LINKING NUMBERS AND THE GROUPS OF LINKS

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Introduction

A link in $S^3$ is a union $L = K_1 \cup \cdots \cup K_\mu$ of finitely many, pairwise disjoint, embedded copies of $S^1$, its components. In this paper we will consider links that are ordered (each component $K_i$ has been assigned a certain index $i$) and oriented (each component has a preferred orientation). Two such links are equivalent iff there is an orientation-preserving auto-homeomorphism of $S^3$ under which the image of the $i$th component of one is the $i$th component of the other, for each $i$; the orientations of the components must also correspond under the automorphism. We will restrict our attention to links that are tame, that is, equivalent to polygonal links.

Among the simplest invariants of such a link are the linking numbers. If $i \neq j \in \{1, \ldots, \mu\}$, the linking number $\mathcal{L}(K_i, K_j) = \mathcal{L}(K_j, K_i)$ can be obtained from a regular projection of $L$ in the plane (that is, a projection in which the only singularities are double points, of which there are only finitely many) by assigning to each crossing of $K_i$ over $K_j$ one of the integers $\pm 1$, according to a suitable orientation convention, and then adding up the integers assigned to the various crossings of $K_i$ over $K_j$. (Other definitions are given in [3, p. 132].) We will call a link null iff the linking number of every pair of its components is zero.

The group of a link $L$ is the fundamental group $G = \pi_1(S^3 - L)$ of its complement. Certain elements of this group, the meridians, are distinguished; if $1 \leq i \leq \mu$ the $i$th meridian $m_i$ of $L$ (or the meridian to $K_i$ in $G$) is represented by a loop in $S^3 - L$ which is simply linked with $K_i$ (with linking number $+1$), and is unlinked from the sublink $L - K_i$. The meridians are well defined only up to conjugacy.

The lower central series subgroups of $G$ are given by: $G_1 = G$, and if $q \geq 1$ then $G_{q+1}$ is the subgroup generated by the set of commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in G_q$ and $h \in G$. As an abuse of language, if $q \geq 2$ we will use "meridians" to refer to the images $m_iG_q$ of the meridians of $L$ in $G/G_q$. In particular, if $\tilde{L} = \tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ is a tame link in $S^3$ with the same number of components as $L$, and $\tilde{G}$ is the group of $\tilde{L}$, then a homomorphism $G/G_p \to \tilde{G}/\tilde{G}_q$, $p, q \geq 2$, is meridian-preserving iff the image of the $i$th meridian $m_iG_p$ of $L$ is the $i$th meridian $m_i\tilde{G}_q$ of $\tilde{L}$, for $1 \leq i \leq \mu$. Similarly, if $p, q \geq 3$ a homomorphism $G_2/G_p \to \tilde{G}_2/\tilde{G}_q$ is
meridian-preserving iff the image of the commutator \([m_i, m_j]G_p\) is \([\tilde{m}_i, \tilde{m}_j]\tilde{G}_\mu_q\) whenever \(1 \leq i, j \leq \mu\). Of course, a meridian-preserving homomorphism \(G/\tilde{G}_p \to \tilde{G}/\tilde{G}_\mu\) induces a meridian-preserving homomorphism \(G_2/\tilde{G}_p \to \tilde{G}_2/\tilde{G}_\mu\) by restriction.

In this paper we are concerned with the relationship between the linking numbers in a link \(L\) and the isomorphism types of the groups \(G/G_3\) and \(G_2/G_3\) under both meridian-preserving and arbitrary isomorphisms. This relationship was first noted by K.-T. Chen [1].

He showed that the group \(G/G_3\) has the presentation
\[
\langle x_1, \ldots, x_\mu; \rho_i, \rho_{ijk} \rangle,
\]
in which there are relators
\[
\rho_{ijk} = [[x_i, x_j], x_k]
\]
whenever \(1 \leq i, j, k \leq \mu\) and
\[
\rho_i = \prod_{j \neq k} [x_i, x_j]^{l(K_i, K_j)}
\]
whenever \(1 \leq i \leq \mu\). Here the generator \(x_i\) represents the meridian \(m_iG_3\). (Also, any one of the relators \(\rho_i\) is a consequence of the other relators, and can be deleted from the presentation without effect.)

He also showed that the abelian group \(G_2/G_3\), written multiplicatively, has the presentation
\[
\langle x_{12}, \ldots, x_{\mu-1, \mu}; \sigma_{ii}, \sigma_{ijk} \rangle
\]
in which there is a generator \(x_{ij}\) whenever \(1 \leq i < j \leq \mu\), a relator
\[
\sigma_{ijk} = [x_{ij}, x_{kk}]
\]
whenever \(1 \leq i < j \leq \mu\) and \(1 \leq h < k \leq \mu\), and a relator
\[
\sigma_i = \left( \prod_{j \neq h} x_{ij}^{l(K_i, K_j)} \right) \cdot \left( \prod_{j \neq l} x_{ij}^{-l(K_i, K_j)} \right)
\]
whenever \(1 \leq i \leq \mu\). Here the generator \(x_{ij}\) represents the commutator \([m_i, m_j]G_3\).

(As before, any one relator \(\sigma_i\) is redundant, and can be deleted.) Equivalently, if we define a \(\mu \times \left( \begin{array}{c} \mu \\ 2 \end{array} \right)\) matrix \(\lambda\), with rows and columns indexed by \(\{1, \ldots, \mu\}\) and \(\{(p, q) | 1 \leq p < q \leq \mu \}\) respectively, by

\[
\lambda_{(i,p,q)} = \begin{cases} 
\lambda(K_i, K_p) & \text{if } i = p \\
-\lambda(K_i, K_q) & \text{if } i = q \\
0 & \text{if } p \neq i \neq q,
\end{cases}
\]

then \(\lambda\) is a presentation matrix for the abelian group \(G_2/G_3\).

