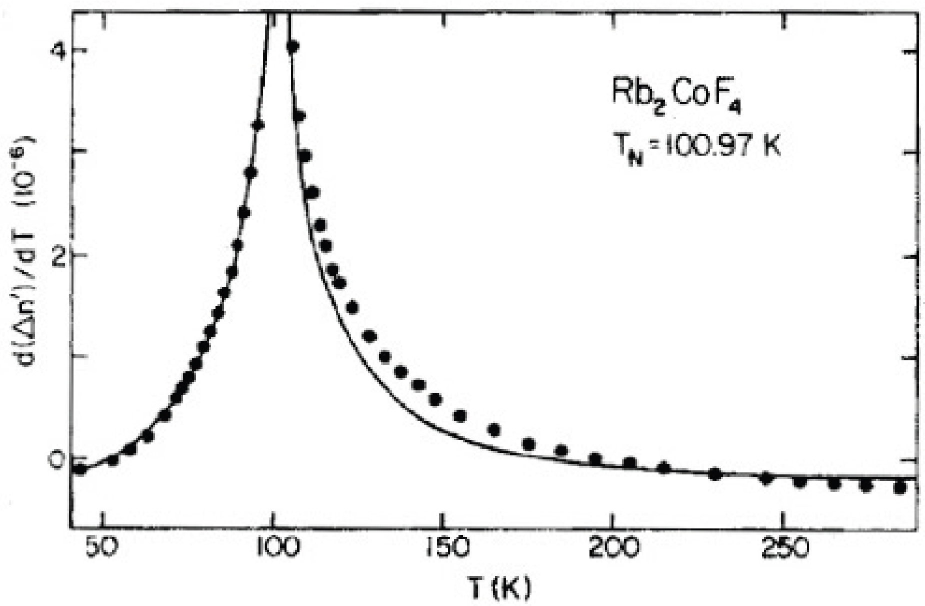


B=0, J=0



B=0, J=0.4406868 B=0, J=0.8

Magnetic Specific Heat of Rb_2CoF_4



$$c_m = -T \left. \frac{\partial^2 G(T,H)}{\partial T^2} \right|_H = A \frac{d(\Delta n)}{dT}$$

$$n = index\ of\ refraction$$

- Rb_2CoF_4 is a layered (2D Ising) antiferromagnet
- Nonmagnetic background (0.16E-6) subtracted
- Amplitude adjusted for the unknown A.

Nordblad, et al., Phys. Rev. B 28. 278 (1983).

