Scaling Approach to Complexity

Mini-talk 3

Widom's Hypothesis

Kadanoff's Block Decimation

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Agenda

- Experimental Data
- Homogeneous Functions, a Review
- Scaling hypothesis of Widom
- Order Parameter of Landau
  - Concept only
- Block Renormalization of Kadanoff
- Next time:
  - Landau's Order Parameter
  - Correlation function of Ornstein-Zernike

Notable Quote

- The principal object of research in any department of knowledge
- is to find the point of view from which
- the subject appears in its Simplicity

J. W. Gibbs

Motivation I: Systems

TABLE I. Partial list of transitions with critical points. In general, the symbol h denotes the conjugate to $\langle\phi\rangle$.

Transition	Meaning of $\langle\phi\rangle$	Free choice in $\langle\phi\rangle$	Thermodynamic conjugate of ϕ
Liquid–gas	$\rho-\rho_c$	$\phi>0$=liquid $\phi<0$=vapor (2 choices)	μ
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Heisenberg model ferromagnet	magnetization $\langle\mathbf{M}\rangle$	direction of $\langle\mathbf{M}\rangle$ [can choose any value on surface of sphere.]	$\mathbf{H}$
Antiferromagnet	sublattice magnetization	if n "easy axes" $2n$ choices	not physical
Ising model	$\langle\sigma_r\rangle$	2 choices	h
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Superfluid	$\langle\psi\rangle$ (condensate wave function)	phase of $\langle\psi\rangle$	not physical
Ferroelectric	lattice polarization	finite number of choices	electric field
Phase separation	concentration	2 choices	a difference of chemical potentials

Motivation II: Parameters

TABLE II. Parameters describing phase transition.

Physical quantity		Range of variables $\epsilon=(T-T_c)/T_c$	h	Behavior of quantity	Parameter describing quantity
$\langle p\rangle$		>0	0	$\langle p\rangle=0$	
		<0	0	$\langle p\rangle\sim\pm\lvert\epsilon\rvert^{\beta}$	β
		0	$\neq0$	$\sim\pm\lvert h\rvert^{1/\delta}$	δ
$\chi=\partial\langle p\rangle/\partial h\rvert_{\epsilon}$		>0	0	$\sim\epsilon^{-\gamma}$	γ
		<0	0	$\sim\lvert\epsilon\rvert^{-\gamma'}$	γ'
$g(r,r')=\langle p_r p_{r'}\rangle-\langle p\rangle^2$		0	0	$\sim\lvert r-r'\rvert^{-d+2-\eta}$	η
$\xi=$range of $g(r,r')$		>0	0	$\sim\epsilon^{-\nu}$	ν
		<0	0	$\sim\epsilon^{-\nu'}$	ν'
$C_h=$specific heat at constant h		>0	0	$a\epsilon^{-\alpha}+b$	α
		<0	0	$a'\lvert\epsilon\rvert^{-\alpha'}+b'$	α'
	or	>0	0	$A\log\epsilon^{-1}+B$	$\alpha=0$
		<0	0	$A'\log\lvert\epsilon\rvert^{-1}+B'$	$\alpha'=0$

1^{st}: Relations among Indices?

<u>Relations among Critical Exponents</u>

$\alpha' + 2\beta + \gamma' \geq 2$

$\alpha' + \beta(\delta + 1) \geq 2$

$(2 - \alpha')\zeta + 1 \geq (1 - \alpha')\delta$

$\gamma'(\delta + 1) \geq (2 - \alpha')(\delta - 1)$

$\gamma' \geq \beta(\delta - 1)$

$\vdots$

$(2 - \alpha)\sigma \geq \delta_s + 1$

$\vdots$

$d\nu \geq 2 - \alpha$

Total of 17 (Griffiths 1965b)
(Stanley (1971))

