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Crapo's β-Invariant and *K*-Terminal Networks

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Abstract

An invariant of K-terminal networks that is analogous to Crapo's β-invariant and the reliability domination of Satyanarayana et al. is introduced. The extent to which it reflects the principal properties of these important invariants is discussed.

1 Introduction

A *K-terminal network* is a (multi-)graph G given with a distinguished subset K of its vertex-set. Such networks are used to model various kinds of real-world networks (e.g., those involved in communications and power distribution) in which the arrangements of internal connections are important only insofar as they do or do not tend to help provide effective service to the networks' customers. They are not often mentioned in introductory texts on graph theory such as [10] (whose basic terminology we follow here), but because of their applicability K-terminal networks have a prominent place in the literature of network reliability; see [2] for a general account. Interesting combinatorial issues are often raised when one tries to generalize familiar properties of ordinary graphs (regarded as K-terminal networks in which all the vertices belong to K) to arbitrary K-terminal networks.

In this paper we discuss generalizing the β-invariant introduced by H. H. Crapo [3] to K-terminal networks. This integer invariant was originally defined for matroids:

$$\beta(M) = (-1)^{r(E)} \cdot \sum_{S \subseteq E} (-1)^{|S|} r(S),$$

where r is the rank function of a matroid M on a set E. Among the important properties of the β-invariant are the following, all but the last of which are due to Crapo [3]. (Also see [9] for information on matroids in general and the β-invariant in particular.)

$(\beta^{(1)}) : \beta(M)$ satisfies the deletion/contraction formula $\beta(M) = \beta(M-e) + \beta(M/e)$ whenever e is not an isthmus or a loop

$(\beta^{(2)}) : \beta(M)$ is unchanged if M is altered by a series or parallel extension

$(\beta^{(3)}) : \beta(M) \geq 0$ for every matroid M

$(\beta^{(4)}) :$ if $\beta(M) = 1$ and $|E| \geq 2$ then for every $e \in E$, either $\beta(M-e) = 0$ or $\beta(M/e) = 0$

$(\beta^{(5)}) : \beta(M) = 0$ iff either M is disconnected (in the matroid sense) or some element of M is a loop

$(\beta^{(6)}) :$ if $|E| \geq 2$ then $\beta(M) = 1$ iff M is the circuit matroid of a series-parallel network [1]

The property $(\beta^{(6)})$ has been extended by Oxley [6], who characterized the matroids with $\beta \leq 4$.

A close relative of Crapo's β-invariant is the reliability domination $d_K(G)$ of a K-terminal network; it was introduced by Satyanarayana and Prabhakar [8] for directed K-terminal networks, and extended to undirected ones by Satyanarayana and Chang [7]. The undirected invariant satisfies the following properties [7].

$(d_K^{(1)}) : d_K(G)$ satisfies the deletion/contraction formula $d_K(G) = d_{K/e}(G/e) - d_K(G-e)$ for any edge e

$(d_K^{(2)}) : d_K(G)$ is unchanged if G is simplified by any parallel reduction, or by a series reduction with respect to K

$(d_K^{(3)}) : (-1)^{|V(G)|-i+|E(G)|+1} \cdot d_K(G) \geq 0$ if G has i isolated vertices $\notin K$

$(d_K^{(4)}) :$ if $d_K(G) = \pm 1$ and $|E(G)| \geq 2$ then for every $e \in E(G)$, either $d_{K/e}(G/e) = 0$ or $d_K(G-e) = 0$

$(d_K^{(5)}) : d_K(G) = 0$ iff G has more than one connected component meeting K, or more than one connected component with at least one edge, or at least one edge that appears in no simple path between elements of K

$(d_K^{(6)}) : d_K(G) = \pm 1$ iff G can be reduced to a K-tree by parallel reductions and series reductions with respect to K

Here K/e contains all the elements of K not incident on e, and also contains the vertex arising from the end-vertices of e if either of them is in K. Also, a *series reduction with respect to* K is one in which a vertex $v \notin K$ of degree 2 is removed.

