CONGRESSUS NUMERANTIUM

WINNIPEG, CANADA

A note on reliability and expected value

Lorenzo Traldi
Department of Mathematics, Lafayette College
Easton, Pennsylvania 18042

Abstract

We observe that several properties of reliability domination can be deduced from elementary properties of expected values.

Suppose E is a finite set, provided with a function $v : 2^E \to [0, \infty)$. We think of the elements of E as being independently subject to failure, with $e \in E$ having probability of successful operation $p(e)$, and for $S \subseteq E$ we think of $v(S)$ as the *value* of the successful operation of the elements of S (and the failure of the elements of $E - S$). The *expected value* of such a *value system* is given by the following formula.

$$Ev(E) = \sum_{S \subseteq E} \left(\prod_{e \in S} p(e) \right) \left(\prod_{e \in E-S} (1 - p(e)) \right) v(S)$$

We presume that the value systems we deal with have the following property: if $S \subseteq E$ and $e_1 \in E$ then either $v(S \cup \{e_1\}) = v(S)$ or $v(S \cup \{e_1\}) = v(S) + v(\{e_1\})$; note that if $v(E) \neq v(\emptyset)$ then $v(\emptyset)$ must be 0. If $v(S \cup \{e_1\}) = v(S)$ we say e_1 is *redundant* with respect to S; we denote by $Red(E, e_1)$ the family of subsets of $E - \{e_1\}$ with respect to which e_1 is redundant. We denote by $red(E, e_1)$ the probability that e_1 is redundant with respect to a randomly chosen subset of $E - \{e_1\}$, i.e.,

$$red(E, e_1) = \sum_{S \in Red(E, e_1)} \left(\prod_{e \in S} p(e) \right) \left(\prod_{e \in E - \{e_1\} - S} (1 - p(e)) \right).$$

Observation 1. *If $e_1 \in E$ then the expected increase in value due to the presence of e_1 in E is*

$$Ev(E) - Ev(E - \{e_1\}) = p(e_1) v(\{e_1\})(1 - red(E, e_1)).$$

CONGRESSUS NUMERANTIUM 133 (1998), pp.95-99

Proof. Clearly $Ev(E) - Ev(E - \{e_1\})$ is the product of three factors: the probability that e_1 is operational, the probability that e_1 is not redundant with respect to a randomly chosen subset of $E - \{e_1\}$, and the increase in value e_1 creates when it is not redundant. $\Box$

Observation 1 implies that the expected value of E is intimately tied to the redundancy probabilities of the elements of E. Not only can we determine $red(E, e_1)$ if we know $Ev(E)$ and $Ev(E - \{e_1\})$, but conversely $Ev(E)$ is determined by redundancy probabilities: if $E = \{e_1, ..., e_m\}$ then

$$Ev(E) = \sum_{j=1}^{m} p(e_j)v(\{e_j\})(1 - red(E - \{e_1, ..., e_{j-1}\}, e_j)).$$

Suppose the elements of E all have the same probability $p = p(e)$ of successful operation, and we regard p as an indeterminate. Then $Ev(E)$ is a polynomial in p of degree at most $|E|$, and $red(E, e_1)$ is a polynomial in p of degree at most $|E| - 1$; we denote the coefficient of $p^{|E|}$ in $Ev(E)$ by $b(E, v)$.

Observation 2.
$$Ev(E) = \sum_{S \subseteq E} b(S, v)p^{|S|}$$

Proof. If $S \subseteq E$ then

$$b(S, v) = \sum_{T \subseteq S} (-1)^{|S-T|} v(T),$$

so

$$\sum_{S \subseteq E} b(S, v)p^{|S|} = \sum_{T \subseteq E} p^{|T|} v(T) \left(\sum_{S \supseteq T} (-p)^{|S-T|} \right).$$

The binomial theorem then implies the asserted equality. $\Box$

Some interesting results of reliability theory can be deduced from these two observations. To discuss these results we need to introduce some terminology.

A *coherent family* of subsets of a set X is a family $R \subseteq 2^X$ such that whenever $S_1 \in R$ and $S_1 \subseteq S_2 \subseteq X$, $S_2 \in R$ too; coherent families occur naturally as the families of operational states of many different kinds of reliability problems. If $Red(E, e_1)$ is a coherent family of subsets of $E - \{e_1\}$ then $red(E, e_1)$ is the *reliability polynomial* of $Red(E, e_1)$, and the coefficient of $p^{|E|-1}$ in $red(E, e_1)$ is the (signed) *reliability domination*

$d(Red(E, e_1))$. Domination was introduced into network reliability theory by Satyanarayana and his coauthors [6, 7, 8], and has since been generalized to arbitrary coherent reliability problems by Huseby [3, 4] and Barlow and Iyer [1].

Observations 1 and 2 directly imply the following.