2^{nd}: "Universal" Equation of State

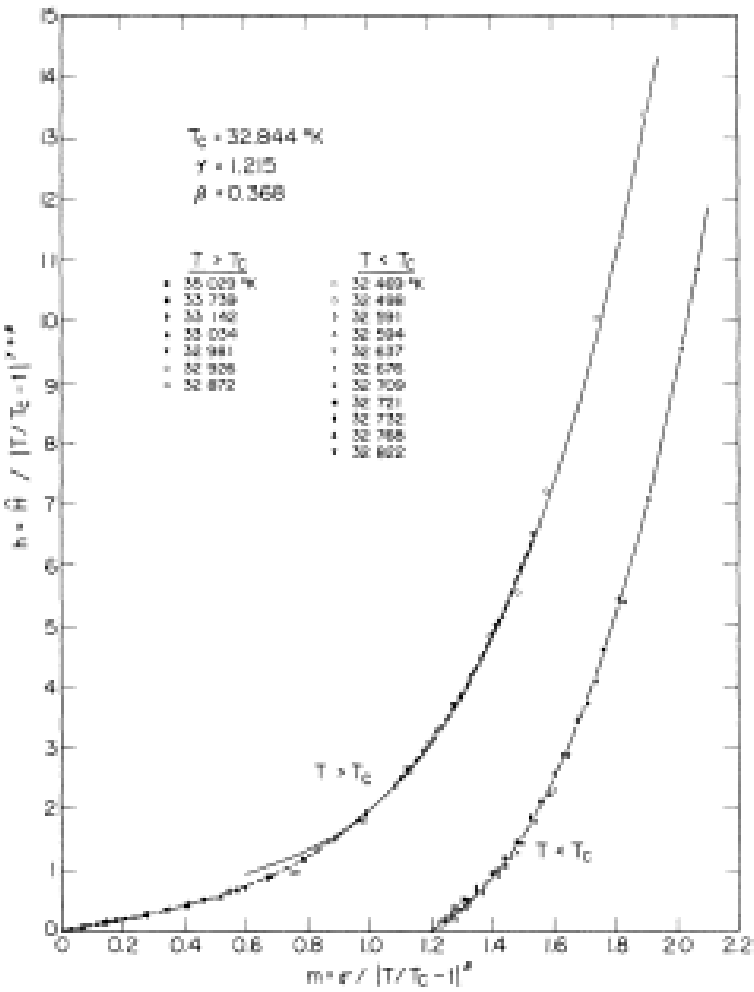


FIG. 1. A linear plot of scaled magnetic field $h(m)$ as a function of scaled magnetization m. The dashed line represents Eq. (1), and fits experimental points from $0 < m < 1.8$ when $T > T_c$. The solid lines are Eq. (2), and the dotted lines show Eq. (4) when it departs significantly from the experimental points. On this plot the origin is the critical isochore, the critical isotherm is at $h = m = \infty$, and the coexistence curve is represented by the point $h = 0$, $m \simeq 1.2$.

$$\frac{M}{\epsilon^{\beta}} = f\left(\frac{H}{\epsilon^{\beta\delta}}\right) ???$$

J.T. Ho and J.D. Litster, PRL, 22, 603 (1969)

Homogeneous Function

- $\mathit{Homogeneous\ function}\ f(x,y)\ \mathit{of}\ \ 2\ \ \mathrm{var}\,\mathit{iables},\ x\ \mathit{and}\ \ y$

- $\mathit{Definition}:\ \mathit{for\ all}\ x\ \mathit{and}\ \ y, x \rightarrow\ \lambda x,$

