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Polynomials for Directed Graphs

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Abstract

Several polynomials are defined on directed graphs and rooted directed graphs which are all analogous to the Tutte polynomial of an undirected graph. Various interrelations between these polynomials are explored. Many of them have the number of maximal directed trees (arborescences) as an evaluation, and are related to the one-variable greedoid polynomial $\lambda_G(y)$. Applications to reliability are also examined.

1. Introduction.

The central theme of this study is the attempt to generalize the Tutte polynomial of an undirected graph to a polynomial invariant of a directed graph, especially one having a distinguished vertex (a rooted digraph). It does not seem that there is any one "right" such generalization; rather, different definitions of the Tutte polynomial suggest different directed polynomials. One definition of the Tutte polynomial of an undirected graph G depends on a rank function on subsets of $E(G)$, the edge-set of G. For $A \subseteq E(G)$ the (circuit) rank of A, $r(A)$, is the cardinality of the largest acyclic subset of A. The *corank-nullity polynomial* $f(G;t,z)$ (also called the *Whitney polynomial*) is defined by

$$\text{(T1)} \qquad f(G;t,z) \equiv \sum_{A \subseteq E(G)} t^{r(E(G))-r(A)} z^{|A|-r(A)}.$$

In Section 2, we use three different rank functions to define polynomials on rooted digraphs via (T1). (By a "rank function" we simply mean any function mapping sets of edges of directed graphs to non-negative integers.) One of these three is the rank function of the directed branching greedoid [1] associated to a rooted digraph D, while the second is the basis rank function of the same greedoid. Thus, these first two rank functions yield two-variable polynomial invariants for arbitrary greedoids. The third rank function considered is not associated with any greedoid, though it is still associated with the notion of reachability of vertices of D from its root. All of these polynomials generalize the one-variable greedoid polynomial of [1] when the greedoid is the directed branching greedoid of a rooted digraph.

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The definition of the corank-nullity polynomial of an undirected graph can be rephrased without using a rank function. For this formulation, define a subgraph A of G to be ***vertex-spanning*** if it contains the entire vertex-set $V(G)$ of G, and let S denote the collection of all such subgraphs; we will often use the same letter A to refer to both a vertex-spanning subgraph and its edge-set. Then if $c(A)$ denotes the number of connected components of A,

$$\text{(T2)} \qquad f(G;t,z) \equiv \sum_{A \subseteq E(G)} t^{c(A)-c(G)} z^{|A|+c(A)-|V(G)|}.$$

Since $r(A) = |V(G)| - c(A)$, (T2) is a direct translation of (T1). We mention it here because it is natural to investigate the notions of "number of components" associated to various rank functions. For instance, the notion associated to the rank function of the directed branching greedoid is that the number of components of a rooted digraph is one more than the number of vertices not reachable from the root, while the notion associated to the third rank function considered in Section 2 is that the number of components of a rooted digraph is the minimum number of edges that can be inserted into the digraph so that all its vertices will be reachable from its root. Also, it often happens that a certain notion of "number of components" can be more natural than the associated rank function. For instance, in Section 3 we mention two polynomials that are associated with strong connectedness rather than reachability, and for these polynomials there are no "natural" descriptions of the associated rank functions.

The recursive definition of the Tutte polynomial is very useful, since it allows for natural inductive proofs. If e is an edge of G then the deletion $G - e$ is simply the graph G with e erased (deleted). The contraction G/e is the graph obtained from $G - e$ by identifying the endpoints of e. The Tutte polynomial can be defined by

(T3) *(a)* $f(G) = 1$ *if* G *has no edges,*
(b) $f(G) = f(G-e) + f(G/e)$ *if* e *is neither an isthmus nor a loop,*
(c) $f(G) = xf(G/e)$ *if* e *is an isthmus, and*
(d) $f(G) = yf(G-e)$ *if* e *is a loop.*

Note that (T3) (b), (c), and (d) can be replaced by the following single condition.

(e) For any e, $f(G) = (y-1)^{|V(G/e)|+1-|V(G)|} f(G/e) + (x-1)^{r(G)-r(G-e)} f(G-e)$.

(T3) is equivalent to (T1) and (T2) under the substitution $x = t+1, y = z+1$. It seems difficult to define a "directed isthmus" that can be used in (T3) to recursively define a polynomial that is independent of the order in which edges are deleted and contracted. In Section 4 we give two examples of recursively defined polynomials which depend on a given ordering of the edges or vertices.

One can also define the ordinary Tutte polynomial in terms of activities (see [2] and [9]) or generalized activities (see [7]). Although several of the polynomials we define for rooted digraphs can be described in terms of analogous activities, these descriptions are complicated and do not seem to contribute much to understanding the polynomials.

In Section 5 we extend the polynomials to weighted digraphs; several can then be evaluated to yield the determinant of the Kirchhoff matrix. We also discuss the application of these polynomials to some reliability problems. The polynomials yield not only information about the likelihood that a directed network with imperfect edges will remain fully operational, but also information about the expected number of edges that will fail, the expected number of repairs that will have to be made to restore a network to operational status, and the expected cost of making those repairs.

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2. Polynomials for rooted digraphs.

In this section, we define three polynomials for rooted digraphs via (T1). We first define the branching greedoid independence rank $r_1(A)$ and basis rank $r_2(A)$, and also the rooted forest rank $r_3(A)$, of a subset A of the edge-set of a rooted digraph $(D, *)$. (Here $*$ represents the root vertex.) A subgraph of D is a $*$ *rooted arborescence* if for every vertex v of T, there is a unique directed path in T from $*$ to v. Thus T is a directed tree, rooted at $*$. (See Figure 1a.)