Using these presentations, it is a simple matter to verify
THEOREM 1. Let $L = K_1 \cup \cdots \cup K_\mu$ and $L' = \bar{K}_1 \cup \cdots \cup \bar{K}_\mu \subseteq S^3$ be tame links with groups $G$ and $\bar{G}$, and suppose that either

$\langle K_i, K_j \rangle = \langle \bar{K}_i, \bar{K}_j \rangle$ for all $i \neq j \in \{1, \ldots, \mu\}$ or $\langle K_i, K_j \rangle = -\langle \bar{K}_i, \bar{K}_j \rangle$ for all $i \neq j \in \{1, \ldots, \mu\}$. Then there are meridian-preserving isomorphisms $G|G_3 \cong \bar{G}|G_3$ and $G_2|G_3 \cong \bar{G}_2|G_3$.

Counterexamples to the converse of THEOREM 1 are constructed in §1, using connected sums of links. Motivated by these, we make the following

DEFINITION: A link $L$ has inseparable linking numbers (loosely, $L$ is inseparable) iff whenever the linking numbers in a connected sum $L' \# L''$ are identical with those in $L$, at least one of $L'$, $L''$ is null (or a knot).

This definition is discussed further in §2. Using it, we prove

THEOREM 2. Let $L = K_1 \cup \cdots \cup K_\mu$ and $L' = \bar{K}_1 \cup \cdots \cup \bar{K}_\mu$ be tame links in $S^3$, with groups $G$ and $\bar{G}$. Then any two of the following are equivalent.

(a) Whenever $K_1 \cup \cdots \cup K_\mu$ has inseparable linking numbers, either $\langle K_{i_1}, K_{i_2} \rangle = \langle \bar{K}_{i_1}, \bar{K}_{i_2} \rangle$ for all $j \neq k \in \{1, \ldots, \eta\}$ or $\langle K_{i_1}, K_{i_2} \rangle = -\langle \bar{K}_{i_1}, \bar{K}_{i_2} \rangle$ for all $j \neq k \in \{1, \ldots, \eta\}$.

(b) There is a meridian-preserving isomorphism $G|G_3 \cong \bar{G}|G_3$.

(c) There is a meridian-preserving isomorphism $G_2|G_3 \cong \bar{G}_2|G_3$.

(d) There are matrices $B$ and $C$ with $\lambda = B\tilde{A}$ and $\bar{\lambda} = C\bar{A}$, where $\tilde{A}$ is the matrix defined as $\lambda$ was above, but using the linking numbers in $L'$ rather than those in $L$.

In particular, then, the converse of Theorem 1 does hold for inseparable links.

Theorem 2 also yields a simple necessary condition for the existence of meridian-preserving isomorphisms, when combined with the observation that any two-component link is inseparable, namely, the following

COROLLARY. Let $L$, $\bar{L} \subseteq S^3$ be tame links with groups $G$ and $\bar{G}$, and suppose there is a meridian-preserving isomorphism $G_2|G_3 \cong \bar{G}_2|G_3$. Then $\langle K_i, K_j \rangle = \pm \langle \bar{K}_i, \bar{K}_j \rangle$ for all $i \neq j \in \{1, \ldots, \mu\}$.

Given a $\mu \times \mu$ integral matrix $A$, let $\tilde{A}$ be the $\left( \begin{array}{c} \mu \\ 2 \end{array} \right) \times \left( \begin{array}{c} \mu \\ 2 \end{array} \right)$ matrix, with rows and columns indexed by $\{(p, q)|1 \leq p < q \leq \mu\}$, whose $(i, j)(p, q)$ entry is $\tilde{a}_{i_1}a_{j_2} - a_{i_2}a_{j_1}$ whenever $1 \leq i < j \leq \mu$ and $1 \leq p < q \leq \mu$. As can easily be verified, if $A$ is nonsingular, $\tilde{A}$ will be too; in fact, $(\tilde{A})^{-1} = (\tilde{A}^{-1})$.

THEOREM 3. Let $L = K_1 \cup \cdots \cup K_\mu$ and $L' = \bar{K}_1 \cup \cdots \cup \bar{K}_\mu$ be tame links in $S^3$, with groups $G$ and $\bar{G}$. Then $G|G_3 \cong \bar{G}|G_3$ iff there are $\mu \times \mu$ matrices $A$, $B$, and $C$, $A$ invertible, with $\lambda = B\tilde{A}\bar{A}^{-1}$ and $\bar{\lambda} = C\bar{A}\tilde{A}$.

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1. Some Counter-Examples to the Converse of Theorem 1.

The class of examples we consider consists of certain links which can be decomposed as "products" or "connected sums". Following Schubert [4, p. 142], this concept is defined as follows. Let $S \subseteq S^3$ be a tame embedded two-sphere; then $S^3 - S$ is the union of two open three-balls, say, $B'$ and $B''$. Let $L' = K_1' \cup \cdots \cup K_\alpha'$ and $L'' = K_1'' \cup \cdots \cup K_\beta''$ be tame links such that $L' \subseteq S \cup B'$, $L'' \subseteq S \cup B''$, and $L' \cap S = L'' \cap S$ consists of a single simple arc (that is, a homeomorphic image of the closed interval $[0, 1]$). Let $I$ be the interior of this arc, and suppose that $I$ receives opposite orientations from the components of $L'$ and $L''$, say, $K_i'$ and $K_j''$, that contain it. Then $L = L' \cup L'' - I$ is a link of $\mu = \alpha + \beta - 1$ components in $S^3$, each of which inherits an orientation from the component of $L'$ or $L''$ that intersects it; the ordering of the components of $L$ can follow any pattern one may choose. We say $L$ is a connected sum of $L'$ and $L''$ obtained by splicing together $K_i'$ and $K_j''$. An example of such a connected sum is illustrated in Figure 1; there $L = K_1 \cup K_2 \cup K_3 \cup K_4$ is a connected sum of $L' = K_1' \cup K_2'$ and $L'' = K_1'' \cup K_2'' \cup K_3''$ obtained by splicing together $K_2'$ and $K_1''$. ($K_2'$ and $K_1''$ have been moved off $S$, in the interest of clarity.)