Superfluid λ-Transition

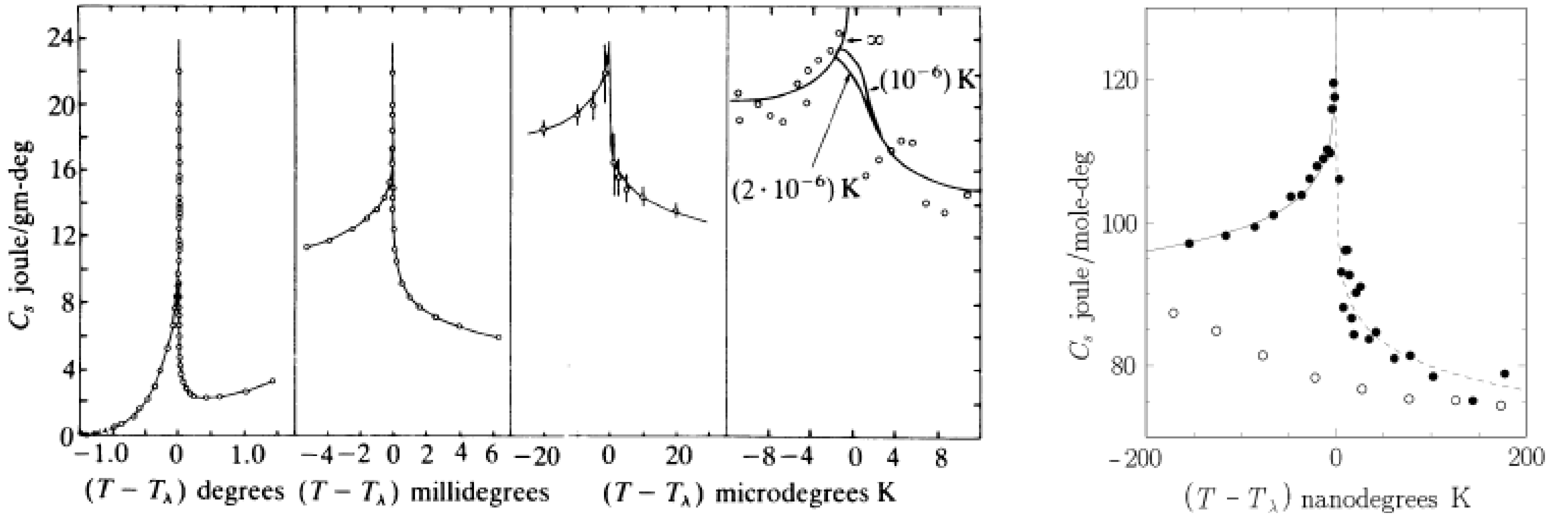


FIGURE 1.1 Specific heat near superfluid transition at $T_\lambda \approx 2.18$ K measured with increasing temperature resolutions. The curve has the typical λ-shape which is the reason for calling it λ-transition. Note that at higher resolutions, the left shoulder of the peak lies above the right shoulder. The data are from Ref. [3]. The forth plot is broadened by the pressure difference between top and bottom of the sample. This is removed by the microgravity experiment in the space shuttle yielding the last plot (open circles are irrelevant here) [4]. They show no pressure broadening even in the nK regime around the critical temperature.