Corollary 1. *If $v(\{e_1\}) \neq 0$ and $Red(E, e_1)$ is coherent then*

$$red(E, e_1) = \sum_{e_1 \in S \subseteq E} d(Red(S, e_1))p^{|S|-1}.$$

Proof. If $e_1 \in S \subseteq E$ and $|S| \geq 2$ then Observation 1, applied to the value system (S, v), implies that $b(S, v) = -v(\{e_1\})d(Red(S, e_1))$. On the other hand, $b(\{e_1\}, v) = v(\{e_1\})$ and $d(Red(\{e_1\}, e_1)) = 0$. Consequently

$$\begin{aligned} 1 - red(E, e_1) &= (pv(\{e_1\}))^{-1} (Ev(E) - Ev(E - \{e_1\})) \\ &= (pv(\{e_1\}))^{-1} \sum_{e_1 \in S \subseteq E} b(S, v)p^{|S|} \\ &= 1 - \sum_{e_1 \in S \subseteq E} d(Red(S, e_1))p^{|S|-1}. \quad \square \end{aligned}$$

A *matroid* M on E can be determined by assigning to each subset $S \subseteq E$ a *rank* $r(S)$. A general introduction to the theory of matroids may be found in [9] or [10, 11]. A matroid rank function is a value function such that for every $e_1 \in E$, $r(\{e_1\}) \in \{0, 1\}$ and $Red(E, e_1)$ is coherent. If r is a matroid rank function then the Crapo β-invariant of M [2] is

$$\beta(M) = (-1)^{r(E)} \sum_{S \subseteq E} (-1)^{|S|} r(S);$$

clearly $(-1)^{|E|-r(E)}\beta(M) = b(E, r)$. Also, for $e_1 \in E$ the minimal elements of $Red(E, e_1)$ constitute an important aspect of the matroid's structure, the *port* of M with respect to e_1.

Focusing our attention on terms of highest possible degree, and taking into account the fact that a matroid M with any element e_1 such that $r(\{e_1\}) = 0$ will have $\beta(M) = 0$, Observation 1 has these immediate consequences.

Corollary 2. *If M is a matroid on E and $e_1 \in E$ then $\beta(M) = (-1)^{1+|E|-r(E)}d(Red(E, e_1))$.*

Corollary 3. *If M is a matroid on E then $d(Red(E, e_1))$ is the same for all $e_1 \in E$.*

Corollary 2 is Theorem 5.5 of [4]. Also, Corollary 3 is a weak version of Theorem 6.3 of [4], which is essentially the theorem that for every $e_1 \in E$, $|d(Red(E, e_1))|$ is the minimum of the β-invariants of the one-element extensions of M; it is a weak version because the word "minimum" is absent. (Huseby's theorem in turn is a generalization of Johnson's theorem [5] that the minimum domination of a graph G is $|d(Red(E, e_1))|$ for every edge e_1 of G.) This weak version arises in our context because a one-element extension of a general value system may give rise to smaller values of $|d(Red(E, e_1))|$ than the original system; it is easy to show that this cannot occur for matroidal value systems, however.

If G is a graph then its polygon matroid $M(G)$ is a matroid on $E(G)$ whose rank function is given by $r(S) = |V(G)| - \omega(G : S)$ for $S \subseteq E(G)$; here $G : S$ is the subgraph of G with $E(G : S) = S$ and $V(G : S) = V(G)$, and $\omega(G : S)$ is the number of connected components of this subgraph. Huseby [3, 4] has shown that if K is a set of two or more vertices of a single component of G then there is a matroid $M(G, K)$ on $E(G, K) = E(G) \cup \{e_1\}$, where e_1 is an "artificial" element that is not an edge of G, such that $M(G, K)$ extends $M(G)$ and $Red(E(G, K), e_1)$ consists of those subsets $S \subseteq E(G)$ such that K lies in a single component of $G : S$. Consequently, $red(E(G, K), e_1)$ is the K-terminal reliability polynomial $rel(G, K)$ and $d(Red(E(G, K), e_1))$ is the K-terminal domination $d_K(G)$. Specializing Corollary 1 to this situation, we deduce the result of Satyanarayana and Khalil [7] that

$$rel(G, K) = \sum_{S \subseteq E(G)} d_K(G : S) p^{|S|}.$$

References

[1] R. E. Barlow and S. Iyer, *Computational complexity of coherent systems and the reliability polynomial*, Prob. Eng. Inf. Sci. **2** (1988), 461-469.

[2] H. H. Crapo, *A higher invariant for matroids*, J. Comb. Theory **2** (1967), 406-417.

[3] A. B. Huseby, *A unified theory of domination and signed domination with applications to exact reliability computations*, Statistical Research Report, Institute of Mathematics, University of Oslo, Oslo, Norway, 1984.

[4] A. B. Huseby, *Domination theory and the Crapo β-invariant*, Networks **19** (1989), 135-149.

[5] R. Johnson, *Network reliability and acyclic orientations*, Networks **14** (1984), 489-505.

[6] A. Satyanarayana and M. K. Chang, *Network reliability and the factoring theorem*, Networks **13** (1983), 107-120.

[7] A. Satyanarayana and Z. Khalil, *On an invariant of graphs and the reliability polynomial*, SIAM J. Alg. Disc. Meth. **7** (1986), 399-403.

[8] A. Satyanarayana and A. Prabhakar, *A new topological formula and rapid algorithm for reliability analysis of complex networks*, IEEE Trans. Reliability **R-27** (1978), 82-100.

[9] D. J. A. Welsh, *Matroid theory*, Academic Press, London, 1976.

[10] N. White, ed., *Theory of matroids*, Cambridge University Press, Cambridge, 1986.

[11] N. White, ed., *Combinatorial geometries*, Cambridge University Press, Cambridge, 1987.