A simple application of the Seifert-van Kampen theorem [3, p. 370] allows one to deduce that if $L$ is a connected sum of $L'$ and $L''$ obtained by splicing together $K_i'$ and $K_j''$, and $G$, $G'$ and $G''$ are the groups of $L$, $L'$, and $L''$ (respectively), then $G$ is isomorphic to the free product with amalgamation

$$G = G' *_{C} G''$$

where $C$ is an infinite cyclic group and the homomorphisms $C \rightarrow G'$ and $C \rightarrow G''$ which define the amalgamation map a generator of $C$ to the $i$th meridian of $L'$ and $j$th meridian of $L''$. Furthermore, the meridians of $L$ are precisely the images in $G$ of the meridians of $L'$ and $L''$.

We may now describe the examples we have in mind. Let $L' = K_1' \cup \cdots \cup K_\alpha'$, $L'' = K_1'' \cup \cdots \cup K_\beta''$, and $L' = K_1' \cup \cdots \cup K_\alpha'$ be tame links in $S^3$, in each of which some pair of components has nonzero linking number, and such that $\lambda(K_i', K_j') = -\lambda(K_i', K_j')$ for all $i \neq j \in \{1, \ldots, \alpha\}$. Let $L$ be a connected sum of $L'$ and $L''$ obtained by splicing together, say, $K_\alpha'$ and $K_j''$, and let $L$ be a connected sum of $L'$ and $L''$ obtained by splicing together $K_\alpha'$ and $K_j''$. Let $G$, $G'$, $G''$, and $G'''$ be the groups of $L$, $L'$, $L''$, and $L'''$, respectively. Then $G/G_3$ is isomorphic to the quotient of


Figure 1

\[(G'/G_3') \ast (G''/G_3')\]

by its third lower central series subgroup, and \(\bar{G}/\bar{G}_3\) is isomorphic to the quotient of

\[(\bar{G}'/\bar{G}_3') \ast (G''/G_3')\]

by its third lower central series subgroup. By Theorem 1, there is a meridian-preserving isomorphism between \(G'/G_3\) and \(\bar{G}'/\bar{G}_3\); hence there is also a meridian-preserving isomorphism between \(G/G_3\) and \(\bar{G}/\bar{G}_3\), even though \(L\) and \(\bar{L}\) do not satisfy the hypothesis of Theorem 1.

2. Links with Inseparable Linking Numbers

Restating a definition given earlier, a link \(L = K_1 \cup \ldots \cup K_\mu\) has inseparable linking numbers iff whenever a connected sum \(L' \# L'' = K_1 \cup \ldots \cup K_\mu\) has \(\ell(K_i, K_j) = \ell(K_i, K_j)\) for all \(i \neq j \in \{1, \ldots, \mu\}\), at least one of \(L'\) and \(L''\) is either
null or a knot. In this section we discuss this concept further, after recalling some definitions from graph theory, taken (with slight modifications) from [5].

A simple graph \( \mathcal{G} = (V, E) \) consists of a set \( V \) (the set of vertices of \( \mathcal{G} \)) and a set \( E \) of doubleton subsets of \( V \) (the set of edges of \( \mathcal{G} \)); either \( E \) or both \( E \) and \( V \), may be empty. Two vertices \( v, w \in V \) are adjacent iff \( \{v, w\} \in E \). The graph \( \mathcal{G} \) is connected iff whenever \( v, w \in V \) there is a finite sequence \( v = v_0, \ldots, v_m = w \) of vertices, each of which is adjacent to its successor. The maximal connected subgraphs of a graph are its connected components.

In a connected graph \( \mathcal{G} \), it may be that the removal of a single vertex (and the edges which contain it) will produce a disconnected graph; such a vertex is sometimes called an articulation point of \( \mathcal{G} \). We will call a connected graph 2-connected iff none of its vertices is an articulation point; for instance, we consider any simple, connected graph with no more than two vertices to be 2-connected. (Though convenient for us, this use of the term “2-connected” is not completely standard [5, p. 131].)

Now, suppose \( L = K_1 \cup \cdots \cup K_n \subseteq S^3 \) is a tame link. We define the linking graph of \( L \) to be the graph \( \mathcal{G}(L) = (V, E) \) with \( V = \{v_i\} \cup (K_1, K_j) \neq 0 \) for some \( j \neq i \in \{1, \ldots, n\} \) and \( E = \{v_i, v_j\} \cup (K_1, K_j) \neq 0 \). For instance, the linking graph of the link \( L \) of Figure 1 is indicated in Figure 2. (As usual in pictorial representations of graphs, dots represent vertices, and line segments represent edges.) Note that this graph is not 2-connected; the vertex \( v_2 \) is an articulation point.

![Figure 2](image)

This is no mere coincidence, as we see in

**Proposition 2.1.** The link \( L \) has inseparable linking numbers iff its linking graph \( \mathcal{G}(L) \) is 2-connected.

**Proof.** If the linking numbers in \( L \) are not inseparable, then there is a connected sum \( L' \# L'' \), whose components have the same linking numbers as those of \( L \), and in which neither \( L' \) nor \( L'' \) is either null or a knot; then \( \mathcal{G}(L) = \mathcal{G}(L' \# L'') \). The graph \( \mathcal{G}(L' \# L'') \) cannot be 2-connected, though, for it is connected, the vertex corresponding to the component along which \( L' \) and \( L'' \) are spliced together will, clearly, be an articulation point.

On the other hand, suppose \( \mathcal{G}(L) \) is not 2-connected. If it is not connected, let \( L' \) be the sublink of \( L \) consisting of those components with the same indices
as the vertices of some single connected component of \( \mathcal{L}(L) \), and \( L'' \) the link consisting of the remaining components of \( L \) together with a single unknot, unlinked from them. Any connected sum of links equivalent to \( L' \) and \( L'' \), in which this unknot in \( L'' \) is spliced together with some component of \( L' \), will, clearly, have the same linking numbers as \( L \).