A *spanning* $*$ *rooted arborescence* is a $*$ rooted arborescence which is incident with every vertex of D. In general, a digraph need not have any spanning $*$ rooted arborescence; even if it doesn't, it is still true that all maximal $*$ rooted arborescences have the same number of edges. A $*$ *rooted forest of arborescences* is a vertex-disjoint union of arborescences rooted at $*$, v_1, v_2, ... for some vertices v_1, v_2, ... (In Figure 1b, the heavy lines give a $*$ rooted forest of arborescences.) We will often fail to distinguish between a $*$ rooted forest of arborescences and its edge-set.

We now define rank functions r_1, r_2, and r_3 by

(2.1) $\qquad r_1(A) \equiv \max\{|T| \mid T \subseteq A \textit{ is a } * \textit{ rooted arborescence}\},$

(2.2) $\qquad r_2(A) \equiv \max\{|T \cap A| \mid T \subseteq E(D) \textit{ is a } * \textit{ rooted arborescence}\}, \textit{ and}$

(2.3) $\qquad r_3(A) \equiv \max\{|F| \mid F \subseteq A \textit{ is a } * \textit{ rooted forest of arborescences}\}.$

Proposition 2.1. *For every $A \subseteq E(D)$, $r_1(A) \leq r_2(A) \leq r_3(A)$; also, $r_1(D) = r_2(D)$.*

Proof. Since any $*$ rooted arborescence can be extended to a maximal $*$ rooted arborescence, $r_1(A) \leq r_2(A)$. Furthermore, since any subset of a maximal $*$ rooted arborescence is a $*$ rooted forest of arborescences, $r_2(A) \leq r_3(A)$.

The above inequalities are not generally reversible. For instance, in Figure 2a if $A = \{a, c, e\}$ then $r_1(A) = 1$, $r_2(A) = 2$, and $r_3(A) = 3$. It is a routine verification to demonstrate that r_1 is a greedoid rank function. In general, it is not a matroid rank function. For example, let $A = \{b, c, d\}$ and $x = a$ in the digraph of Figure 2a. Then $r_1(A) = 0$, but $r_1(A \cup \{x\}) = 4$. It is also easy to verify that each of r_2 and r_3 satisfies the definition of a matroid rank function *except* for semimodularity. For example, in Figure 2b for $i = 2$ or 3 we have $r_i(\{c\}) = r_i(\{c, d\}) = r_i(\{a, c\}) = 1$, but $r_i(\{a, c, d\}) = 2$.

Imitating definitions of matroid theory (see [10] for an account), for $1 \leq i \leq 3$ we define $A \subseteq E(D)$ to be *i-spanning* if $r_i(A) = r_i(E(D))$, *i-independent* if $r_i(A) = |A|$, and an *i-basis* if it is both i-spanning and i-independent. The following proposition is an immediate consequence of Definitions (2.1), (2.2), and (2.3).

Proposition 2.2. *Suppose $A \subseteq E(D)$.*

(a) A is 1-independent iff it is a $*$ rooted arborescence, and is 1-spanning iff it contains a maximal $*$ rooted arborescence.

(b) A is 2-independent iff it is a subset of a $*$ rooted arborescence, and is 2-spanning iff it is 1-spanning.

(c) A is 3-independent iff it is a $*$ rooted forest of arborescences, and is 3-spanning iff it contains a $*$ rooted forest of arborescences of maximum size.

The i-independent and i-spanning sets have several other interesting properties, all of which are easy to establish. If $r_3(D) = r_1(D)$ then the 3-spanning sets will coincide with the 1- and 2-spanning sets. Also, $r_i(A)$ is the maximum cardinality of an i-independent subset of A. In particular, since the 2-independent sets are simply the subsets of 1-independent sets this implies that the 1-bases and 2-bases coincide. (If $r_3(D) = r_1(D)$ then the 3-bases will also coincide with the 1- and 2-bases.) Every 1- or 2-independent set can be extended to a 1-basis, but the same is not true of 3-independent sets. (Consider $\{e\}$ in Figure 2a.) We also note that the 1-independent sets are precisely the feasible sets in the directed branching greedoid. The 2- and 3-independent sets constitute hereditary families of subsets of $E(D)$ (i.e., subsets of 2- and 3-independent sets are 2- and 3-independent), but in general are not the feasible sets of any greedoid, or (consequently) the independent sets of any matroid.

We now denote by $f_i(D; t, z)$ the polynomial associated to r_i, $1 \leq i \leq 3$, via (T1):

$$f_i(D; t, z) \equiv \sum_{A \subseteq E(D)} t^{r_i(E(D)) - r_i(A)} z^{|A| - r_i(A)}.$$

Proposition 2.3. *For $1 \leq i \leq 3$,*

(a) $f_i(D; 0, 0) =$ the number of i-bases of D,

(b) $f_i(D; 0, 1) =$ the number of i-spanning sets,

(c) $f_i(D; 1, 0) =$ the number of i-independent sets,

(d) $f_i(D; z^{-1}, z) = z^{-r_i(E(D))}(z + 1)^{|E(D)|}$,

(e) $f_i(D; -1, -1) = 0$ (when $E(D) \neq \emptyset$), and

(f) $f_i(D; 1, 1) = 2^{|E(D)|}$.

A striking difference among these polynomials is given in the next proposition. Part (a) gives a nice application of f_1 which was proven in [6]. Part (b) follows from noting that every subset of a $*$ rooted arborescence is 2-independent (and 3-independent).

Proposition 2.4. *(a) Let $(D_1, *_1)$ and $(D_2, *_2)$ be $*_i$ rooted arborescences. Then $f_1(D_1) = f_1(D_2)$ if and only if D_1 and D_2 are isomorphic.*

(b) If $(D, *)$ is a $*$ rooted arborescence then for $i = 2$ or 3, $f_i(D) = (t+1)^n$, where $n = |E(D)|$.