The resemblance between these properties of d_K and the aforementioned properties of Crapo's β-invariant has been explained and strengthened by A. B. Huseby [4, 5], who proved that d_K is given by a formula reminiscent of the definition of β:

$$d_K(G) = \sum(-1)^{|E(G)-S|},$$

the sum indexed by the collection of subsets $S \subseteq E(G)$ which suffice to connect the elements of K. More importantly, Huseby showed that there is a way of associating a matroid M to a K-terminal network so that $\beta(M)$ is the same as $d_K(G)$, up to sign. Huseby's description of the situation is rather general and less explicit than it might be. In fact, such an M is easy to describe explicitly; M is the matroid on $E = E(G) \cup \{K\}$ whose rank function is the following: if $S \subseteq E(G)$ then $r(S)$ is the same as the rank of S in the circuit matroid of G, $r(S \cup \{K\}) = r(S)$ if $c(G : S, K) = 1$, and $r(S \cup \{K\}) = r(S) + 1$ if $c(G : S, K) > 1$. (Here for $S \subseteq E(G)$, $G : S$ is the graph with $V(G : S) = V(G)$ and $E(G : S) = S$, and $c(G : S, K)$ is the number of connected components of $G : S$ that meet K; in particular, if $K = V(G)$ then $c(G : S, K)$ is simply the total number of connected components of $G : S$.) Equivalently, the circuits of M are the usual circuits of G together with the sets $E(T) \cup \{K\}$ obtained by adjoining K to the edge-sets of K-trees of G. This construction clarifies the relationship between $\beta(M)$ and $d_K(G)$ by facilitating the translation between properties $(\beta^{(i)})$ and $(d_K^{(i)})$. For instance, in comparing $(\beta^{(5)})$ and $(d_K^{(5)})$ it is useful to observe that if G has more than one connected component meeting K then K is an isthmus of M, and if G has a connected component which has at least one edge and does not meet K then the edge-set of this connected component constitutes a component (in the matroid sense) of M; in either case M is a disconnected matroid.

2 A β-invariant for K-terminal networks

We consider the generalization of Crapo's β-invariant to K-terminal networks given by the formula

$$\hat{\beta}(G,K) = \sum_{S \subseteq E(G)} (-1)^{|S|} \cdot (|K| - c(G:S,K)).$$

Note that if $K = V(G)$ then $\hat{\beta}(G,K)$ differs from the β-invariant of the polygon matroid of G only by sign. Also, observe that $\hat{\beta}(G,K)$ is unchanged by the insertion or removal of isolated vertices, whether or not they lie in K. Finally, note that when $E(G) \neq \emptyset$

$$\hat{\beta}(G,K) = \sum_{S \subseteq E(G)} (-1)^{|S|+1} \cdot c(G:S,K),$$

because the appearances of $(-1)^{|S|} \cdot |K|$ in the original definition of $\hat{\beta}$ add up to 0. We will refer to this equation as the *second definition of* $\hat{\beta}$.

In the balance of the paper we discuss the extent to which the $\hat{\beta}$-invariant satisfies properties $(\hat{\beta}^{(i)})$ similar to $(\beta^{(i)})$ and $(d_K^{(i)})$.

Theorem 1. *Let G be a K-terminal network with more than one edge, and let e be an edge of G. Then* $\hat{\beta}(G,K)=\hat{\beta}(G-e,K)-\hat{\beta}(G/e,K/e)$.

Proof. Partition the sum of the second definition of $\hat{\beta}(G,K)$ into two sums, one consisting of the summands corresponding to sets S with $e \in S$, and the other consisting of the summands corresponding to sets S with $e \notin S$. The first sum equals $-\hat{\beta}(G/e, K/e)$, and the second equals $\hat{\beta}(G-e,K)$. Q. E. D.

Observe that unlike $(\beta^{(1)})$, $(\hat{\beta}^{(1)})$ is true for all edges, including loops and isthmuses. (Even the restriction that e not be the only edge of G could be done away with, if one were willing to use the second definition to define $\hat{\beta}$ for edgeless graphs.) This is not a very surprising observation, though, because the original β -invariant satisfies the equation $\beta(M) = \beta(M-e) - \beta(M/e)$ when e is a loop or isthmus and $|E| > 1$, for a trivial reason: $\beta(M) = 0$ and $M - e \cong M/e$.