If \( \mathcal{L}(L) \) is connected but not 2-connected, it must have an articulation point \( v_i \). Let \( L' \) be the sublink of \( L \) consisting of \( K_i \) and those components of \( L \) with the same indices as the vertices of some single connected component of the graph obtained from \( \mathcal{L}(L) \) by deleting \( v_i \) (and all edges containing \( v_i \)). Let \( L'' \) be the sublink of \( L \) containing \( K_i \) and, in addition, all the components of \( L \) not contained in \( L' \). Then the connected sum of links equivalent to \( L' \) and \( L'' \), in which the two copies of \( K_i \) are spliced together, will have the same linking numbers as \( L \).

Either way, we conclude that if \( \mathcal{L}(L) \) is not 2-connected \( L \) cannot have inseparable linking numbers.

Q.E.D.

3. The Maximal Purely Inseparable Sublinks of a Link

We will say that the linking numbers in a link are purely inseparable (or, loosely, that the link itself is purely inseparable) iff they are inseparable and, in addition, every component of the link has nonzero linking number with some other component. An arbitrary tame link \( L \) in \( S^3 \) may then be expressed in an essentially unique way as \( L = L_{(0)} \cup \cdots \cup L_{(v)} \), where \( L_{(0)} \) is the (possibly empty) sublink of \( L \) consisting of those components whose linking numbers with all other components are zero, and \( L_{(1)}, \ldots, L_{(v)} \) are the maximal purely inseparable sublinks of \( L \); in particular, \( v = 0 \) iff \( L \) is null. Also, note that \( L_{(0)} \cap L_{(j)} \) is empty if \( j \leq 1 \), and if \( j > 1 \), \( L_{(i)} \cap L_{(j)} \) is either empty or a single component of \( L \), for if \( L_{(i)} \) and \( L_{(j)} \) were to have two or more components in common \( L_{(i)} \cup L_{(j)} \) would be purely inseparable, violating the maximality of \( L_{(i)} \) and \( L_{(j)} \).

**Lemma 3.1.** \( L_{(1)}, \ldots, L_{(v)} \) can be indexed in such a way that for \( j \geq 2 \), \( L_{(j)} \cap (L_{(1)} \cup \cdots \cup L_{(j-1)}) \) is either empty or a single component of \( L \).

**Proof.** Observe, first, that there can be no sequence \( L_{(i_1)}, \ldots, L_{(i_n)} \) of pairwise distinct, maximal purely inseparable sublinks of \( L \), \( n \geq 3 \), such that \( L_{(i_k)} \cap L_{(i_l)} \neq \emptyset \neq L_{(i_k)} \cap L_{(i_m)} \) for \( 1 \leq k < n \); for if there were such a sequence, \( L_{(i_1)} \cup \cdots \cup L_{(i_n)} \) would be purely inseparable. We inductively index the set of all maximal purely inseparable sublinks of \( L \) as follows. First of all, let \( L_{(1)} \) be any maximal purely inseparable sublink of \( L \). Suppose that, for some integer \( j (1 \leq j < v) \), we have already chosen \( L_{(1)}, \ldots, L_{(j)} \). If there is a maximal purely inseparable sublink \( L' \) of \( L \) distinct from \( L_{(1)}, \ldots, L_{(j)} \), such that \( L' \cap \{L_{(1)} \cup \cdots \cup L_{(j)}\} \neq \emptyset \), then let \( L_{(j+1)} \) be any such sublink. If there are no such sublinks, then let \( L_{(j+1)} \) be any maximal purely inseparable sublink of \( L \) which is distinct from
L_{(1)}, \ldots, L_{(j)}$. By this process, we obtain an indexing $L_{(1)}, \ldots, L_{(v)}$ of all maximal purely inseparable sublinks of $L$. From the observation of the preceding paragraph, we can see that this indexing has the required properties. \hfill Q.E.D.

The reader may have recognized that this proof is essentially graph-theoretic: in the first paragraph it is observed that a certain graph with $v$ vertices is a forest (that is, its connected components are trees), and the remainder of the argument is based on this observation.

Given a link $L$ and its sublinks $L_{(0)}, \ldots, L_{(v)}$ defined above, we define a sequence $L_{(0)}, \ldots, L_{(v)}$ of links in the following manner: if $L_{(j)} \cap (L_{(0)} \cup \cdots \cup L_{(j-1)}) = \emptyset$ (in particular, if $j = 0$), $L_{(j)}$ consists of $L_{(j)}$ and a single unknotted, unlinked from $L_{(j-1)}$, if $L_{(j)} \cap (L_{(0)} \cup \cdots \cup L_{(j-1)}) \neq \emptyset$, $L_{(j)} = L_{(j-1)}$. A link $\hat{L}$ can be obtained from links equivalent to $L_{(0)}, \ldots, L_{(v)}$ by repeated use of connected sums, arranged so that in the formation of the $j$th connected sum either the extra unknotted component of $L_{(j)}$ is spliced together with some component of the result of the first $j-1$ connected sums (if $L_{(j)} \neq L_{(j-1)}$), or else the two copies of the component $L_{(j)}$ has in common with $L_{(0)} \cup \cdots \cup L_{(j-1)}$ are spliced together (if $L_{(j)} = L_{(j-1)}$). Clearly this link $\hat{L}$ has the same linking numbers as $L$.

Let $G_{(j)} = \pi_1(S^3 - L_{(j)})$ for $0 \leq j \leq v$. Let $P$ be the group obtained from by repeated use of free products with amalgamation, arranged so that in the formation

$$G_{(0)}/G_{(0)3}, \ldots, G_{(v)}/G_{(v)3}$$

of the $j$th free product with amalgamation the amalgamated subgroups are either both trivial (if $L_{(j)} \neq L_{(j-1)}$), or are both infinite cyclic subgroups, generated by the meridians to the copies of the component $L_{(j)}$ and $L_{(0)} \cup \cdots \cup L_{(j-1)}$ have in common (if $L_{(j)} = L_{(j-1)}$). Then we have

**Proposition 3.2.** The quotients $P/P_3$ and $G/G_3$ are isomorphic and furthermore, such an isomorphism exists, under which the meridian to $K_i$ in $G/G_3$ corresponds to the image in $P/P_3$ of the meridian to $K_i$ in $G_{(j)}/G_{(j)3}$ whenever $K_i \subseteq L_{(j)}$.