Fairbank et al, Proc. 1965 Washington Conference on Critical Phenomena

C_v of ^{4}He near T_c

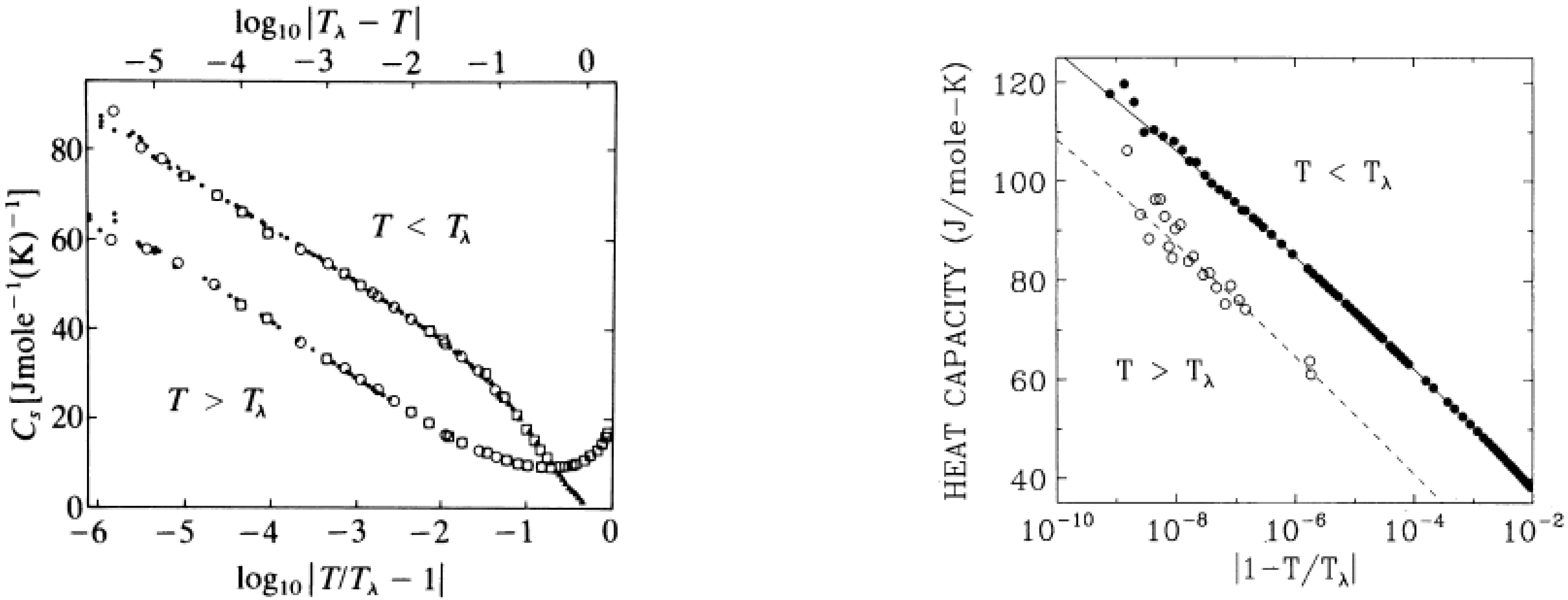


FIGURE 1.2 Specific heat of ^{4}He near superfluid transition plotted against $\log_{10}|T/T_\lambda - 1|$. The early data on the left-hand side by G. Ahlers [5] yield $\alpha \approx -0.026 \pm 0.004$ (for these an upper scale shows $\log_{10}|T_\lambda - T|$ with T measured in units of K). The right-hand side shows the space shuttle data of J.A. Lipa et al. in Refs. [4, 6], which yield $\alpha = -0.01056 \pm 0.0004$.

Ahlers, PRA 3, 696 (1971); Lipa et al, PRL 76, 944 (1996)

Superfluid Density near T_c

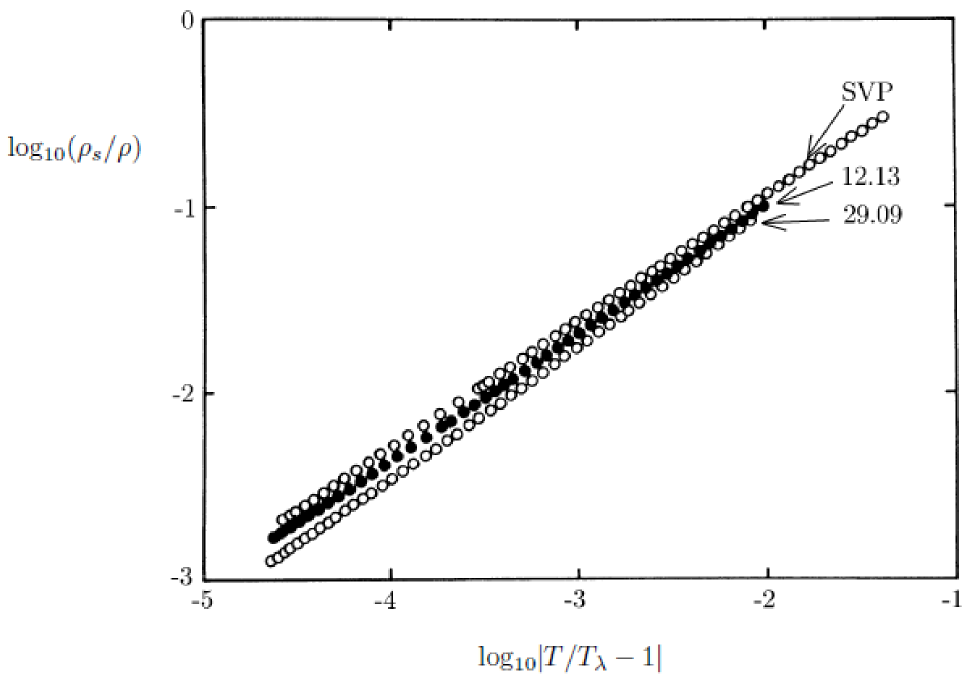


FIGURE 1.3 Doubly logarithmic plots of superfluid density ρ_s divided by the total density as a function of temperature. The slope is 2/3 [7]. The labels on the curves indicate the pressures in bar. The label SVP refers to the saturated vapor pressure.

