Polygon matroids for hypergraphs and K-terminal networks

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Abstract

We propose definitions of "polygon matroids" associated to hypergraphs and K-terminal networks, and we discuss some of their properties.

1. Introduction

Let G be a graph with edge-set $E(G)$. For $S \subseteq E(G)$ let $G : S$ be the subgraph of G with $V(G : S) = V(G)$ and $E(G : S) = S$, and let $c(G : S)$ be the number of connected components of $G : S$. Then $c(G : S)$ has three obvious properties: (i) $c(G : S) \leq c(G : \emptyset) = |V(G)|$, (ii) if $e \notin S$ then $c(G : (S \cup \{e\})) \in \{c(G : S), c(G : S) - 1\}$, and (iii) if $c(G : S) = c(G : (S \cup \{e_1\})) = c(G : (S \cup \{e_2\}))$ then $c(G : S) = c(G : (S \cup \{e_1, e_2\}))$. These properties imply that the function $r(S) = |V(G)| - c(G : S)$ defines a *matroid* structure on $E(G)$; r is the *rank function* of the *polygon* or *circuit* matroid $M(G)$. We presume the reader is familiar with the basic ideas and terminology of matroid theory, as set forth in any of [12, 20, 21]; we also use the standard terminology of graph theory, as given for instance in [5].

For ease of reference we explicitly state two obvious properties of the polygon matroids of graphs: (a) if G and G' are graphs then $M(G)$ and $M(G')$ are isomorphic if and only if there is a bijection $i : E(G) \to E(G')$ such that $|V(G)| - c(G : S) = |V(G')| - c(G' : i(S))$ for every $S \subseteq E(G)$, and (b) if G and G' have the same blocks then $M(G) \cong M(G')$ no matter how these blocks may be connected to each other in G and G'.

Suppose now that G is a *K-terminal network*, i.e., a nonempty subset K of $V(G)$ has been specified. K-terminal networks are used to model communication and distribution networks in which there are two distinct classes of service points (e.g., high- and low-priority customers of a power-distribution network), and there is a large body of research literature about them, focused for the most part on their reliability properties; see [7] for instance. For $S \subseteq E(G)$ let $c(G : S, K)$ be the number of connected components of $G : S$ that meet K. In applications $c(G, K)$ may measure the "value" or "effectiveness" of G — for instance if G represents a communications network whose customers are represented by the elements of K, and whose other vertices represent switching loci or some other aspect of the internal structure of the network, then the lower $c(G, K)$ is, the better the service the network provides to its customers.

The function $c(G : S, K)$ has properties analogous to the first two properties of $c(G : S)$ mentioned above, but in general the K-terminal analogue of (iii) is not valid unless every simple path in G between elements of K also has all its internal vertices in K. It may seem therefore that matroids have no place in the theory of K-terminal networks, but this does not turn out to be the case.

If $j \geq 1$ and $*_1, \ldots, *_j$ are not elements of $E(G)$ then we define a function r on $E(G) \cup \{*_1, \ldots, *_j\}$ in the following way: if $S \subseteq E(G)$ then $r(S) = |V(G)| - c(G : S)$ and for $i_1, \ldots, i_a \in \{1, \ldots, j\}$, $r(S \cup \{*_{i_1}, \ldots, *_{i_a}\}) = r(S) + \min\{a, c(G : S, K) - 1\}$.

Theorem 1. *For any $j > 1$ this defines a matroid, which we will denote $M_j(G, K)$.*

Huseby [9, 10] introduced the matroid $M_1(G, K)$, and proved that it provides a connection between important ideas of matroid theory and K-terminal network reliability theory: unless $E(G) = \emptyset$ the (unsigned) K-terminal reliability domination of G coincides with the β-invariant of $M_1(G, K)$. The β-invariant was introduced by Crapo [8] (see also [20, 22]); it is related to the chromatic polynomial and the Tutte polynomial, and it is useful in the study of matroids related through series-parallel transformations [6, 11]. Reliability domination was introduced by Satyanarayana and his co-authors [15, 16, 17, 18] (see also [1, 2, 4, 7]), and is related to the reliability polynomial; it is recognized as one of the most important recent innovations in network reliability theory.

The matroids $M_j(G, K)$ are examples of a larger family of matroids, associated with hypergraphs; we discuss this family in Section 2. These hypergraph matroids in turn are examples of matroids *induced* from the polygon matroids of graphs; see [13] or Chapter 12 of [12] for a general discussion of this technique.