$f(x,y) \Rightarrow f(\lambda x, \lambda y) \equiv \lambda^p f(x,y)$

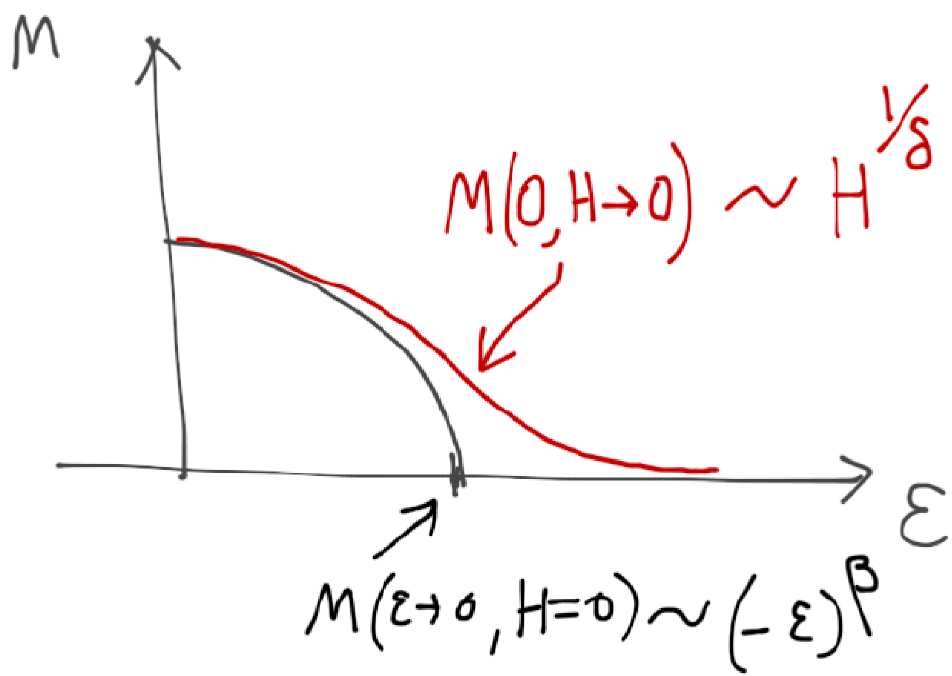
- $\text{Set } \lambda x \equiv 1 \Rightarrow f(1, y/x) = x^{-p} f(x,y)$
- $f(x,y) = x^p f(1, y/x)\ \mathit{or}$
- $\dfrac{f(x,y)}{x^p} = F\left(\dfrac{y}{x}\right)$
-

$\mathit{e.g.\ magnetic\ equation\ of\ \ state}\ M = M(\epsilon, H)$

$\epsilon \equiv (T_c - T)/T_c \Rightarrow ?$

Widom’s Scaling Hypothesis (1965)

2 Critical exponents for $M(T, H)$: $\varepsilon \equiv \frac{T - T_c}{T_c}$



Widom's Scaling Hypothesis (1965)

- $Gibb's$ potential $G(\epsilon \equiv (T - T_c)/T_c, H)$ is a homogeneous function:

$$G(\lambda^a \epsilon, \lambda^b H) \triangleq \lambda G(\epsilon, H)$$

- $Only\ 2\ \exp onents: a\ and\ b$

$$Equation\ of\ State\ is\ given\ by: \left.\frac{\partial G(T,H)}{\partial H}\right|_T = M(T,H) \rightarrow$$

$$\lambda^b M(\lambda^a \epsilon, \lambda^b H) = \lambda M(\epsilon, H)$$

- Relation to β: $H = 0,\ \lambda \rightarrow (-1/\epsilon)^{1/a}$

$$M(\epsilon, 0) = (-\epsilon)^{(1-b)/a} M(-1,0) \sim (-\epsilon)^{\beta} \Rightarrow \beta = \frac{1-b}{a}$$

- Relation to δ: $\epsilon = 0,\ \lambda \rightarrow H^{-1/b}$

$$M(0,H) = H^{(1-b)/b} M(0,1) \sim H^{1/\delta} \Rightarrow \delta = \frac{b}{1-b}$$

$$a = \frac{1}{\beta}\frac{1}{1+\delta},\ b = \frac{\delta}{1+\delta}$$

Widom's Scaling Hypothesis (1965)

- Taking *Additional* $\dfrac{\partial^k}{\partial H^k}$ *or* $\dfrac{\partial^k}{\partial T^k} \Rightarrow$ *more* exp*onents* : $\gamma, \ \gamma'$
- $\left.\dfrac{\partial G^2(T,H)}{\partial H^2}\right|_T \equiv \chi_T \to \lambda^{2b}\chi_T(\lambda^a\epsilon, \lambda^b H) = \lambda\chi_T(\epsilon, H)$

- Relation to γ': $H = 0, \ \lambda \to (-\epsilon)^{-1/a}$

$$\chi_T(\epsilon, 0) = (-\epsilon)^{(2b-1)/a}\chi_T(-1,0) \sim (-\epsilon)^{-\gamma'} \Rightarrow \gamma' = \frac{2b-1}{a}$$

- Relation to γ : $H = 0, \ \lambda \to \epsilon^{-1/a} \Rightarrow \gamma = \dfrac{2b-1}{a} = \gamma'$

All prime indices are equal.