Greywall and Ahlers, PRA 7, 2145 (1973)

Fluid: Specific Heat of $NiCl_2 \cdot 6H_2O$

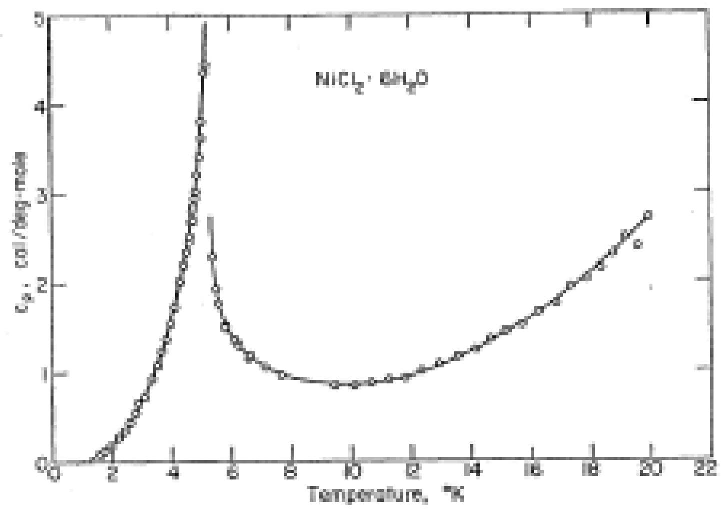


Fig. 1. Molar specific heat of $NiCl_2 \cdot 6H_2O$ as a function of temperature.

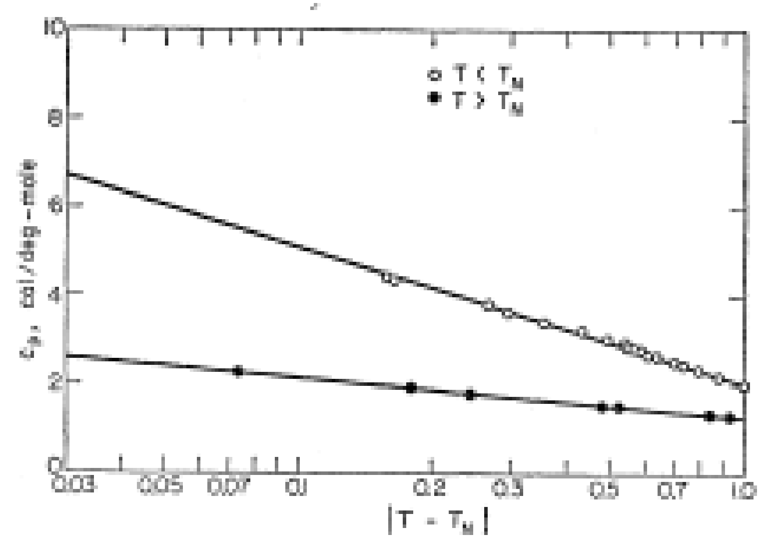


Fig. 5. Plot of C_{Mag} vs $\log|T - T_N|$ for $NiCl_2 \cdot 6H_2O$.

- $t \equiv \dfrac{T - T_c}{T_c}$

- $C_p \sim t^{-\alpha} \qquad t > 0$

  $\sim (-t)^{-\alpha'} \quad t < 0$

- $\alpha < \alpha'$?

Robinson et al, Phys. Rev. 117, 402–408 (1960)

1st: Relations among Indices?