Let G be a K-terminal network and let $c = c(G, K)$. We associate to G the following sequence of matroids on $E(G)$, which we refer to as the *sequence of K-terminal polygon matroids of G.*

$$\begin{aligned}
\widetilde{M}_0(G, K) &= M(G), \\
\widetilde{M}_1(G, K) &= M_c(G, K)/\{*_1, \ldots, *_c\}, \\
\widetilde{M}_2(G, K) &= M_{c+1}(G, K)/\{*_1, \ldots, *_{c+1}\}, \ldots
\end{aligned}$$

We will say that two K-terminal networks G and G' have *isomorphic sequences of polygon matroids* if there is a single bijection $i : E(G) \to E(G')$ which gives an isomorphism between $\widetilde{M}_j(G, K)$ and $\widetilde{M}_j(G', K')$ for every $j \geq 0$.

Theorem 2. *Let G and G' be graphs, and suppose $K \subseteq V(G)$ and $K' \subseteq V(G')$ are nonempty. Then G and G' have isomorphic sequences of polygon matroids if and only if there is a bijection $i : E(G) \to E(G')$ such that $|V(G)| - c(G : S) = |V(G')| - c(G' : i(S))$ and $|K| - c(G : S, K) = |K'| - c(G' : i(S), K')$ for every $S \subseteq E(G)$.*

Theorem 2 implies that the sequence of K-terminal polygon matroids has properties analogous to the properties (a) and (b) of the ordinary polygon matroid mentioned above. The theorem's statement is of course very similar to property (a). In order to discuss the appropriate analogue of property (b) we should specify that by a *K-terminal block* of a K-terminal network we mean a subgraph of G which is minimal among the unions of blocks of G that have the property that the "boundary" vertices of the subgraph — those adjacent to vertices outside the subgraph — belong to K. With this definition in mind, we can say that if G and G' have the same K-terminal blocks then the polygon matroid sequences of G and G' are isomorphic no matter how these blocks may be connected to each other in G and G'. (Though by definition they can only be connected to each other at terminal vertices.)

The reader may well doubt that an infinite sequence of K-terminal polygon matroids will be a useful analogue of the single conventional polygon matroid. It will perhaps be of some solace to find out that the sequence of K-terminal polygon matroids is eventually stationary.

Theorem 3. *If $j_1, j_2 \geq |K| - c(G,K)$ then $\widetilde{M}_{j_1}(G,K) = \widetilde{M}_{j_2}(G,K)$.*

The converse of Theorem 3 is also true: if $\widetilde{M}_{j_1}(G,K) = \widetilde{M}_{j_2}(G,K)$ then it must be that $j_1, j_2 \geq |K| - c(G,K)$.

An ordinary graph G may be thought of as an *all-terminal* network, i.e., a K-terminal network with $K = V(G)$. For such a network it turns out that the sequence of K-terminal polygon matroids is completely determined by the first one.

Theorem 4. *If $K = V(G)$ then for $j > 0$, $\widetilde{M}_j(G,K)$ is simply the truncation of $\widetilde{M}_{j-1}(G,K)$. Consequently, if $j \geq |V(G)| - c(G)$ then $\widetilde{M}_j(G,V(G))$ is a trivial matroid, i.e., its rank function is identically zero.*

Indeed, it turns out that if G is a K-terminal network and any $\widetilde{M}_j(G,K)$ is a trivial matroid then G is "essentially all-terminal" in an appropriate sense.

Theorem 5. *Let G be a K-terminal network. The following are equivalent: (a) some $\widetilde{M}_j(G,K)$ is a trivial matroid, (b) every $\widetilde{M}_j(G,K)$ with $j \geq |K| - c(G,K)$ is a trivial matroid, and (c) no non-loop edge of G is incident on a vertex not in K.*

As we mentioned above, Huseby [9, 10] proved that $\beta(M_1(G,K))$ is the K-terminal reliability domination of G. Satyanarayana and Tindell [19] generalized the K-terminal reliability domination to a family of invariants $D(G,K,j)$, $0 < j < |K|$; they called them the (unsigned) *(K,j)-dominations* of G. $D(G,K,1)$ is the K-terminal reliability domination of G, and $D(G,K,j) = 0$ if $j < c(G,K)$. It turns out that these invariants are connected with the sequence of K-terminal polygon matroids.

Theorem 6. *If $0 \leq j < |K| - c(G,K)$ then*

$$D(G,K,c(G,K)+j) = \beta(\widetilde{M}_j(G,K)) + \beta(\widetilde{M}_{j+1}(G,K)).$$

Theorem 6 can be restated so that only one matroid is involved, instead of two. The matroid $M = M_{j+c(G,K)}(G,K)/\{*_2, \dots, *_{j+c(G,K)}\}$ has $M - *_1 = \widetilde{M}_j(G,K)$ and $M/*_1 = \widetilde{M}_{j+1}(G,K)$; it follows that $\beta(M) = \beta(\widetilde{M}_j(G,K)) + \beta(\widetilde{M}_{j+1}(G,K)) = D(G,K,c(G,K)+j)$. The matroid M was introduced in [14], where this restated version of Theorem 6 was also proven.