- $\dfrac{b}{a} = \Delta = \beta\delta$ *etc*

Widom's Scaling Hypothesis (1965)

Equation of State is given by with

$(1-b)/a = \beta$ *and* $(b/a) = \beta\delta$:

$$\left.\frac{\partial G(T,H)}{\partial H}\right|_T = M(T,H) \rightarrow \lambda^b M(\lambda^a \epsilon, \lambda^b H) = \lambda M(\epsilon, H)$$

- $\lambda \rightarrow |\epsilon|^{1/a} \Rightarrow M(\epsilon, H) = |\epsilon|^{(1-b)/a} M\left(\frac{\epsilon}{|\epsilon|}, \frac{H}{|\epsilon|^{b/a}}\right)$
- $m \equiv \frac{M(\epsilon, H)}{|\epsilon|^{\beta}}$ *and* $h = \frac{H(\epsilon, M)}{|\epsilon|^{\beta\delta}} \Rightarrow$

$m = F(\pm 1, h)$ *or* $h = f(m)$

- *Scaling thus predicts 2 universal curves for all* T,

one for $T > T_c$, *and one for* $T < T_c$.

Experimental Verification

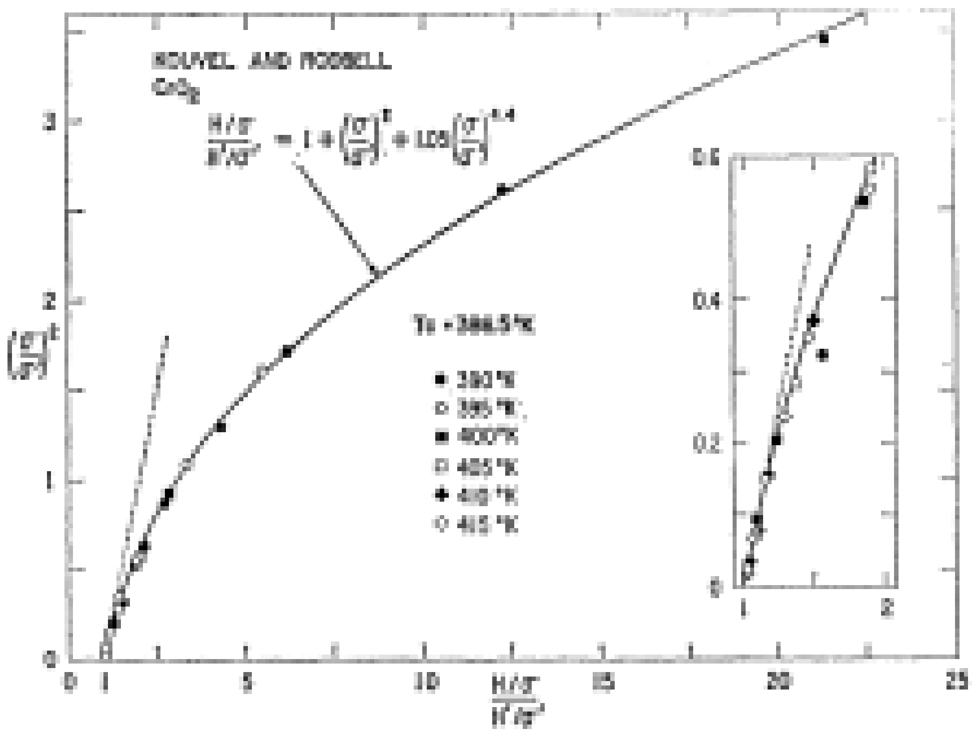


FIG. 4. Magnetization vs field for CrO_2. In our notation, $\sigma = M$, $\sigma' \sim \epsilon^{\beta}$, $H'/\sigma' \sim \epsilon^{\gamma} = \epsilon^{\beta(\delta-1)}$. Points for different ϵ fall on the same curve, which verifies the scaling law prediction, Eq. (4.1).