The following lemma will come in handy in discussing $(\hat{\beta}^{(2)})$.

Lemma 1. *If G has a K-loop, i.e., an edge which appears in no simple path between elements of K, then $\hat{\beta}(G,K) = 0$. If G has a K-isthmus, i.e., an isthmus whose end-vertices both lie in K, then $\hat{\beta}(G,K) = 0$ unless this K-isthmus is the only edge of G, in which case $\hat{\beta}(G,K) = -1$.*

Proof. If e appears in no simple path between elements of K then clearly $c(G : S, K)$ is never affected if e is adjoined to, or removed from, S. Either definition of $\hat{\beta}$ makes it clear that this implies that $\hat{\beta}(G,K) = 0$.

If e is a K-isthmus then whenever $e \notin S \subseteq E(G)$, $c(G : S, K) = c(G : (S \cup \{e\}), K) + 1$; the second definition of $\hat{\beta}$ implies that consequently

$$\hat{\beta}(G,K) = \sum_{e \notin S \subseteq E(G)} (-1)^{|S|+1}.$$

If e isn't the only edge of G then this sum is 0; if e is the only edge of G then this sum is -1. Q. E. D.

Theorem 2. *(a) If G has a family of parallel edges, and G' is obtained from G by replacing this family with a single edge, then $\hat{\beta}(G,K) = \hat{\beta}(G',K)$.*

(b) If G has a vertex $v \notin K$ of degree 2, and e is either edge incident on v, then $\hat{\beta}(G,K) = -\hat{\beta}(G/e,K)$.

(c) Suppose G has a vertex $v \in K$ of degree 2, and one of the vertices adjacent to v is $w \in K$. Let e be the other edge incident on v. Then $\hat{\beta}(G,K) = -\hat{\beta}(G/e,K/e)$ unless $|E(G)| = 2$, in which case $\hat{\beta}(G,K)$ is $-1-\hat{\beta}(G/e,K/e)$.

Proof. In each case one of G/e, $G-e$ has a K-loop or a K-isthmus, and the assertion follows from Theorem 1 and Lemma 1. Q. E. D.

It does not seem that $\hat{\beta}$ satisfies any property $(\hat{\beta}^{(3)})$ or $(\hat{\beta}^{(4)})$. This observation is justified by considering the first K-terminal network G drawn in Figure 1, which has $\hat{\beta}(G,K) = -1$, $\hat{\beta}(G-e,K) = 2$, and $\hat{\beta}(G/e,K/e) = 3$. The second network drawn in Figure 1 provides an example in which $\hat{\beta}(G,K) = -1$, $\hat{\beta}(G-e,K) = -2$, $\hat{\beta}(G/e,K/e) = -1$, and G cannot be subjected to any of the series or parallel reductions of Theorem 2. (In the figures we follow the convention that elements of K are represented by filled-in circles, while other vertices are represented by open circles.)

By the way, we know of no example that shows that $\hat{\beta}$ does not satisfy the weaker property: if $\hat{\beta}(G,K) = \pm 1$ and $|E(G)| \geq 2$ then there is at least one edge e with $\hat{\beta}(G/e, K/e) = 0$ or $\hat{\beta}(G-e,K) = 0$.

It seems that $\hat{\beta}$ satisfies only the following rather weak property ($\hat{\beta}^{(5)}$).

Theorem 3. *If G has a K-loop, i.e., an edge contained in no simple path between elements of K, then $\hat{\beta}(G,K) = 0$. If G has more than one connected component with at least one edge then $\hat{\beta}(G,K) = 0$. Also, if there is a $v \in K$ such that G-v has more connected components than G then $\hat{\beta}(G,K) = 0$.*

Proof. The first assertion appeared in Lemma 1.

Suppose G has more than one connected component that isn't edgeless, and let C be any such component. If C has only one edge, e, then either this edge is a K-loop or else it is a K-isthmus. Since e isn't the only edge of G (for G has at least one other connected component that isn't edgeless) Lemma 1 guarantees that $\hat{\beta}(G,K) = 0$. If C has more than one edge, pick any one edge e of C, and observe that induction implies that $\hat{\beta}(G/e,K/e) = 0 = \hat{\beta}(G-e,K)$.