Relations among Critical Exponents

$$\alpha' + 2\beta + \gamma' \geq 2$$

$$\alpha' + \beta(\delta+1) \geq 2$$

$$(2-\alpha')\zeta + 1 \geq (1-\alpha')\delta$$

$$\gamma'(\delta+1) \geq (2-\alpha')(\delta-1)$$

$$\gamma' \geq \beta(\delta-1)$$

$$\vdots$$

$$(2-\alpha)\sigma \geq \delta_s + 1$$

$$\vdots$$

$$d\nu \geq 2-\alpha$$

Total of 17 (Griffiths 1965b)
(Stanley (1971)

Scaling laws: predictions and Falsification

Table IV. Check of scaling law equalities for Ising model. The scaling laws predict that, for a given transition, all the numbers listed in the table (except for Δ and y) should be the same.

	$2-\alpha$	$2-\alpha'$	$d\nu$	$d\nu'$	$d\gamma/(2-\eta)$	$\gamma+2\beta$	$\gamma'+2\beta$	$\beta(\delta+1)$	y	$\Delta=2x/y$
Two-dimensional Ising model	2	2	2	2	2	2	2	2	1	3.75
Three-dimensional Ising model	1.87 $\pm$0.12	1.93+0.04, −0.16	1.93 $\pm$0.01	?	1.933 $\pm$0.008	1.87 $\pm$0.01	1.94 $\pm$0.05	1.93 $\pm$0.05	1.55 $\pm$0.01	3.22 $\pm$0.02

Table X. Scaling law comparison.

	$2-\alpha$	$2-\alpha'$	$d\nu$	$d\nu'$	$d\gamma/2-\eta$	$\gamma+2\beta$	$\gamma'+2\beta$	$\beta(\delta+1)$	$y=1/\nu$	Δ^{b}
Ferromagnets	1.92 $\pm$0.08	1.92 $\pm$0.08	1.95 $\pm$0.09	…	2.08 $\pm$0.12	1.99 $\pm$0.09	…	1.7 $\pm$0.2	1.54 $\pm$0.07	3.2 $\pm$0.2
Antiferromagnets	1.92 $\pm$0.08	1.92 $\pm$0.08	1.95 $\pm$0.09	…	…	1.96 $\pm$0.10[a]	…	…	1.54 $\pm$0.07	3.2 $\pm$0.2

Kadanoff et al, RMP 39, 395 (1967)

Fluid Compressibility

TABLE XVI. Value of γ.

Fluid	Value of γ	Reference	Range of ϵ	Comments
Xe	1.3 ± 0.2	99	$3\times10^{-4}<\epsilon<3\times10^{-2}$	$\epsilon\lesssim10^{-4}$ neglected
Ar	0.6 ± 0.2	110	$3\times10^{-3}<\epsilon<6\times10^{-2}$	Is extrapolation procedure right?
CO_2	1.37 ± 0.2	105, 104, 111	$10^{-5}<\epsilon<10^{-2}$	
CO_2	1.0 ± 0.1	106	$10^{-4}<\epsilon<2\times10^{-2}$	Perhaps pressure gradients are important
"Best value"	1.37 ± 0.2			
Lattice gas	1.250 ± 0.001	Table III		

TABLE XVII. Values of γ'.