$$\frac{M}{\epsilon^{\beta}} = f\left(\frac{H}{\epsilon^{\beta\delta}}\right)$$

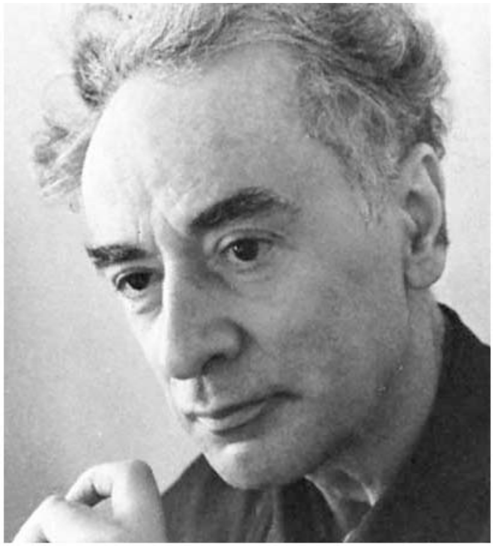
L.P. Kadanoff, et. RMP, 39, 395 (1967).

Landau (~1937) Order Parameter Concept

• 1st order Phase transitions
are manifestations of a Broken Symmetry of an

• Order parameter
  • fluid density, magnetization
  • measures the extent of symmetry breaking

• Spatial and time dependence implicates
  • Correlation (scattering experiments) and
  • Dimensionality