Proposition 2.4 should not give the reader the impression that f_1 is invariably more discriminating than f_2 or f_3. For instance, the two rooted digraphs of Figure 3 have $f_1(D_1) = f_1(D_2) = (z+1)(t^2z + t^2 + t + 1)$, $f_2(D_1) = f_2(D_2) = (z+1)(t+1)^2$, $f_3(D_1) = (z+1)(t+1)^2$, and $f_3(D_2) = (t + z + 2)(t+1)$. Thus, f_3 can distinguish digraphs which cannot be distinguished by f_1 or f_2. (N.b. D_1 and D_2 have isomorphic directed branching greedoids.) A more extensive study of the polynomial f_1 for rooted graphs and digraphs is undertaken in [8], while the application of this polynomial to rooted trees is explored in more detail in [3].

As noted in the Introduction, we can rephrase the definition of $f_i(D)$ in terms of "connected component functions": $c_i(A) = v - r_i(A)$. Direct descriptions of these functions can be given. For example, if we let $d_1(A)$ be the number of vertices which cannot be reached from $*$ by a directed path in A, then $c_1(A) = d_1(A) + 1$. The reader can formulate a similar interpretation for $c_2(A)$. Perhaps a more useful measure of connectivity is given by $c_3(A)$. Let $d(A)$ be the minimum cardinality of a set R of directed edges such that $R \cup A$ contains a spanning arborescence rooted at $*$. (The elements of R may or may not be edges of D.)

Proposition 2.5. *If $A \subseteq E(D)$ then $c_3(A) = d(A) + 1$.*

Proof. Let $R = \{e_1, e_2, ..., e_d\}$ be a set of directed edges (not necessarily in D) of minimum size so that $R \cup A$ contains a spanning $*$ rooted arborescence, where $d = d(A)$. Let T be a spanning $*$ rooted arborescence, $T \subseteq R \cup A$. Then $R \subseteq T$, since d is minimum. Since $|T| = v - 1$, we have $|T \cap A| = |T| - |R| = v - d - 1$. But $T \cap A$ is a $*$ rooted forest of arborescences, so $r_3(A) \geq |T \cap A| = v - d - 1$.

On the other hand, let $F \subseteq A$ be any $*$ rooted forest of arborescences having k roots in addition to $*$. Let $S = \{b_1, b_2, ..., b_k\}$ be the collection of directed edges emanating from $*$ whose terminal vertices are these k roots. Clearly $F \cup S$ is a spanning $*$ rooted arborescence, so $|F \cup S| = v - 1$. But by the definition of d, $d \leq k$, so $|F| + d \leq v - 1$. Since F is arbitrary, $r_3(A) \leq v - d - 1$.

Hence $r_3(A) = v - d - 1$, so $c_3(A) = d(A) + 1$. This completes the proof.

The polynomials $f_i(D)$ have different recursive properties. We require directed versions of isthmuses and loops for the following propositions. A directed edge e of $(D, *)$ is an *i-isthmus* $(1 \leq i \leq 3)$ if it has the property that I is i-independent iff

$I \cup \{e\}$ is i-independent. That is, adding or deleting an i-isthmus to or from an i-independent set does not affect i-independence. An edge e is an *i-loop* if it is in no i-independent set. Equivalently, e is an i-isthmus if $r_i(A \cup \{e\}) = r_i(A - e) + 1$ for every subset A of $E(D)$, and e is an i-loop if $r_i(A \cup \{e\}) = r_i(A)$ for every A. For example, in Figure 2a the edge a is an i-isthmus for $i > 1$ but not for $i = 1$, and the edge b is a 2-isthmus but neither a 1-isthmus nor a 3-isthmus. The next proposition summarizes the recursive properties of the $f_i(D)$. The proof uses the usual idea of partitioning the subsets of $E(D)$ into two classes, those that contain e and those that do not. We leave the details to the reader. (For $i = 1$, the proof appears in [6].)

Proposition 2.6. *(a) If e is an i-loop, then $f_i(D) = (z+1)f_i(D-e)$, $1 \le i \le 3$.*
(b) If e is an i-isthmus, then $f_i(D) = (t+1)f_i(D/e)$, $1 \le i \le 3$.
(c) Let e be any edge directed out of $*$ (a loop or not). Then for $i = 1$ or 3, $f_i(D) = z^{|V(D/e)|+1-|V(D)|}f_i(D/e) + t^{r_i(D)-r_i(D-e)}f_i(D-e)$. Moreover, if e is an i-isthmus then $f_i(D-e) = f_i(D/e)$.

Parts (a) and (b) of this proposition show that the given definitions of i-loop and i-isthmus are "correct" for the given polynomials. Other definitions may also be of some interest, however. For instance an *i-coloop* is defined to be an edge which is in every maximal i-independent set; i-isthmuses and i-coloops coincide for $i = 2$ or 3, but a 1-coloop (= 2-coloop) is not the same as a 1-isthmus. (Coloops and isthmuses also coincide for undirected graphs and matroids.) For example, a in Figure 2a is a 1-coloop but not a 1-isthmus; note that 2.6(b) fails for a.

For a non-loop e directed out of $*$, the proof of part (c) of 2.6 depends on the fact that if $e \notin A$, $r_i(A)$ in $D - e$ is the same as $r_i(A)$ in D, and if $e \in A$, $r_i(A - e)$ in D/e is one less than $r_i(A)$ in D (for $i = 1$ or 3). The latter fact is also true for $r_2(A)$, but the former is not. Consider, for example, the digraph of Figure 4. Then if $A = \{c, d\}$ we have $r_2(A) = 2$ in D but $r_2(A) = 1$ in $D - a$, even though $r_2(D) = r_2(D-a) = 3$. Thus, the recursion of 2.6(c) will not work for f_2. In Section 4 we will give a polynomial closely related to f_2 that does satisfy a complete recursion.