Suppose now that $v \in K$ and $G - v$ has more connected components than G; note that necessarily the degree of v is greater than 1. If one of the "new" connected components contains no element of K, then an edge between this component and v cannot appear in any simple path between elements of K; consequently $\hat{\beta}(G,K) = 0$ by Lemma 1.

Suppose now that every "new" component contains at least one vertex from K. Consider such a component, C. If C consists entirely of a single vertex from K, then either there is a single edge connecting v to this single vertex or else there is a family of parallel edges; by Theorem 2 we might as well assume that there is only a single edge. This edge is a K-isthmus, so Lemma 1 implies that $\hat{\beta}(G,K) = 0$. If C consists of more than merely a single vertex, consider the process of calculating $\hat{\beta}(G,K)$ by applying Theorem 1 to the internal edges of C. If Theorem 1 is used to eliminate all of these edges, then $\hat{\beta}(G,K)$ will be expressed as a sum of $\hat{\beta}$-invariants of graphs each of which has some vertex which is adjacent only to v. Consequently, $\hat{\beta}(G,K)$ will be expressed as a sum of terms each of which is 0. Q. E. D.

Corollary 1. *If G is a tree then $\hat{\beta}(G,K)=0$ unless K is precisely the set of vertices of degree 1 in G, in which case $\hat{\beta}(G,K)$ is $(-1)^{|K|+|E(G)|}$.*

Proof. If any vertex of degree 1 is not in K, then the edge incident on that vertex is a K-loop. If any vertex $v \in K$ is of degree > 1, then $G-v$ has more connected components than G does. In either case $\hat{\beta}(G,K)=0$ by Theorem 2.

Suppose now that K is precisely the set of vertices of degree 1 in G, and consider any $v \in K$; let $e=\{v,w\}$ be the one edge of G incident on v. If e is the only edge of G then $\hat{\beta}(G,K)=-1=(-1)^{2+1}$. If w is of degree 2, then $G-e$ has a vertex of degree 1 not in K, so $\hat{\beta}(G-e,K)=0$. By induction, then, $\hat{\beta}(G,K)=-\hat{\beta}(G/e,K/e)=-(-1)^{|K/e|+|E(G)|-1}$; since $|K/e|=|K|$, this agrees with the statement. If w is of degree > 2, then G/e has a vertex of degree > 1 that's in K, so $\hat{\beta}(G/e,K/e)=0$; hence $\hat{\beta}(G,K)=\hat{\beta}(G-e,K)$. In this situation $G-e$ is not itself a tree, but consists of a tree together with an isolated vertex from K; by induction, then, $\hat{\beta}(G-e,K)=(-1)^{|K|-1+|E(G)|-1}$, as required. Q. E. D.

There are many connected K-terminal networks with $\hat{\beta}(G,K)=0$ that have no K-loops or cutpoints in K. For instance, suppose G has a cutpoint $v \notin K$, at least one connected component of $G-v$ has internal edges, and all the vertices in that component that are adjacent to v lie in K. Then contracting any edge e connecting v to that component will produce a network with a cutpoint in K/e, so $\hat{\beta}(G/e,K/e)=0$ and $\hat{\beta}(G,K)=\hat{\beta}(G-e,K)$. Doing this repeatedly will disconnect that component from G without changing $\hat{\beta}$, but the $\hat{\beta}$ of the resulting K-terminal network must be 0 because it will have at least two connected components with edges; consequently $\hat{\beta}(G,K)=0$ too. (Note that by Corollary 1, not every K-terminal network with a cutpoint not in K has $\hat{\beta}(G,K)=0$.)

We do not know whether or not there is any interesting characterization of the K-terminal networks with $\hat{\beta}(G,K)=0$. In Figure 2 the reader will find all the connected K-terminal networks on six or fewer vertices that have no cutpoints at all, cannot be subjected to any of the series or parallel reductions of Theorem 2, and have $\hat{\beta}=0$.

It also seems difficult to characterize the K-terminal networks with $\hat{\beta}=\pm 1$. The proofs of $(\beta^{(6)})$ and $(d_K^{(6)})$ use $(\beta^{(5)})$ and $(d_K^{(5)})$ together with $(\beta^{(4)})$

and $(d_K^{(4)})$, so such arguments are doubly doubtful in connection with $\hat{\beta}$. In Figures 1 and 3 the reader will find four networks with $\hat{\beta}(G, K) = \pm 1$.