The Order Parameters

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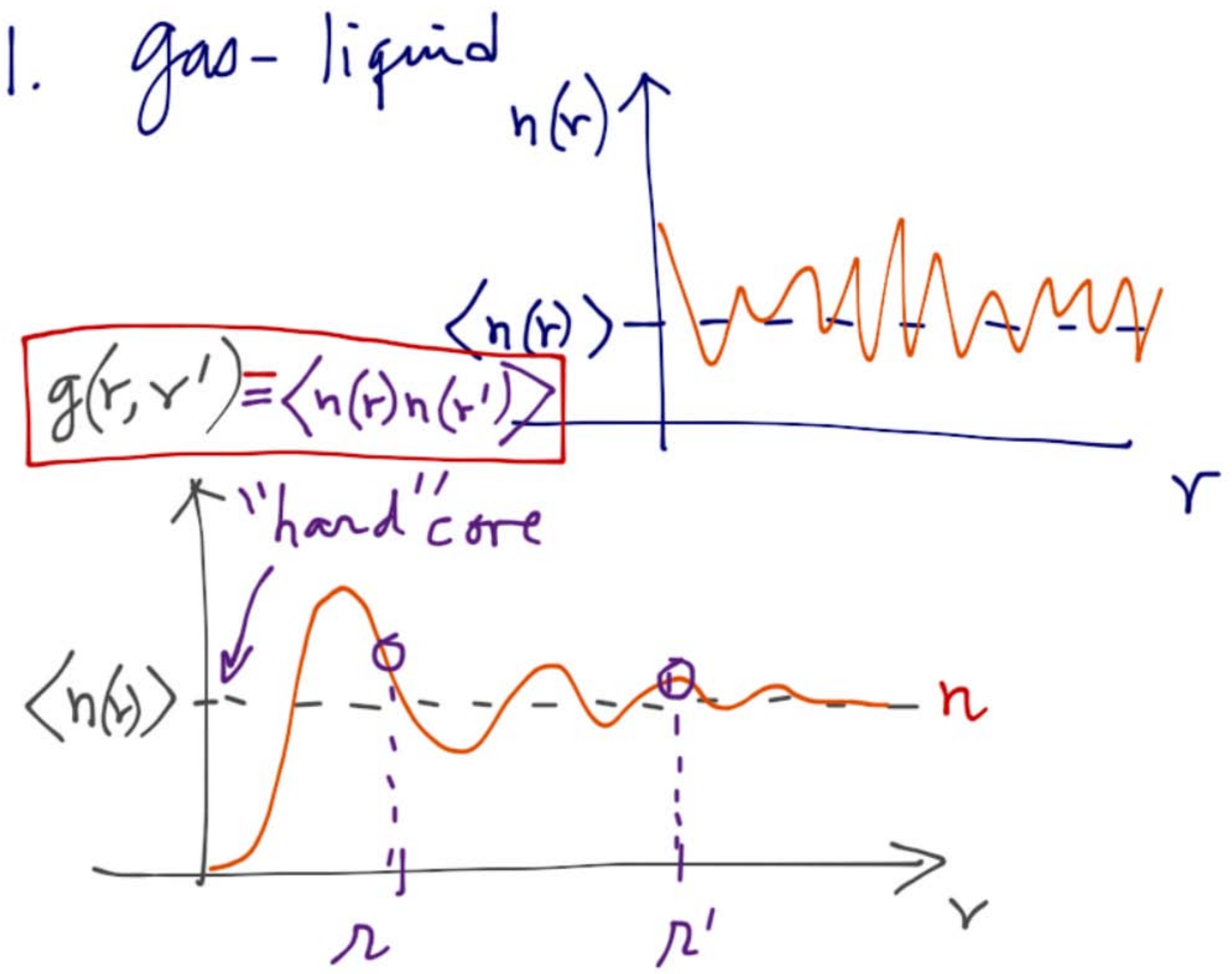
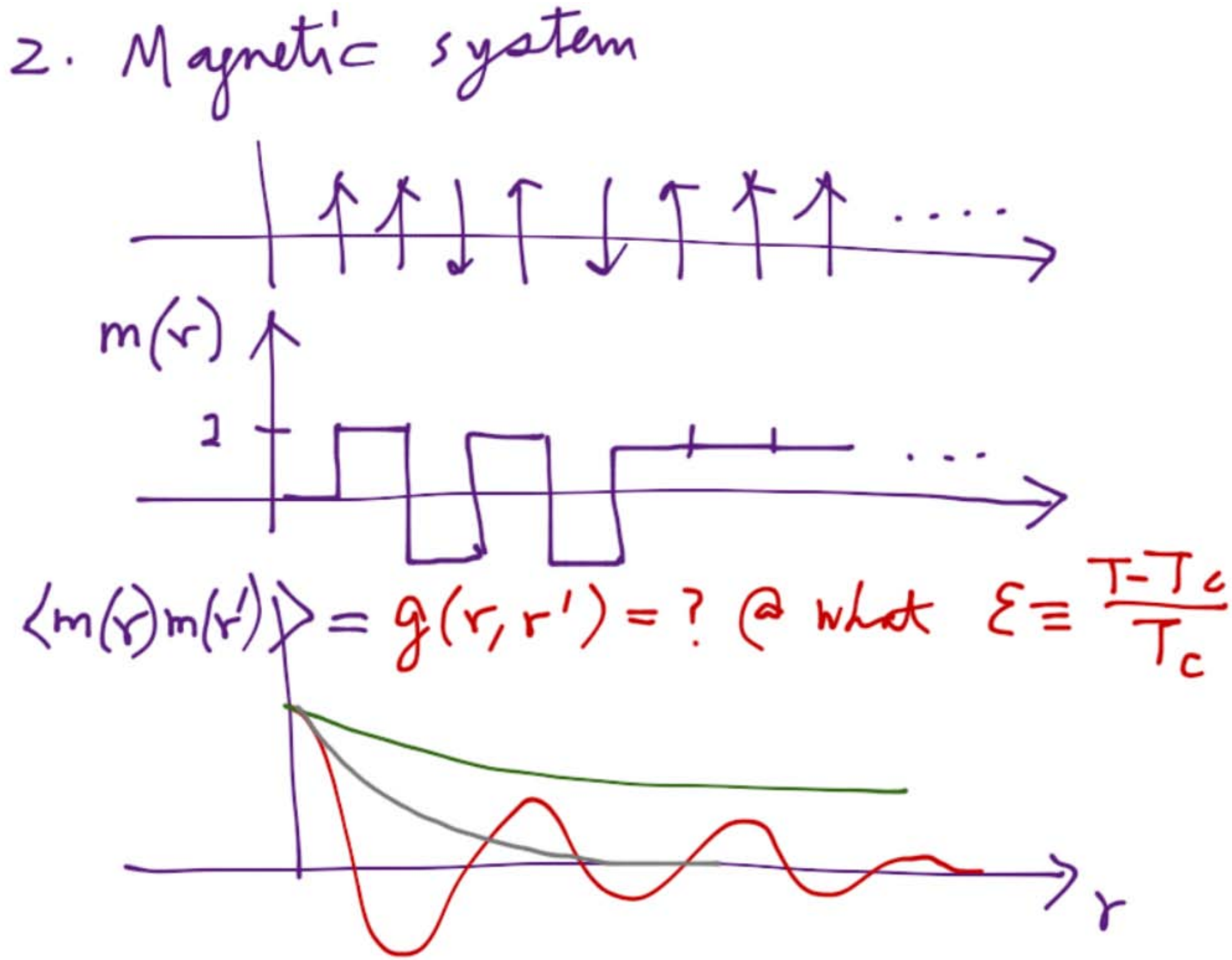
Spatial Correlation Function