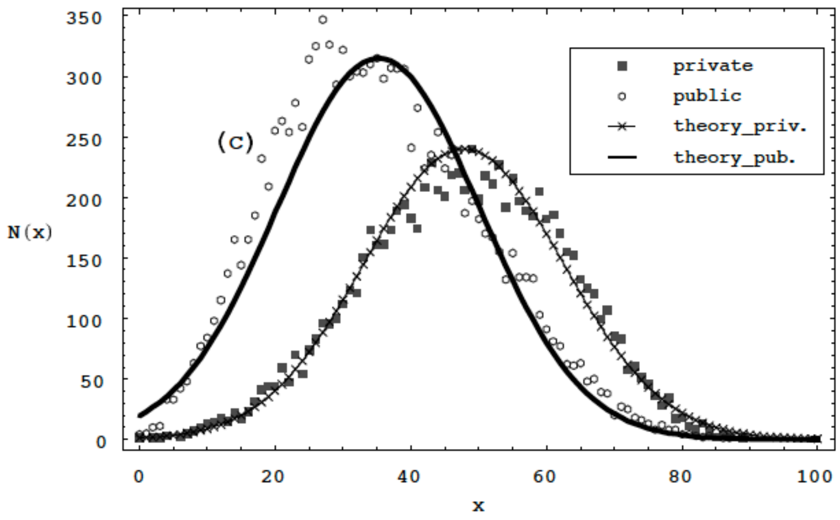
Fluid	Value of γ'	Reference	Range of ϵ	Comments
Ar	1.1 ± 0.2	110	$6\times10^{-4}<-\epsilon<6\times10^{-2}$	Is extrapolation procedure right?
CO_2	1.0 ± 0.3	105, 104, 111	$\epsilon\approx3\times10^{-2}$ $\epsilon\approx3\times10^{-3}$ $\epsilon\approx3\times10^{-4}$	If pressure gradients are important, this analysis gives no information.
"Best value"	1.0 ± 0.3			Very little information
Lattice gas	1.31 ± 0.05	Table III		

Kadanoff et al, RMP 39, 395 (1967)