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References

[1] T. Brylawski, A combinatorial model for series-parallel networks, Trans. Amer. Math. Soc. 154 (1971), 1-22.

[2] C. J. Colbourn, The combinatorics of network reliability, Oxford Univ. Press, Oxford, 1987.

[3] H. H. Crapo, A higher invariant for matroids, J. Comb. Theory 2 (1967), 406-417.

[4] A. B. Huseby, A unified theory of domination and signed domination with applications to exact reliability computations, Statistical Research Report, Institute of Mathematics, University of Oslo, Oslo, Norway, 1984.

[5] A. B. Huseby, Domination theory and the Crapo β-invariant, Networks 19 (1989), 135-149.

[6] J. G. Oxley, On Crapo's β-invariant for matroids, Stud. Appl. Math. 66 (1982), 267-277.

[7] A. Satyanarayana and M. K. Chang, Network reliability and the factoring theorem, Networks 13 (1983), 107-120.

[8] A. Satyanarayana and A. Prabhakar, A new topological formula and rapid algorithm for reliability analysis of complex networks, IEEE Trans. Reliability R-27 (1978), 82-100.

[9] D. J. A. Welsh, Matroid theory, Academic Press, London, 1976.

[10] R. J. Wilson, Introduction to graph theory (third edition), Longman Scientific and Technical, Harlow, 1985.

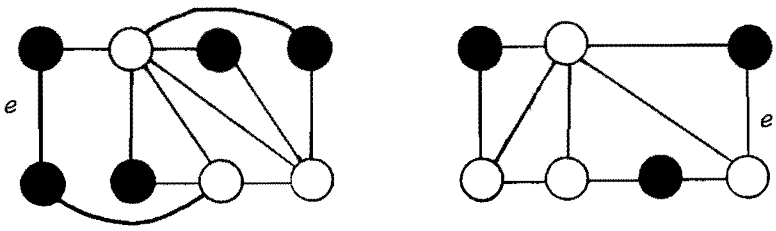

Figure 1

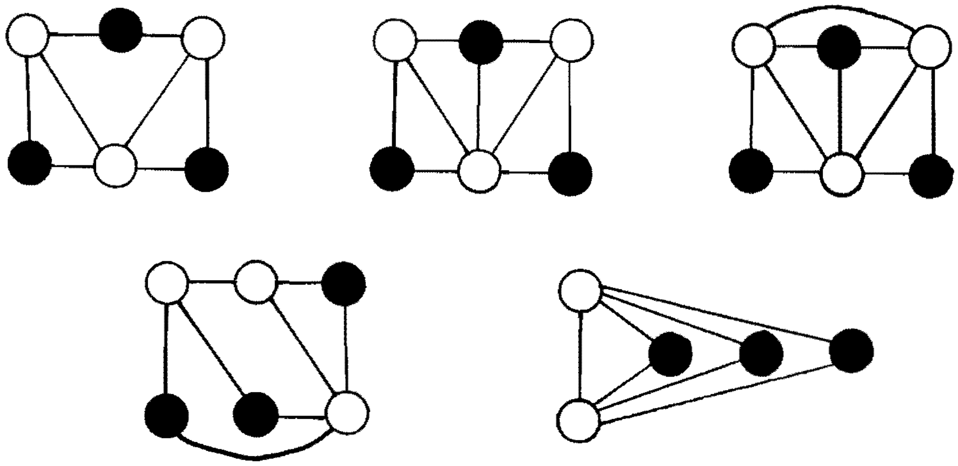

Figure 2

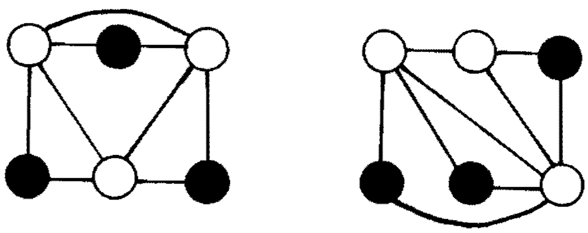

Figure 3