We also note that 2.6(c) gives a complete recursive description of $f_1(D)$, since by 2.6(a), $f_1(D-e) = (z+1)^k$ when $*$ is isolated in $D - e$ and k is the number of edges in $D - e$. The recursion is incomplete for $f_3(D)$, however, since $f_3(D-e)$ depends on the structure of $D - e$, even when $*$ is isolated in $D - e$. The difference is that when $*$ is isolated in D every edge of D is a 1-loop, but not necessarily a 3-loop. In Section 4 we will give a polynomial related to f_3 that satisfies a complete recursion.

We can generalize 2.6(a) and 2.6(b) in the following way. If D_1 and D_2 are rooted digraphs with disjoint vertex-sets and respective roots $*_1$ and $*_2$, then we define the *direct sum* $D_1 \oplus D_2$ to be the rooted digraph obtained by identifying $*_1$ and $*_2$ to form a new root $*$. $D_1 \oplus D_2$ is also called the *one-point union* of D_1 and D_2. It is clear from this definition that a set A of edges of $D_1 \oplus D_2$ is i-independent iff both $A \cap D_1$ and $A \cap D_2$ are i-independent.

Proposition 2.7. For $1 \le i \le 3$, $f_i(D_1 \oplus D_2) = f_i(D_1)f_i(D_2)$.

In [1], Björner and Ziegler discuss a one-variable polynomial $\lambda(G;y)$ associated to a greedoid G. In the case of the directed branching greedoid corresponding to a rooted digraph $(D,*)$, $\lambda(G;y)$ is characterized by the following.

I. $\lambda(D;y) = \lambda(D/e;y)$ *if* e *is a 2-isthmus directed out of* $*$.
II. $\lambda(D;y) = y^k$ *if* D *consists of* k *2-loops.*
III. $\lambda(D;y) = \lambda(D-e;y)+\lambda(D/e;y)$ *if* e *is directed out of* $*$ *and is not a 2-isthmus.*

Note that in case I, $r_1(D-e) < r_1(D)$, and in case III, $r_1(D-e) = r_1(D)$. By 2.6(c), $f_1(D;0,y-1)$ satisfies I and III. Furthermore, by 2.6(a), $f_1(D;0,y-1) = y^k$ if D consists of k 2-loops. Thus $f_1(D;0,y-1) = \lambda(D;y)$. (Since the definition of f_1 is valid for an arbitrary greedoid G, we also get $f_1(G;0,y-1) = \lambda(G;y)$ [6].) Also, note that $f_2(D;0,y-1) = f_1(D;0,y-1)$, since the 1- and 2-spanning sets in $E(D)$ coincide. Similarly, if D has a $*$ rooted spanning arborescence then $f_3(D;0,y-1) = \lambda(D;y)$, although this equality does not hold in general. See Section 4 for other polynomials related to λ.

3. Unrooted digraphs.

In this section we briefly discuss several polynomials associated to unrooted digraphs.

Let D be an unrooted digraph. We say $F \subseteq E(D)$ is a *forest of rooted arborescences* if it is a vertex-disjoint union of arborescences rooted at $v_1, v_2, \ldots$ for some vertices $v_1, v_2, \ldots$. (Thus, a $*$ rooted forest of arborescences in a rooted digraph is simply a forest in which $*$ is one of the roots selected.) We can now define $r_4(A)$ to be the maximum size of a forest of rooted arborescences contained in A. Clearly $r_3(A) \leq r_4(A)$ for every A and every choice of a root vertex $*$. We can imitate the definitions of Section 2 to obtain definitions of 4-independent and 4-spanning sets, and use (T1) to define a polynomial $f_4(D)$. It is straightforward to formulate propositions regarding f_4 that are analogous to 2.3 and 2.5. We leave the details to the reader.

Recall that a digraph D is *strongly connected* if, for every pair of vertices u and v, there exists a directed path from u to v. We denote by $c_5(D)$ the number of strong components of D, and we define $f_5(D)$, the *strong-connected digraph polynomial*, via (T2). We leave it to the reader to verify that $f_5(D)$ is always a polynomial, i.e., that $|E(A)| + c_5(A) - |V(D)| \geq 0$ for every $A \in S$. The reader can also verify the following two families of examples. If D is a rooted arborescence (or any digraph whose strong components are all singletons) then $f_5(D) = (z+1)^n$, where $n = |E(D)|$. Also, if D is a directed circuit then $f_5(D) = z + t^{n-1}((z+1)^n - z^n)$.

It is amusing to note that if r_5 is the rank function defined by $r_5(A) = |V(D)| - c_5(A)$ then $\emptyset$ is the only 5-independent set.

Another polynomial related to strong connectedness can be defined via (T2) using the following: for each vertex-spanning subdigraph A of D let $c_6(A)-1$ be the smallest cardinality of a set R of directed edges such that $R \cup A$ is strongly connected. (The edges in R may or may not be edges of D.) Then c_5 and c_6 are related to the notion of strong connectedness much as c_1 and c_3 are related to the notion of reachability

Proof. We argue by induction on $|R(D,*)|$. If $|R(D,*)| = 1$, then $H = *$ and $G/H = D$, and the result is trivial. Assume $|R(D,*)| > 1$, and let e be the first edge emanating from $*$ in the ordering O. Now e is a 2-isthmus in D if and only if e is a 2-isthmus in H. Further, the induced rooted subdigraph of $R(D/e,*)$ is just H/e. Thus if e is a 2-isthmus, by induction $f_7(D) = xf_7(D/e) = xf_7(H/e)f_7(G/H) = f_7(H)f_7(G/H)$. If e is a loop, then by a similar argument and induction, we have $f_7(D) = yf_7(D-e) = yf_7(H-e)f_7(G/H) = f_7(H)f_7(G/H)$. If e is neither a 2-isthmus nor a loop, then by induction and 4.1 (b) 3., $f_7(D) = f_7(D-e) + f_7(D/e) = f_7(H-e)f_7(G/H) + f_7(H/e)f_7(G/H) = f_7(H)f_7(G/H)$.