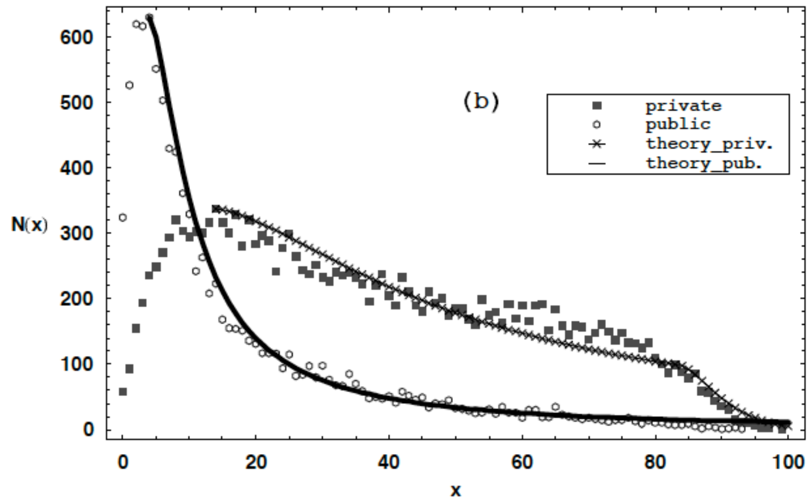
Even

- More Pictures
- Less equations

Scaling in Sociology



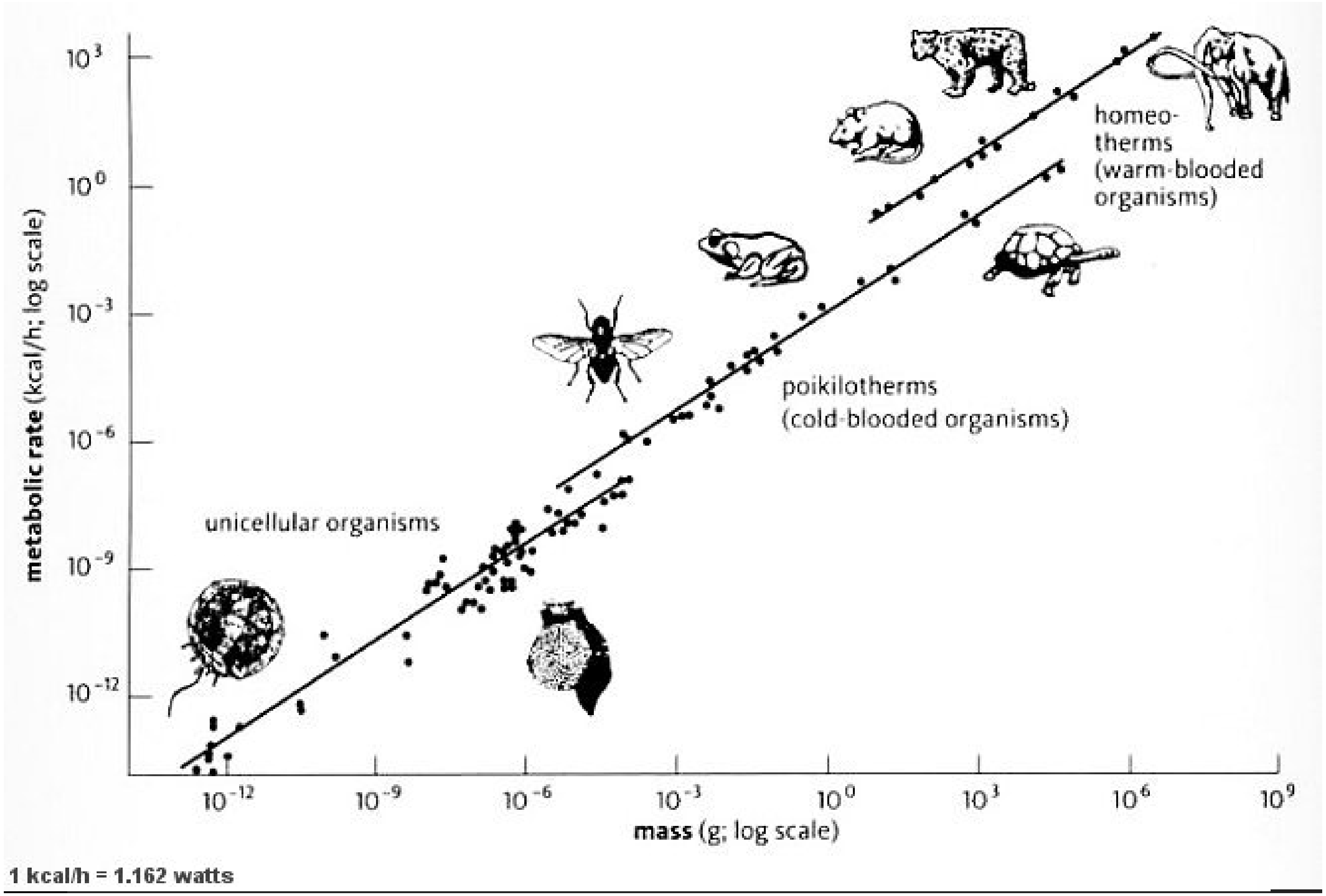
$$p(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$$p(x;\beta,T)=\frac{(\beta-1)T^{\beta-1}}{(x+T)^{\beta}}$$