$S(k) \Leftrightarrow$



Spatial Correlation Function



Time Correlation Function

3) Electrical conduction of charge.

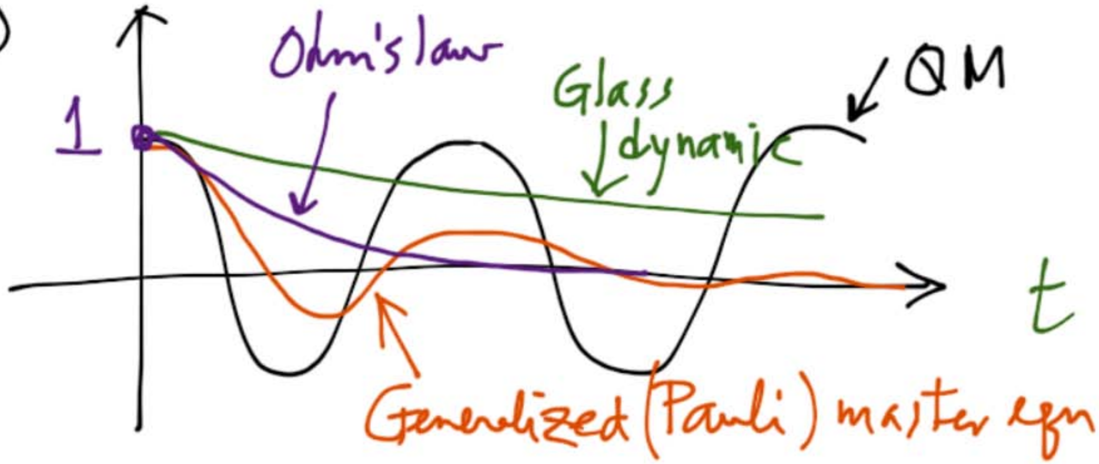
$i(t) \equiv q * v(t)$

(a) $v(t) = \langle v(t) \rangle + \varepsilon \eta(t)$

- $\varepsilon \ll 1$
- $\eta(t)$ noise

(b) $C(t,t') \equiv \langle v(t) v(t') \rangle = ?$

(c)