**Proof.** By Theorem 1, if $\hat{G}$ is the group of $\hat{L}$ then there is a meridian-preserving isomorphism $G/G_3 \cong \hat{G}/\hat{G}_3$. Thus it suffices to show that there is an isomorphism, as described, between $P/P_3$ and $G/G_3$. This follows immediately from the relationship between the group of a connected sum and the groups of the links which are spliced together to form it (see §1). \hfill Q.E.D.

4. **Proof of Theorem 2**

Suppose $L = K_1 \cup \cdots \cup K_\mu$ and $\hat{L} = \hat{K}_1 \cup \cdots \cup \hat{K}_\mu$ are tame links in $S^3$
satisfying condition (a) of the statement of Theorem 2. Since any two-component link is inseparable, it follows from (a) that \( \langle K_i, K_j \rangle = \pm \langle \tilde{K}_i, \tilde{K}_j \rangle \) for all \( i \neq j \in \{1, \ldots, \mu\} \), and so if \( i_1, \ldots, i_n \in \{1, \ldots, \mu\} \) then \( K_{i_1} \cup \cdots \cup K_{i_n} \) has (purely) inseparable linking numbers iff \( \tilde{K}_{i_1} \cup \cdots \cup \tilde{K}_{i_n} \) does.

In particular, then, if \( L = L_{(0)} \cup \cdots \cup L_{(v)} \) is the decomposition of \( L \) defined in §3, and for \( 0 \leq j \leq v \) we let \( L_{(j)} \) be the sublink of \( L \) consisting of those components \( \tilde{K}_i \) such that \( K_i \subseteq L_{(j)} \), then \( L = L_{(0)} \cup \cdots \cup L_{(v)} \) is the decomposition of \( L \) defined in §3. That is, \( L_{(0)} \) consists of those components of \( L \) whose linking numbers with all other components of \( L \) are zero, and \( L_{(1)} \), \( \ldots, L_{(v)} \) are the maximal purely inseparable sublinks of \( L \). For \( 0 \leq j \leq v \) let \( G_{(j)} = \langle S^3 - L_{(j)} \rangle \) and \( \tilde{G}_{(j)} = \pi_1(S^3 - L_{(j)}) \). By (a) and Theorem 1, for each \( j \) there is a meridian-preserving isomorphism between \( G_{(j)} / G_{(j)}^3 \) and \( \tilde{G}_{(j)} / \tilde{G}_{(j)}^3 \). The existence of a meridian-preserving isomorphism between \( G / G_3 \) and \( \tilde{G} / \tilde{G}_3 \) now follows from Proposition 3.2.

Thus if \( L \) and \( \tilde{L} \) satisfy (a), they also satisfy (b). Obviously, if \( L \) and \( \tilde{L} \) satisfy (b) they must satisfy (c).

If \( L \) and \( \tilde{L} \) satisfy (c), then the elements \( \sigma_1, \ldots, \sigma_\mu \) of the (multiplicative) free abelian group on \( \{x_{ij}|1 \leq i < j \leq \mu\} \) (given by the formula appearing in the presentation of \( G_2 / G_3 \) mentioned in the introduction) must be elements of the subgroup generated by the elements \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_\mu \) (given by the analogous formula); that is, there must be a \( \mu \times \mu \) integral matrix \( B \) with \( \lambda = B\lambda \). Similarly, there must be a \( \mu \times \mu \) integral matrix \( C \) with \( \lambda = C\lambda \). Thus if \( L \) and \( \tilde{L} \) satisfy (c), they also satisfy (d).

To complete the proof of Theorem 2, then, it suffices to show that if \( L \) and \( \tilde{L} \) satisfy (a) too. Note that since the sum of the rows of \( \lambda \) is zero, for any \( j \in \{1, \ldots, \mu\} \) there is a matrix \( C \), with \( \lambda = C\lambda \), whose \( j \)th column is without nonzero entries. Similarly, for any \( j \in \{1, \ldots, \mu\} \) there is a matrix \( B \) with \( \lambda = B\lambda \), whose \( j \)th column consists entirely of zeroes.

Suppose \( i_1, \ldots, i_n \in \{1, \ldots, \mu\} \) and \( K_{i_1} \cup \cdots \cup K_{i_n} \) has inseparable linking numbers. We distinguish three cases: this link may be null, or purely inseparable, or neither.

Suppose, first, that \( K_{i_1} \cup \cdots \cup K_{i_n} \) is null. For convenience' sake, we may assume that \( i_1 < \cdots < i_n \). If \( B = (b_{ij}) \) and \( C = (c_{ij}) \) are matrices with \( \lambda = H\lambda \) and \( \tilde{\lambda} = C\lambda \) then, in particular, for \( 1 \leq j < k \leq \eta \) we have

\[
\langle \tilde{K}_{ij}, \tilde{K}_{ik} \rangle = \lambda_{ij} \lambda_{ik} = \sum_{l=1}^{\eta} c_{ij} c_{ik} \lambda_{il} \lambda_{lk} = 0,
\]

so since \( \langle K_{ij}, K_{ik} \rangle = 0 \). Thus \( \tilde{K}_{i_1} \cup \cdots \cup \tilde{K}_{i_n} \) is also null, so