The next result describes the general behavior of the variable z. The proof, which we omit, can be established by induction, using the two previous results.

Proposition 4.4. *$f_7(D,O;x,y,z) = z^k g(x,y)$, for some polynomial g and $k \geq 0$.*

In general, the exponent k depends on the chosen ordering, although $k = 0$ for any ordering when D has a $*$ rooted spanning arborescence (4.2). If we let s be the minimum exponent k which occurs, the minimum being taken over all possible edge orderings, then s is just the minimum number of edges in D whose reversal will create a $*$ rooted spanning arborescence.

It is possible to recover $f_7(D,O;x,y,z)$ completely from the evaluation $f_7(D,O;x,y,(x-1)y)$. More precisely, assume $f_7(D,O;x,y,z) = z^k g(x,y)$ and set $z = (x-1)y$ in $f_7(D,O;x,y,z)$. Then the highest power of $x-1$ which divides this evaluation is the correct power of z for $f_7(D,O;x,y,z)$. To see this, simply note that contracting the k edges which contributed z to the original polynomial leaves a digraph which has a $*$ rooted arborescence. Furthermore, $g(x,y) = f_7(D/A,O/A;x,y,z)$, where A is this set of k edges and O/A is the inherited ordering of $E(D) - A$. Hence by 4.5 (a) below, $x-1$ does not divide $g(x,y)$, so we can uniquely determine k from $f_7(D,O;x,y,(x-1)y)$. In spite of this evaluation, we will continue to write f_7 as a three-variable polynomial for simplicity.

The following properties of $f_7(D,O;x,y,z)$ are all independent of the chosen ordering and are straightforward to verify using induction and 4.1.

Proposition 4.5. *Let O be any ordering of the edges of D.*

(a) $f_7(D,O;1,1,0) =$ the number of spanning arborescences rooted at $*$.

(b) $f_7(D,O;2,2,2) = 2^e$, where e is the number of edges of D.

(c) The highest power of x appearing in $f_7(D,O;x,y,(x-1)y)$ is $v-1$, where v is the number of vertices of D.

(d) If D has a spanning arborescence rooted at $*$, then the lowest power of x appearing in $f_7(D,O;x,y)$ equals the number of 2-isthmuses in D.

(e) The highest power of y appearing in $f_7(D,O;x,y,(x-1)y)$ is $e-v+1$.

Example 4.6. To show how $f_7(D)$ depends on the ordering O, consider the rooted ordered digraphs D_1 and D_2 of Figure 5. (The edge-ordering is given by the numbering of the edges.) The reader can check the following computations.

$$f_7(D_1) = x^2y + 2xy^2 + y^3 \qquad\qquad f_7(D_2) = x^2y^2 + xy + y^2 + y^3$$

We remark that $f_7(D,O;1,y,1)$ is independent of the chosen ordering O. In fact, when D has a $*$ rooted spanning arborescence, $f_7(D,O;1,y) = \lambda(D;y)$, where $\lambda(D;y)$ is the greedoid polynomial of [1]. This can be shown by demonstrating that $f_7(D,O;1,y)$ and $\lambda(D;y)$ both follow the same recursion, as was done with f_1 in Section 2.

The polynomial f_8 is related to f_3 in the same way that f_7 is related to f_2, i.e., D is assumed to be given with an ordering that is used to define a complete recursion for a polynomial resembling f_3. In this case, though, D is to be given with an ordering O on $V(D)$, not $E(D)$; we presume that in this ordering $V(D) = \{v_0, ..., v_{v-1}\}$ with $v_0 = *$. Given such an ordering, we define $c_8(D,O)$ as follows. Define a sequence $*_1, ..., *_c$ of vertices of D by: $*_1$ is $*$, and for $i > 1$, $*_i$ is the least vertex not reachable in D from any of $*_1, ..., *_{i-1}$, if there is any such; then $c_8(D,O) = c$. This definition gives rise to a partition of $V(D)$ into cells we will call the *8-components* of (D,O): $C_1 = R(D,*)$ consists of those vertices reachable from $*$, and for $i > 1$ C_i consists of those vertices reachable from $*_i$ that are not reachable from any of $*_1, ..., *_{i-1}$.

The associated rank function is given by $r_8(A,O) = v - c_8(A,O)$ for each $A \subseteq E(D)$; $r_8(A,O)$ is the maximum size of a subset of A that is a $*$ rooted forest of arborescences in which each arborescence is rooted at its least vertex. Clearly $r_1(A) \leq r_8(A) \leq r_3(A)$ for every $A \subseteq E(D)$. It is worth mentioning that among the seven rank functions we have defined r_1, r_5, and r_8 have the property of being associated with well-defined partitions of $V(D)$; and r_3, r_4, r_6, and r_8 have the property that for each A, $c_i(A) - 1$ is the minimum cardinality of a set R of edges with $c_i(A \cup R) = 1$. The only one that shares both these properties with the circuit rank (and the notion of connectedness) in undirected graphs is r_8.

The polynomial f_8 resulting from the use of r_8 in (T1) is completely characterized by the following.

Proposition 4.7. *Let D be a rooted digraph with vertices $* = v_0, v_1, ..., v_{v-1}$.*

(a) If D has no edges then $f_8(D,O) = 1$.

(b) If the initial vertex of e is $*$ then $f_8(D,O) = t^{r_8(D,O)-r_8(D-e,O)} f_8(D-e,O) + z^{|V(D/e)|+1-|V(D)|} f_8(D/e,O/e)$.

(c) If the terminal vertex of e is $*$, or e is a loop, then $f_8(D,O) = (z+1) f_8(D-e,O)$.