Before proceeding to discuss some details of these ideas we would like to thank our colleague Gary Gordon for many enlightening conversations about matroid theory.

2. Polygon matroids for hypergraphs

We begin with a generalization of the definition of $M_j(G,K)$ given above. Let V and H be sets, and let $f : H \to 2^V$ be a function which associates a subset of V to each element of H. We call such an H a *hypergraph*, even though this definition is not exactly the usual one [3]; the reader will have no trouble comparing the two definitions. Let $K(V)$ be the complete graph with $V = V(K(V))$, and let r be the rank function of the polygon matroid of $K(V)$. Suppose $S \subseteq H$, and let $S_2 = \{h \in S : |f(h)| \geq 2\}$. We call a set E of edges of $K(V)$ a *realization* of S if there is a function $\rho : E \to S_2$ such that $e \subseteq f(\rho(e))$ for every $e \in E$, and we define the *rank* $r(S)$ to be the maximum rank $r(E)$ of a realization of S. It is a simple exercise to verify that this definition of r defines a matroid structure on H. The reader who is familiar with the notion of inducing a new matroid from a given one through some bipartite graph whose vertices are the elements of the ground-sets of the two matroids [12, 13] will recognize that our definition is an example of such a construction: the bipartite graph in question has the disjoint union of H and $E(K(V))$ as its vertex-set, and it has an edge connecting an element $h \in H$ to an edge e of $K(V)$ if $e \subseteq f(h)$.

We hope this definition of rank seems natural. We think of the elements of H as connectors, with h having the capability of connecting any two elements of $f(h)$; of course if $|f(h)| \leq 1$ then h does not really have any connecting capability. A realization of a subset $S \subseteq H$ is a subgraph of $K(V)$ in which each $h \in S_2$ has chosen which two vertices it is going to connect, and $|V| - r(S)$ is the smallest possible number of connected components in a realization of S.

If $j \geq 1$ then the matroid $M_j(G,K)$ mentioned in Theorem 1 of the Introduction is an example of this definition. It has $H = E(G) \cup \{*_1, \dots, *_j\}$, and the function $f : H \to 2^{V(G)}$ is defined so that if $e \in E(G)$ then $f(e)$ contains the vertex or vertices incident on e, while $f(*_i) = K$ for $1 \leq i \leq j$. In two special cases $M_j(G,K)$ can be described particularly easily: if $|K| = 2$ then $M_j(G,K)$ is isomorphic to the polygon matroid of the graph obtained from G by inserting j parallel edges between the elements of K, and if $K = V(G)$ then $M_j(G,K)$ is the free j-point extension of $M(G)$.

3. Theorems 2 — 6

Suppose G is a K-terminal network, and for $j \geq 0$ let r_j be the rank function of $\widetilde{M}_j(G,K)$; also let $c = c(G,K)$. By the definition of matroid contraction, if $j \geq 1$ and $S \subseteq E(G)$ then $r_j(S) = r(S \cup \{*_1, \ldots, *_{c+j-1}\}) - r(\{*_1, \ldots, *_{c+j-1}\})$, where r is the rank function of $M_{c+j-1}(G,K)$. Consequently, $r_j(S)$ is given by the following.

$$r_j(S) = \begin{cases} r(S) & \text{if } 0 \leq j \leq c(G:S,K) - c \\ r(S) + c(G:S,K) - c - j & \text{if } c(G:S,K) - c \leq j \leq |K| - c \\ r(S) + c(G:S,K) - |K| & \text{if } j \geq |K| - c \end{cases} \tag{3.1}$$

Theorem 3 of the Introduction follows immediately from the fact that $r_j(S)$ is independent of j for $j \geq |K| - c(G,K)$. The converse of Theorem 3 is also clear, for if $\widetilde{M}_{j_1}(G,K) = \widetilde{M}_{j_2}(G,K)$ then $r_{j_1}(E(G)) = r_{j_2}(E(G))$, and this can only happen if $j_1, j_2 \geq |K| - c(G,K)$.

It is also a simple matter to verify Theorem 5. If no non-loop edge of G is incident on a vertex not in K then $c(G:S,K) = c(G:S) - |V(G) - K|$ for every $S \subseteq E(G)$, and hence if $j \geq |K| - c(G,K)$ then $r_j(S) = r(S) + c(G:S,K) - |K| = |V(G)| - c(G:S) + c(G:S,K) - |K| = 0$ for every $S \subseteq E(G)$. This shows that (c) implies (b) in Theorem 5. To show that (a) implies (c), suppose some $\widetilde{M}_j(G,K)$ is a trivial matroid; then it must be that $r_j(E(G)) = 0$. Clearly this is impossible if $c(G:S,K) - c(G,K) < j < |K| - c(G,K)$, and if $j = 0$ then the edges of G must all be loops. If $j \geq |K| - c(G,K)$ then it must be that $0 = r(E(G))) + c(G,K) - |K| = |V(G)| - c(G) + c(G,K) - |K|$, and consequently $|V(G)| - |K| = c(G) - c(G,K)$; therefore no non-terminal vertex of G can be in a connected component that contains any other vertex of G.