\[
\langle K_{ij}, K_{ik} \rangle = \langle \tilde{K}_{ij}, \tilde{K}_{ik} \rangle = - \langle \tilde{K}_{ij}, \tilde{K}_{ik} \rangle \quad \text{for all} \quad j \neq k \in \{1, \ldots, \eta\}.
\]
Second, suppose that $K_{i_1} \cup \cdots \cup K_{i_n}$ is purely inseparable; as before, we may assume that $i_1 < \cdots < i_n$. As noted earlier, there are matrices $B=(b_{ij})$ and $C=(c_{ij})$ such that $\lambda = B\bar{\lambda}$, $\bar{\lambda} = C\lambda$, and $b_{i_{l_{n}}i_l} = c_{i_{l_{n}}i_l}$ for all $i \in \{1, \ldots, \mu\}$. Then for $1 \leq j < k < n$ we have

$$/(K_{i_j}, K_{i_k}) = \lambda_{i_j(i_j,i_k)} = \sum_{i=1}^{\mu} b_{i_{j_{k}}} \lambda_{i(i_j,i_k)} = (b_{i_{j_{k}}} - b_{i_{j_k}})//(\bar{R}_{i_j}, \bar{R}_{i_k})$$

$$= -\lambda_{i_k(i_j,i_k)} = -\sum_{i=1}^{\mu} b_{i_{k_{j}}} \lambda_{i(i_j,i_k)} = (b_{i_{k_{j}}} - b_{i_{k_j}})//(\bar{R}_{i_j}, \bar{R}_{i_k})$$

and, similarly,

$$(4.1) \quad /(\bar{R}_{i_j}, \bar{R}_{i_k}) = \lambda_{i_j(i_j,i_k)} = (c_{i_{j_{k}}} - c_{i_{j_k}})//(K_{i_j}, K_{i_k})$$

$$= -\lambda_{i_k(i_j,i_k)} = (c_{i_{k_{j}}} - c_{i_{k_j}})//(K_{i_j}, K_{i_k}).$$

Also, for $1 \leq j < n$ we have

$$/(K_{i_j}, K_{i_n}) = b_{i_{j_{n}}} //(\bar{R}_{i_j}, \bar{R}_{i_n}) = -b_{i_{n_{j}}} //(\bar{R}_{i_j}, \bar{R}_{i_n})$$

and

$$(4.2) \quad /\bar{R}_{i_j}, \bar{R}_{i_n}) = c_{i_{j_{n}}} //(K_{i_j}, K_{i_n}) = -c_{i_{n_{j}}} //(K_{i_j}, K_{i_n}).$$

In particular, note that for $j \neq k \in \{1, \ldots, \mu\}$ the integers $/(K_{i_j}, K_{i_k})$ and $/(\bar{R}_{i_j}, \bar{R}_{i_k})$ are multiples of each other, and hence $/(K_{i_j}, K_{i_k}) = \pm /(\bar{R}_{i_j}, \bar{R}_{i_k})$. If $\eta = 2$, this is all we need prove.

We claim that necessarily $c_{i_{j_k}} = 0$ whenever $j \neq k \in \{1, \ldots, \eta - 1\}$. Since $K_{i_1} \cup \cdots \cup K_{i_n}$ is purely inseparable, its linking graph $/(\rho(K_{i_1} \cup \cdots \cup K_{i_n})$ is connected and has no articulation points, by Proposition 2.1. In particular, if $j \neq k \in \{1, \ldots, \eta - 1\}$ then since $v_{i_j}$ is not an articulation point of this graph, there must be some sequence $v_{i_n} = v_{i_{n_0}}, v_{i_{n_1}}, \ldots, v_{i_{n_n}} = v_{i_{n_0}}$ of vertices, each of which is adjacent to its successor, such that $k_{p} \neq j$ for $0 \leq p \leq n$. By replacing this sequence by an initial segment, if necessary, we may assume that $k_{p} \neq \eta$ for $0 \leq p < n$. For $1 \leq p \leq n$ $v_{i_{k_{p-1}}} = v_{i_{k_{p}}}$ so $/(K_{i_{k_{p-1}}}, K_{i_{k_{p}}}) \neq 0$.

Now, if $1 \leq p < n$ then since $k_{p-1} \neq j \neq k_{p}$,

$$0 = \lambda_{i_j(i_{k_{p-1}}, i_{k_{p}})} = \sum_{i=1}^{\mu} c_{i_{j_{k}}} \lambda_{i(i_{k_{p-1}}, i_{k_{p}})} = (c_{i_{j_{k_{p-1}}}} - c_{i_{j_{k_{p-1}}}})//(K_{i_{k_{p-1}}}, K_{i_{k_{p}}})$$

if $k_{p-1} < k_{p}$, while if $k_{p-1} > k_{p}$

$$0 = \lambda_{i_j(i_{k_{p}}, i_{k_{p-1}})} = (c_{i_{j_{k_{p}}}} - c_{i_{j_{k_{p-1}}}})//(K_{i_{k_{p}}}, K_{i_{k_{p-1}}}).$$

In either case, since $/(K_{i_{k_{p}}}, K_{i_{k_{p-1}}}) = /\bar{R}_{i_{k_{p}}}, \bar{R}_{i_{k_{p-1}}}) \neq 0$, necessarily $c_{i_{j_{k_{p}}}} = c_{i_{j_{k_{p-1}}}}$. Since this holds for $1 \leq p < n$, we conclude that $c_{i_{j_{k_{n-1}}}} = c_{i_{j_{k}}}$ Also, since $k_{n-1} \neq j \neq k_{n} = \eta$,
Linking numbers and the groups of links

\[
\tilde{\tau}_{ij(k_{i,j-1}, k_{i,j})} = c_{ij(k_{i,j-1}, K_{i,j})} \neq 0,
\]
so since \(\sim(K_{i,k-1}, K_{i,k})\neq 0\), necessarily \(c_{ij(k_{i,j-1})} = 0\). This verifies our claim that \(c_{ij} = 0\) whenever \(j \neq k \in \{1, \ldots, \eta - 1\}\).