(d) If no edge of D is incident on $*$ and D' is the digraph obtained from D by identifying v_0 with v_1, then $f_8(D,O) = f_8(D',O')$.

Here O/e is the order on $V(D/e)$ obtained from O by ignoring the terminal vertex of e, and O' is the obvious order on $V(D')$.

It is a simple matter to formulate analogues of Propositions 2.3 and 2.7 that apply to f_8.

For most digraphs $f_8(D,O)$ is very sensitive to changes in O. An exception to this general rule involves the following notion. Suppose G is an undirected graph, and let $D = D(G)$ be the directed graph in which each edge of G is replaced by a pair of oppositely directed edges. (Each loop of G is to be replaced by two loops in D.) Let O be any ordering of $V(G)$, with any vertex playing the role of $*$. Then Proposition 4.7 and the recursive description (T3) of the Tutte polynomial imply

Proposition 4.8. *In this situation*

$$t^{c_8(D,O)} f_8(D,O;t,z) = t^{c(G)}(z+1)^{|E(G)|} t(G;t,z).$$

Let $h_3(D;z)$ and $h_8(D,O;z)$ be the respective polynomials obtained by evaluating at $t=0$. The next proposition is easily deduced from Propositions 4.6 and 2.6.

Proposition 4.9. *Let D be a rooted digraph, and let O be any ordering of $V(D)$ with $* = v_0$. Then $h_3(D;z) = h_8(D,O;z)$.*

If D has a $*$ rooted spanning arborescence then the common evaluation discussed in the proposition is the one-variable greedoid polynomial $\lambda_D(y)$ of [1], as was discussed in Section 2. Therefore f_1, f_2, f_3, f_7, and f_8 may all be regarded as strengthened versions of λ, with different properties: f_1 satisfies a complete recursion and is largely insensitive to the portion of a directed graph not reachable from the root (it only notices the number of edges in this portion); f_2 satisfies no recursion at all in general and is also largely insensitive to the unreachable portion of a digraph; f_3 satisfies only a partial recursion and is much more sensitive to the unreachable portion of a digraph; and f_7 and f_8 satisfy complete recursions and are sensitive to the unreachable portion of a digraph, but require the imposition of edge- and vertex-orders to define their recursions. (In Section 5 we will mention a realistic context in which the imposition of such orders may not seem completely unnatural.)

5. Weighted and doubly weighted digraphs.

A *weighted digraph* is a directed graph D together with a function $w : E(D) \to R$, where R is some commutative ring with unity. Any one of the polynomials $f_1, \ldots, f_6, f_8$ can be modified to reflect the edge-weights in D by defining

$$f_i(D) \equiv \sum_{A \subseteq E(D)} \left(\prod_{e \in A} w(e) \right) t^{r_i(D) - r_i(A)} z^{|A| - r_i(A)}.$$

For convenience' sake we will often suppress the O when discussing f_8 in this section and the next. A common feature of several of these polynomials is their relationship with the Kirchhoff matrix $K(D)$ associated to D. (See [9, Chap. VI] for a discussion of this matrix).

Proposition 5.1. *Let D be a weighted, rooted digraph that has a $*$ rooted spanning arborescence, and let $K_r(D)$ be obtained from $K(D)$ by deleting the row and column corresponding to $*$. Then $f_1(D)$, $f_2(D)$, $f_3(D)$, and $f_8(D)$ all yield the determinant of $K_r(D)$ when evaluated at $t = z = 0$.*

Proof. Observe that the 1-, 2-, 3-, and 8-bases of D are precisely the $*$ rooted spanning arborescences in D. The proposition follows from this observation and the matrix-tree theorem [9, Theorem VI.27].

A *doubly weighted digraph* is a weighted digraph which has a *complementary weight function* $c : E(D) \to R$ in addition to its weight function. Once again, it is a simple matter to extend the polynomials given by (T1) to this context; we define

$$f_i(D) \equiv \sum_{A \subseteq E(D)} \left(\prod_{e \in A} w(e) \right) \left(\prod_{e \notin A} c(e) \right) t^{r_i(D) - r_i(A)} z^{|A| - r_i(A)}.$$

One can consider an ordinary digraph as a doubly weighted digraph in which both weights are identically 1, and a weighted digraph as a doubly weighted digraph with c identically 1. Many of the properties of the f_i generalize to this context, with simple modifications. For instance, if $i \neq 7$ then the following holds.

Proposition 5.2. *(a) If e is an i-loop, $f_i(D) = (zw(e) + c(e)) f_i(D - e)$.*
(b) If e is an i-isthmus, $f_i(D) = (w(e) + tc(e)) f_i(D/e)$.
(c) If e has $w(e) = 0$ and $c(e) = 1$ then $f_i(D) = t^{r_i(D) - r_i(D-e)} f_i(D - e)$.
(d) Let e be any edge with initial vertex $*$, a loop or not. Then for $i = 1$, 3, or 8

$$f_i(D) = w(e) z^{|V(D/e)| + 1 - |V(D)|} f_i(D/e) + c(e) t^{r_i(D) - r_i(D-e)} f_i(D - e).$$

Moreover, if e is an i-isthmus then $f_i(D - e) = f_i(D/e)$.

Proposition 5.2 is of special significance for the polynomial f_8. If D_0 is a directed graph with no edge incident on $*$, consider the digraph D obtained from D_0 by inserting an edge e directed from $v_0 = *$ to v_1, with $w(e) = 0$ and $c(e) = 1$. Then $D_0 = D - e$ and $r_8(D) = 1 + r_8(D_0)$, so by 5.2 (c) $t f_8(D_0) = f_8(D)$. Moreover, e is an 8-isthmus in D, so 5.2 (b) gives $f_8(D) = t f_8(D/e)$; consequently $f_8(D_0) = f_8(D/e)$. Thus Proposition 5.2 implies 4.7 (d).