Note that if $K = V(G)$ then for $S \subseteq E(G)$, $c(G:S,K) - c(G,K) = r(E(G)) - r(S)$ and $c(G:S,K) - |K| = -r(S)$. Consequently, $r_j(S) = r(S)$ for $j \leq r(E(G)) - r(S)$, $r_j(S) = r(S) - j + r(E(G)) - r(S)$ for $r(E(G)) - r(S) \leq j \leq r(E(G))$, and $r_j(S) = 0$ for $j \geq r(E(G))$. This verifies the assertion of Theorem 4, that when $K = V(G)$ the matroids in the K-terminal polygon matroid sequence are the successive truncations of the polygon matroid of G.

We turn now to the proof of Theorem 2. Suppose G and G' are graphs and $K \subseteq V(G)$ and $K' \subseteq V(G')$ are nonempty. If $i : E(G) \to E(G')$ is a bijection such that $|V(G)| - c(G:S) = |V(G')| - c(G':i(S))$ and $|K| - c(G:S,K) = |K'| - c(G':i(S),K')$ for every $S \subseteq E(G)$, then $r(S) = r'(i(S))$ for every $S \subseteq E(G)$, where r and r' are the rank functions of the polygon matroids of G and G' respectively. Also,

$$\begin{aligned} c(G : S,K) - c(G,K) &= |K| - c(G:E(G),K) - (|K| - c(G:S,K)) \\ &= |K'| - c(G':E(G'),K') - (|K'| - c(G':i(S),K')) \\ &= c(G':i(S),K') - c(G',K') \end{aligned}$$

for every $S \subseteq E(G)$. Since $|K| - c(G,K) = |K'| - c(G',K')$, it follows from this and (3.1) that i defines an isomorphism between the K-terminal polygon matroid sequences of G and G'.

Conversely, suppose $i : E(G) \to E(G')$ is a bijection that defines an isomorphism between the K-terminal polygon matroid sequences of G and G', i.e., $r_j(S) = r'_j(i(S))$ for every $S \subseteq E(G)$ and every $j \geq 0$. Then $r(S) = |V(G)| - c(G : S) = |V(G')| - c(G' : i(S)) = r'(i(S))$ for every $S \subseteq E(G)$. Also, if $S \subseteq E(G)$ then the smallest j with $r(S) \neq r_j(S)$ is equal to the smallest j with $r'(i(S)) \neq r'_j(i(S))$, and (3.1) implies that consequently $c(G : S, K) - c(G, K) = c(G' : i(S), K') - c(G', K')$. Finally, it must be that the smallest j with $\widetilde{M}_j(G,K) = \widetilde{M}_{j+1}(G,K)$ is equal to the smallest j with $\widetilde{M}_j(G',K') = \widetilde{M}_{j+1}(G',K')$, and the converse of Theorem 3 then implies that $|K| - c(G,K) = |K'| - c(G',K')$. Consequently if $S \subseteq E(G)$ then $|K| - c(G : S, K) = |K| - c(G,K) - (c(G : S,K) - c(G,K)) = |K'| - c(G',K') - (c(G' : i(S),K') - c(G',K')) = |K'| - c(G' : i(S), K')$.

To prove Theorem 6, recall that by definition [8] if $j \geq 0$ then

$$\beta(\widetilde{M}_j(G,K)) = (-1)^{r_j(E(G))} \sum_{S \subseteq E(G)} (-1)^{|S|} r_j(S).$$

Note that by (3.1), if $0 \leq j < |K| - c(G,K)$ then $r_j(E(G)) = r_{j+1}(E(G)) + 1$. It follows that

$$\begin{aligned}
\beta(\widetilde{M}_j(G,K)) + \beta(\widetilde{M}_{j+1}(G,K)) &= (-1)^{r_j(E(G))} \sum_{S \subseteq E(G)} (-1)^{|S|} (r_j(S) - r_{j+1}(S)) \\
&= (-1)^{r_j(E(G))} \sum_{\substack{S \subseteq E(G) \\ c(G:S,K) \leq c(G,K)+j}} (-1)^{|S|}.
\end{aligned}$$

This last formula is easily seen to equal $D(G, K, c(G,K) + j)$ [14].

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