A second claim we make is that \(c_{ij} = c_{i,i+1}\) for all \(j \in \{2, \ldots, \eta - 1\}\). Since \(\sim(K_{i,j} \cup \cdots \cup K_{i,k})\) is connected and \(v_{i,k}\) is not an articulation point, if \(1 < j < \eta\) there must be some sequence \(v_{i,j} = v_{i,j+1}, \ldots, v_{i,m} = v_{i,k}\) of vertices, each of which is adjacent to its successor, such that \(j \neq \eta\) for \(0 \leq p \leq m\). For \(1 \leq p \leq m\), \(\sim(K_{i,p-1}, K_{i,p}) \neq 0\) since \(v_{i,p-1}\) is adjacent to \(v_{i,p}\). By (4.1) and the first claim,

\[
\sim(\tilde{\tau}_{i,j-1}, \tilde{\tau}_{i,j}) = c_{i,j-1,i,j-1} \cdot \sim(K_{i,j-1}, K_{i,j}) = c_{i,j-1,i,j} \cdot \sim(K_{i,j-1}, K_{i,j}),
\]
and hence \(c_{i,j-1,i,j-1} = c_{i,j-1,i,j}\), for \(1 \leq p \leq m\). Thus \(c_{ij} = c_{i,i+1}\), verifying our second claim.

By (4.1), (4.2), and our two claims,

\[
\sim(\tilde{\tau}_{i,j}, \tilde{\tau}_{i,k}) = c_{i,j,i,k} \neq 0
\]
for all \(j \neq k \in \{1, \ldots, \eta\}\). As noted earlier, \(\pm \sim(\tilde{\tau}_{i,j}, \tilde{\tau}_{i,k}) = \sim(K_{i,j}, K_{i,k})\) for all \(j \neq k \in \{1, \ldots, \eta\}\). Since \(K_{i,j} \cup \cdots \cup K_{i,k}\) is purely inseparable, some linking number \(\sim(K_{i,p}, K_{i,k})\) is certainly nonzero; from this we may conclude that \(c_{i,i} = \pm 1\).

This completes our consideration of the second case.

Recall that, in the third case, \(K_{i,j} \cup \cdots \cup K_{i,k}\) is supposed to be inseparable, but neither null nor purely inseparable. If \(L'\) is the maximal purely inseparable sublink of \(K_{i,j} \cup \cdots \cup K_{i,k}\), then \(\sim(K_{i,p}, K_{i,k}) = 0\) whenever either of \(K_{i,p}, K_{i,k}\) is not contained in \(L'\). As in the second case above, there is a \(c = \pm 1\) with \(\sim(\tilde{\tau}_{i,j}, \tilde{\tau}_{i,k}) = c \cdot \sim(K_{i,j}, K_{i,k})\) whenever \(L'\) contains both \(K_{i,p}\) and \(K_{i,k}\). On the other hand, if \(j \neq k \in \{1, \ldots, \eta\}\) and either of \(K_{i,j}, K_{i,k}\) is not contained in \(L'\), then by the first case above (applied to the null sublink \(K_{i,j} \cup K_{i,k}\)), \(0 = \sim(\tilde{\tau}_{i,j}, \tilde{\tau}_{i,k}) = c \cdot \sim(K_{i,j}, K_{i,k})\).

Thus \(\sim(\tilde{\tau}_{i,j}, \tilde{\tau}_{i,k}) = c \cdot \sim(K_{i,j}, K_{i,k})\) for all \(j \neq k \in \{1, \ldots, \eta\}\).

Combining our three cases, we see that if \(L\) and \(L'\) satisfy the condition (d) of the statement of Theorem 2, then they must also satisfy (a). This completes our proof of Theorem 2.

5. Proof of Theorem 3

As is well known, if \(K\) is a group and \(x_1, y_1, x_2, y_2 \in K\) satisfy \(x_1 K_2 = x_2 K_2\) and \(y_1 K_2 = y_2 K_2\), then \([x_1, y_1] K_3 = [x_2, y_2] K_3\). (This is a simple consequence of [2, Theorem 5.3].) Thus if \(x, y \in K/K_2\) there is a single element of \(K_2/K_3\), which we will denote \([x, y] K_3\), with the property that \([x, y] K_3 = [x_1, y_1] K_3\) whenever \(x = x_1 K_2\) and \(y = y_1 K_2\). Also, if \(K = G/G_3\) then the groups \(G/G_3\) and \(K/K_3\) are canonically isomorphic, as are \(G_2/K_2\) and \(K_2/K_3\); we will identify these groups with each other. Using these notational conventions, we have
LEMMA 5.1. Let $L$ and $\tilde{L}$ be tame links in $S^3$, with groups $G$ and $\bar{G}$. Then $G/G_3 \cong \bar{G}/\bar{G}_3$ iff there are isomorphisms $g: G/G_2 \rightarrow \bar{G}/\bar{G}_2$ and $h: G_2/G_3 \rightarrow \bar{G}_2/\bar{G}_3$ such that

$$h([x, y]G_3) = [g(x), g(y)]\bar{G}_3$$

for all $x, y \in G/G_2$.

PROOF. If $f: G/G_3 \rightarrow \bar{G}/\bar{G}_3$ is an isomorphism, the isomorphisms $g$ and $h$ defined by $f$ certainly satisfy the statement.

Conversely, suppose $g$ and $h$ are isomorphisms satisfying the statement. Let $\Phi$ be the free group on $\{x_1, \ldots, x_\mu\}$. As mentioned in the introductory section of this paper, K.-T. Chen has shown [1] that there are epimorphisms $\psi: \Phi \rightarrow G/G_3$ and $\bar{\psi}: \Phi \rightarrow \bar{G}/\bar{G}_3$ whose kernels are the normal subgroups of $\Phi$ generated by $\{\rho_i, \rho_{1k}\}$ and $\{\bar{\rho}_i, \rho_{1k}\}$, respectively (the $\bar{\rho}_i$ are defined in analogy with the $\rho_i$). Note that $\ker \psi$, $\ker \bar{\psi} \subseteq \Phi_2$.

For $1 \leq i \leq \mu$ let $g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \bar{\psi}(x_i)^{a_{ij}} \bar{G}_2$.