An important example of a doubly weighted digraph is a *probabilistic digraph*, i.e., a digraph given with a function $p = w : E(D) \to [0, 1]$. If we regard D as representing some directed network (e.g., a municipal water supply system), then for $e \in E(D)$, $p(e)$ can be taken to be the probability that the element of the network represented by e (e.g., a water main) will operate successfully for a certain period of time; the complementary weight $c(e) = 1 - p(e)$ measures the probability that this element will fail sometime during this period.

The general notion of the *reliability* of a probabilistic digraph involves the assessment of the likelihoods of various possible attributes of the portion of D that has not failed after the passage of a single time period. A commonly studied special case (see

[5], for instance) is the assessment of the probability that if all the vertices of D are reachable from $*$, then the portion of D that survives after a single time period still has this property. The polynomials $f_1, ..., f_6, f_8$ contain a great deal of general reliability information, for the coefficient of $t^a z^b$ in $f_i(D; t, z)$ is the probability that after the passage of a single period of time, the remaining network will have $c_i = c_i(D) + a$ and will have $b - a - c_i(D) + |V(D)|$ operative edges.

For instance, suppose every vertex of D is reachable from $*$ and it is desired that after the passage of a single period of time this property be restored to what remains of D by adjoining edges. (We presume for the moment that it is just as easy to adjoin new edges as it is to restore edges of D that have failed.) Then the coefficient of t^a in $f_1(D; t, 1)$ measures the likelihood that precisely $a - 1$ vertices of D need to have their reachability from $*$ restored, while the coefficient of t^a in $f_3(D; t, 1)$ gives the probability that $a - 1$ is the minimum number of edges that must be adjoined to accomplish this restoration. Suppose a repair crew tours $V(D)$ according to a vertex-order O, and whenever it encounters a vertex not reachable from $*$ the crew installs an edge to that vertex from some vertex that is reachable from $*$. Then the coefficient of t^a in $f_8(D, O; t, 1)$ is the probability that this crew will install precisely $a - 1$ edges.

In real-world situations it is often not practical to simply ignore failed elements of a network; broken water mains must be repaired or sealed off, for instance. Information about the number of edges likely to fail in a given period of time is contained in any one of the polynomials f_i with $i \neq 7$, as was mentioned above. It may be, however, that certain edges represent elements of a network that are more difficult or expensive to repair than others, and the polynomials can easily be modified to record this kind of information. Suppose there are functions r and s mapping $E(D)$ into the field $\mathbf{R}$ of real numbers that give the repair cost and value of the individual edges of D, respectively. (The value could be used, for instance, in calculating the network's worth as collateral for a loan.) Let x and y be indeterminates, and define two weight functions $w, c : E(D) \to \mathbf{R}[x, y]$ by $w(e) = p(e)y^{s(e)}$ and $c(e) = (1 - p(e))x^{r(e)}$. The resulting polynomials $f_i(D; t, z, x, y)$ will have the property that the coefficient of $t^a z^b x^c y^d$ is the likelihood that after the passage of a single period of time the remnant of D will have $c_i = c_i(D) + a$, this remnant will have $b - a - c_i(D) + |V(D)|$ operative edges, it will cost c to repair all the edges of D that will have failed, and the edges of the remnant will have a total value d. (We are assuming that the total repair cost and total value are simply the sums of the repair costs of the failed edges and the values of the operative edges, respectively.)

Suppose D is a probabilistic digraph and $K \subseteq V(D)$ is a set of k distinguished vertices. For $i = 1$, 5, or 8 the value of $c_i(D)$ is the number of cells in a particular partition of $V(D)$, and so we can define the number of *K-terminal i-components* of D, $c_i(D, K)$, to be the number of these cells that meet K. We could also define $c_3(A, K) - 1$ to be the minimum cardinality of a set R of edges such that $c_1(A \cup R, K) = 1$, and $c_6(A, K) - 1$ to be the minimum cardinality of a set R with $c_5(A \cup R, K) = 1$. The associated K-terminal reliability problems involve the assessment of the probabilities of the various possible numbers of K-terminal i-components D might have after the passage of a period of time, and the various possible minimum numbers of edges that will have to be installed to bring the number of K-terminal i-components down to

1. Any one of f_1, f_3, f_5, f_6 and f_8 can be easily modified so as to be of value in this K-terminal context; we need only replace $r_i(A)$ by $r_i(A, K) = k - c_i(A, K)$ and $r_i(D)$ by $k - c_i(D, K)$ in the definition of the polynomial of a doubly weighted digraph.

Proposition 5.3. *Let e be any edge with initial vertex $*$, a loop or not. Then for $i = 1$ or 8,*

$$f_i(D, K) = (1 - p(e))t^{r_i(D,K) - r_i(D-e,K)} f_i(D - e, K) + p(e)z^{|K/e|+1-k} f_i(D/e, K/e)$$

where K/e is the obvious subset of $V(D/e)$.

References

[1] A. Björner and G. Ziegler, *Introduction to greedoids*, in Matroid Applications (N. White, ed.), Encyclopedia of Mathematics and Its Application v. 40, Cambridge Univ. Press, 1992, 284-357.

[2] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, in Matroid Applications (N. White, ed.), Encyclopedia of Mathematics and Its Application v. 40, Cambridge Univ. Press, 1992, 123-225.

[3] S. Chaudhary and G. Gordon, *Tutte polynomials for trees*, J. Graph Theory **15** (1991), 317-331.

[4] F. Chung and R. Graham, *The cover polynomial of a digraph*, preprint, Bell Communications Research and AT&T Bell Laboratories, New Jersey, 1992.

[5] C. J. Colbourn, *The Combinatorics of Network Reliability*, Oxford Univ. Press, Oxford, 1987.