Kadanoff's heuristic argument for Scaling hypothesis

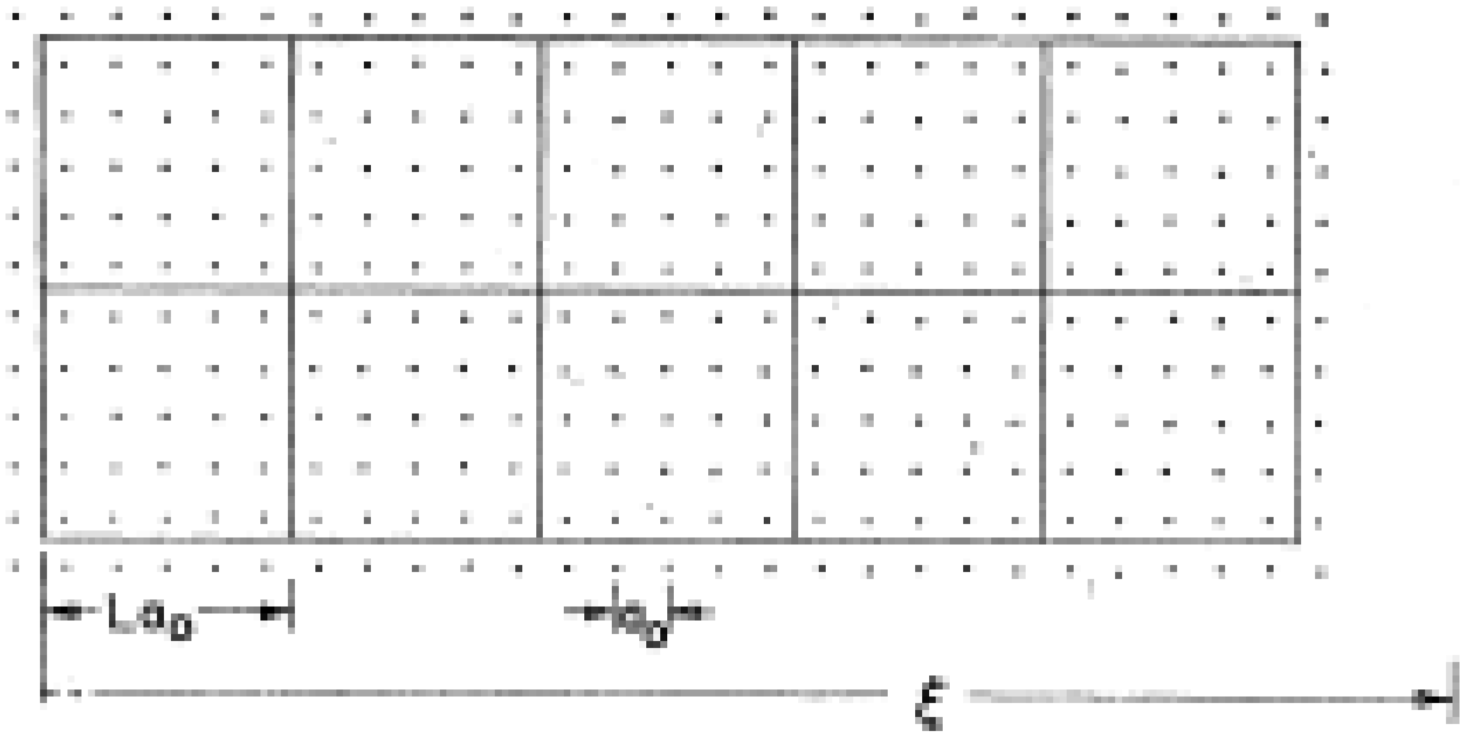


FIG. 2. Division of Ising model lattice into cells. $L \gg 1$ but $La_0 \ll \xi$.

Kadanoff's heuristic arguments for Scaling hypothesis

At or very near critical point, Correlation length, ξ,

1a. very long, and

1b. independent of details

2. Block of size La construction is possible: $1 << L << \xi/a$

3. With a cell of size La, most spins align in one direction

4. Site: σ_r and $(\epsilon,\ h)$ and Cell: μ_a and $(\tilde{\epsilon}, \tilde{h})$

5. $G(\tilde{\epsilon}, \tilde{h}) = L^d G(\epsilon,\ h)$

6. Kadanoff: $\tilde{h} \triangleq L^x h \textit{ and } \tilde{\epsilon} \triangleq L^y \epsilon \Rightarrow G(L^y \epsilon, L^x h) = L^d G(\epsilon,\ h)$

7. Assuming true for all L, $G(L^{y/d}\epsilon, L^{x/d} h) = LG(\epsilon,\ h)$

8. $a \equiv y/d \textit{ and } b \equiv x/d \Leftrightarrow \textit{Scaling Hypothesis}$

Agenda-Summary

- Experimental Data to
  - Landau, Widom, Kadanoff, many others
- Order Parameter of Landau
- Scaling hypothesis (homogeneous function) of Widom
- Block Renormalization of Kadanoff
- Next time:
  - Landau's Mean Field Theory
  - Correlation function of Ornstein-Zernike