There is certainly a homomorphism $F: \Phi \rightarrow \bar{G}/\bar{G}_3$ with

$$F(x_i) = \prod_{j=1}^{\mu} \bar{\psi}(x_i)^{a_{ij}}$$

for each $i$; note that then $F(x)\bar{G}_2 = g(\psi(x)G_2) \in \bar{G}/\bar{G}_2$ for all $x \in \Phi$. Consequently,

$$F([x, y]) = [F(x)\bar{G}_2, F(y)\bar{G}_2] = [g(\psi(x)G_2), g(\psi(y)G_2)]\bar{G}_3 = h([\psi(x), \psi(y)])$$

for all $x, y \in \Phi$, so $F(c) = h\psi(c)$ for all $c \in \Phi_2$. Since $h$ is a monomorphism, it follows that $\Phi_2 \cap \ker F = \Phi_2 \cap \ker \psi$.

If $x \in \Phi_2$, then $\psi(x)G_2 \neq G_2$, so since $g$ is a monomorphism, $g(\psi(x)G_2) = F(x)\bar{G}_2 \neq \bar{G}_2$; certainly then $x \in \ker F$. Thus $\ker F \subseteq \Phi_2$. Also, $\ker \psi \subseteq \Phi_2$, as noted earlier. Thus $\ker F = \Phi_2 \cap \ker F = \Phi_2 \cap \ker \psi = \ker \psi$; it follows that there is a monomorphism $f: G/G_3 \rightarrow \bar{G}/\bar{G}_3$ with $f\psi = F$.

If $z \in \bar{G}/\bar{G}_3$, then since $g$ is an epimorphism $z\bar{G}_2 = g(\psi(x)G_2) = F(x)\bar{G}_2$ for some $x \in \Phi$. Then $zF(x^{-1}) \in \bar{G}_2/\bar{G}_3$, so since $h$ is an epimorphism $zF(x^{-1}) = h\psi(c) = F(c)$ for some $c \in \Phi_2$; then $z = F(cx) = f\psi(cx)$. Thus $f$ is an epimorphism, and hence an isomorphism.

Q.E.D.

Suppose that $L, \tilde{L} \subseteq S^3$ are tame links, and $G/G_3$ is isomorphic to $\bar{G}/\bar{G}_3$. Let $g$ and $h$ be isomorphisms satisfying the lemma, and let $A = (a_{ij})$ be the $\mu \times \mu$ integral matrix with

$$g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \bar{\psi}(x_i)^{a_{ij}} \bar{G}_2,$$

as in the proof of the lemma. Since $g$ is an isomorphism, $A$ must be invertible.

Recall that $[ab, cd]\bar{G}_3 = [a, c][a, d][b, c][b, d]\bar{G}_3$ and $[a^n, b]\bar{G}_3 = [a, b]^n \bar{G}_3$.
= \left[ a, b^n \right] \overline{G}_3 \text{ for any } a, b, c, d \in \overline{G}/\overline{G}_2 \text{ and any integer } n \text{ [2, Corollary 5.3]. Using these facts repeatedly, if } 1 \leq i < j \leq \mu \text{ we have}

h([\psi(x_i), \psi(x_j)]G_3) = [g(\psi(x_i)G_2), g(\psi(x_j)G_2)]\overline{G}_3

= \left[ \prod_{p=1}^{\mu} \tilde{\psi}(x_p)^{a_{ip}}\overline{G}_2, \prod_{q=1}^{\mu} \tilde{\psi}(x_q)^{a_{jq}}\overline{G}_2 \right] \overline{G}_3

= \prod_{p=1}^{\mu} \prod_{q=1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}}\overline{G}_3

= \left( \prod_{p=1}^{\mu} \prod_{q=p+1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}}\overline{G}_3 \right) \cdot \left( \prod_{p=1}^{\mu} \prod_{q=1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}}\overline{G}_3 \right)

= \left( \prod_{p=1}^{\mu} \prod_{q=p+1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}}\overline{G}_3 \right) \cdot \left( \prod_{q=1}^{\mu} \prod_{p=q+1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}}\overline{G}_3 \right)

= \prod_{p=1}^{\mu} \prod_{q=p+1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}a_{ip}a_{jq}}\overline{G}_3.

Recalling that \( a_{(i,j)(p,q)} = a_{ip}a_{jq} - a_{iq}a_{jp} \) is the \((i,j)(p,q)\) entry of \( \tilde{A} \), it follows from the fact that \( h \) is an isomorphism, and from the presentations of \( G_2/G_3 \) and \( \overline{G}_2/\overline{G}_3 \) mentioned in the introductory section, that there must be integral matrices \( B \) and \( C \) such that \( \tilde{A} = C\lambda \tilde{A} \) and \( \lambda = B\lambda \tilde{A}^{-1} \).

Conversely, suppose there are matrices \( A, B, \) and \( C \) with \( \tilde{A} = C\lambda \tilde{A} \) and \( \lambda = B\lambda \tilde{A}^{-1} \), where \( A \) is invertible. Then we may define \( g: G/G_2 \rightarrow \overline{G}/\overline{G}_2 \) and \( h: G_2/G_3 \rightarrow \overline{G}_2/\overline{G}_3 \) by the equations

\[
g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \tilde{\psi}(x_j)^{a_{ij}}\overline{G}_2
\]

for \( 1 \leq i \leq \mu \), and

\[
h([\psi(x_i), \psi(x_j)]G_3) = \prod_{p=1}^{\mu} \prod_{q=p+1}^{\mu} \left[ \tilde{\psi}(x_p), \tilde{\psi}(x_q) \right]^{a_{ip}a_{jq}a_{ip}a_{jq}}\overline{G}_3
\]

for \( 1 \leq i < j \leq \mu \). It is a simple matter to show that \( g \) and \( h \) are isomorphisms that satisfy the condition of Lemma 5.1, and so \( G/G_3 \cong \overline{G}/\overline{G}_3 \).

This completes our proof of Theorem 3.

References


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