[6] G. Gordon and E. McMahon, *A greedoid polynomial which distinguishes rooted arborescences*, Proc. Amer. Math. Soc. **107** (1989), 287-298.

[7] G. Gordon and L. Traldi, *Generalized activities and the Tutte polynomial*, Discrete Math. **85** (1990), 167-176.

[8] E. McMahon, *On the greedoid polynomial for rooted graphs and rooted digraphs*, to appear in J. Graph Theory.

[9] W. T. Tutte, *Graph Theory*, Cambridge Univ. Press, Cambridge, 1984.

[10] N. White, ed., *Theory of Matroids*, Cambridge Univ. Press, Cambridge, 1986.

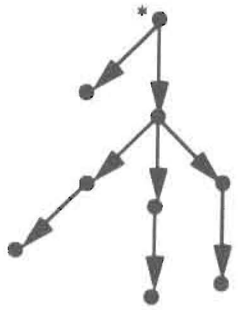

Figure 1a

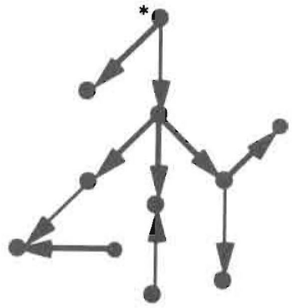

Figure 1b

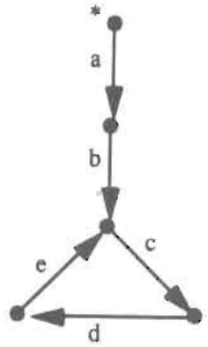

Figure 2a

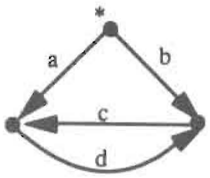

Figure 2b

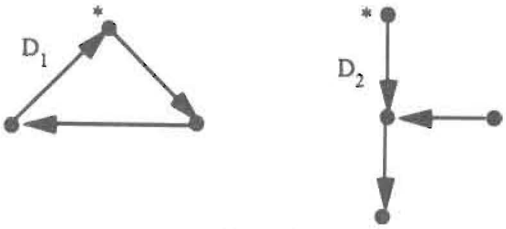

Figure 3

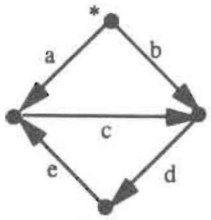

Figure 4

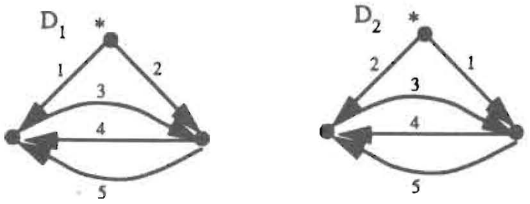

Figure 5

Addendum to "Polynomials for Directed Graphs"

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We regret that an incomplete manuscript of the article *Polynomials for directed graphs* appeared in the last volume of this journal [Congressus Numerantium **94** (1993), pp. 187-201]. The following page should have appeared between those numbered 193 and 194 in the published paper.

CONGRESSUS NUMERANTIUM **100**(1994), pp.5-6

from a root. (See Proposition 2.5.)

Chung and Graham [4] have introduced an interesting polynomial invariant of unrooted digraphs, the *cover polynomial*. Unlike the polynomials already mentioned in this section, the cover polynomial satisfies a deletion/contraction property somewhat similar to (T3), though it does have the property that the appropriate notion of contraction is not symmetric: the initial and terminal vertices of the edge being contracted are treated differently.

4. Two order-dependent polynomials.

In the discussion subsequent to Proposition 2.6, it is noted that complete recursive descriptions of f_2 and f_3 are not possible. Nevertheless, the notions of 2-isthmus and 2-loop are similar to the notions of isthmus and loop in a graph or matroid. Recall that an edge e in a rooted digraph D is a 2-isthmus iff e is in every maximal $*$ rooted arborescence (i.e., e is in every 2-basis), and is a 2-loop iff it is in no maximal $*$ rooted arborescence. We will define a polynomial on rooted digraphs recursively, using these notions of isthmus and loop. We will also distinguish between 2-loops which are *ordinary loops* (i.e., those in which the initial and terminal vertices coincide), and 2-loops which are not, which we will call *reversed loops*. (The term "loop" means "ordinary loop".)

We now define a polynomial $f_7(D) = f_7(D, O; x, y, z)$ associated to a rooted digraph D whose underlying undirected graph is connected, with respect to an ordering O on $E(D)$. This definition is based on the recursive definition (T3).

Definition 4.1.
(a) If $D = \{*\}$, then $f_7(D) = 1$.
(b) Let e be the first edge (in the ordering O) which emanates from $*$.
  1. $f_7(D) = x f_7(D/e)$ if e is a 2-isthmus.
  2. $f_7(D) = y f_7(D - e)$ if e is a loop.
  3. $f_7(D) = f_7(D - e) + f_7(D/e)$ otherwise.
(c) If no edge emanates from $*$, then let e be the first edge directed into $*$, i.e., e is a reversed loop. Then $f_7(D) = z f_7(D/e)$.

The next proposition can be proven using the definition and induction.

Proposition 4.2. *Suppose D has a spanning arborescence rooted at $*$. Then for any ordering O, $f_7(D)$ is a polynomial in x and y (i.e., no z term appears).*

If D has no spanning $*$ rooted arborescence, then it is still easy to describe the behavior of the variable z. Let $R(D, *)$ denote the set of vertices in D which are reachable from $*$. Let H be the induced rooted subdigraph on $R(D, *)$ and G/H the rooted subdigraph obtained from D by contracting all of H to the single vertex $*$.

Lemma 4.3. $f_7(D) = f_7(H) f_7(G/H)$.