Gupta et al., arXiv.0301523

Scaling in Biology



$$k \sim m^{\alpha} \quad (\alpha = 3/4)$$

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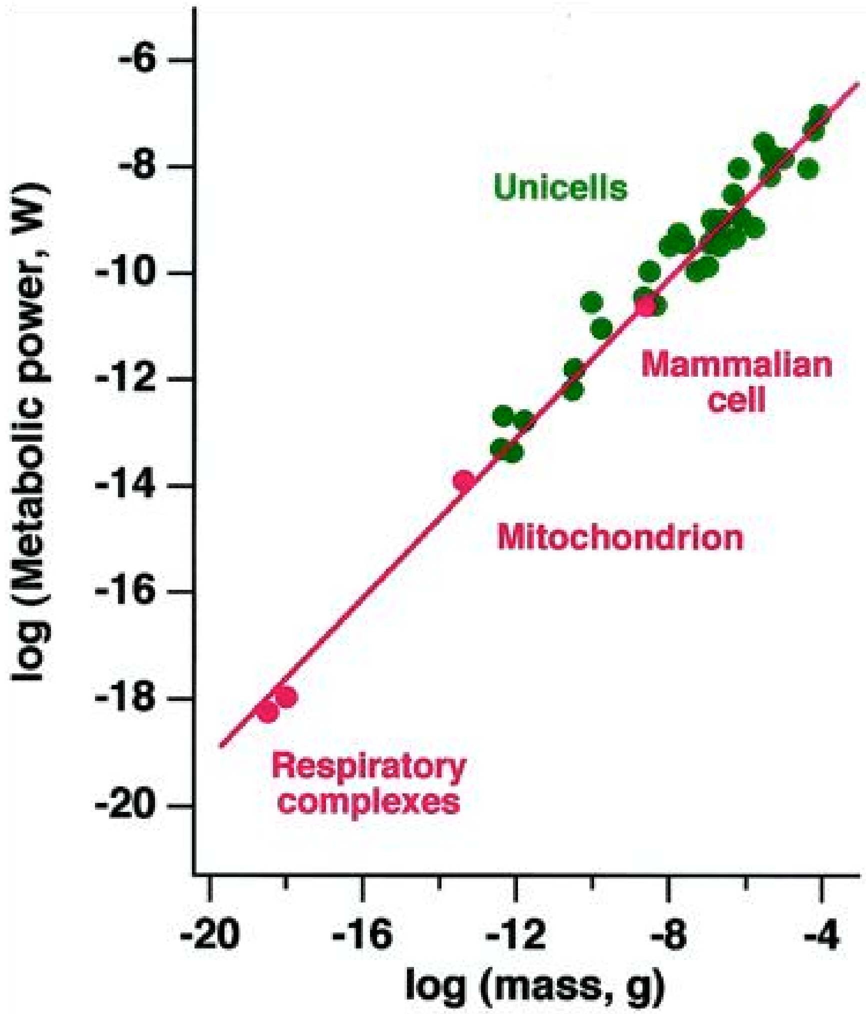
Scaling in Microbiology



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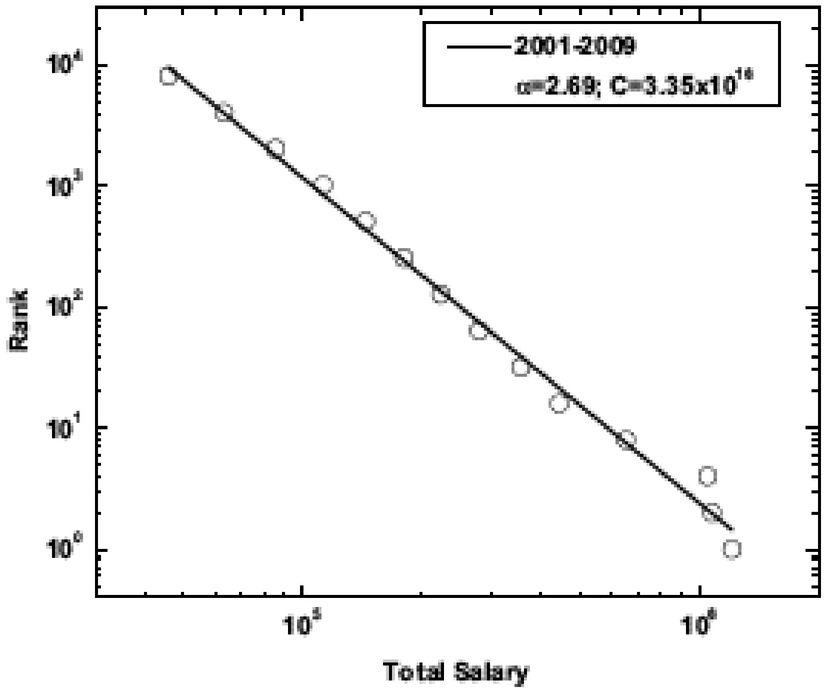
Pareto: Income



$$cdf(x) = cx^{-\alpha}$$

$$p(x) \sim \frac{1}{x^{1+\alpha}}$$

Italian economist Vilfredo Pareto, is a power law probability distribution that coincides with social, scientific, geophysical, actuarial, and many other types of observable phenomena.

End Show and Tell

- More Pictures
- Less equations