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LINKING NUMBERS AND THE GROUPS OF LINKS

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Introduction

A *link* in S^3 is a union $L=K_1 \cup \cdots \cup K_\mu$ of finitely many, pairwise disjoint, embedded copies of S^1, its *components*. In this paper we will consider links that are ordered (each component K_i has been assigned a certain index i) and oriented (each component has a preferred orientation). Two such links are *equivalent* iff there is an orientation-preserving auto-homeomorphism of S^3 under which the image of the ith component of one is the ith component of the other, for each i; the orientations of the components must also correspond under the atuo-homeomorphism. We will restrict our attention to links that are *tame*, that is, equivalent to polygonal links.

Among the simplest invariants of such a link are the *linking numbers*. If $i \neq j \in \{1,\dots,\mu\}$, the linking number $\ell(K_i, K_j)=\ell(K_j, K_i)$ can be obtained from a regular projection of L in the plane (that is, a projection in which the only singularities are double points, of which there are only finitely many) by assigning to each crossing of K_i over K_j one of the integers ± 1, according to a suitable orientation convention, and then adding up the integers assigned to the various crossings of K_i over K_j. (Other definitions are given in [3, p. 132].) We will call a link *null* iff the linking number of every pair of its components is zero.

The *group* of a link L is the fundamental group $G=\pi_1(S^3-L)$ of its complement. Certain elements of this group, the *meridians*, are distinguished; if $1\leq i\leq \mu$ the *ith meridian* m_i *of* L (or the *meridian to* K_i *in* G) is represented by a loop in S^3-L which is simply linked with K_i (with linking number $+1$), and is unlinked from the sublink $L-K_i$. The meridians are well defined only up to conjugacy.

The *lower central series subgroups of* G are given by: $G_1=G$, and if $q\geq 1$ then G_{q+1} is the subgroup generated by the set of commutators $[g, h]=ghg^{-1}h^{-1}$ with $g\in G_q$ and $h\in G$. As an abuse of language, if $q\geq 2$ we will use "meridians" to refer to the images m_iG_q of the meridians of L in G/G_q. In particular, if $\tilde{L}=\tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ is a tame link in S^3 with the same number of components as L, and $\tilde{G}$ is the group of $\tilde{L}$, then a homomorphism $G/G_p\to\tilde{G}/\tilde{G}_q$, $p, q\geq 2$, is *meridian-preserving* iff the image of the ith meridian m_iG_p of L is the ith meridian $\tilde{m}_i\tilde{G}_q$ of $\tilde{L}$, for $1\leq i\leq\mu$. Similarly, if $p, q\geq 3$ a homomorphism $G_2/G_p\to\tilde{G}_2/\tilde{G}_q$ is *meridian-preserving* iff the image of the commutator $[m_i, m_j]G_p$ is $[\tilde{m}_i, \tilde{m}_j]\tilde{G}_q$ whenever $1 \leq i, j \leq \mu$. Of course, a meridian-preserving homomorphism $G/G_p \to \tilde{G}/\tilde{G}_q$ induces a meridian-preserving homomorphism $G_2/G_p \to \tilde{G}_2/\tilde{G}_q$ by restriction.

In this paper we are concerned with the relationship between the linking numbers in a link L and the isomorphismtypes of the groups G/G_3 and G_2/G_3 under both meridian-preserving and arbitrary isomorphisms. This relationship was first noted by K.-T. Chen [1].

He showed that the group G/G_3 has the presentation

$$\langle x_1, \dots, x_\mu;\ \rho_i,\ \rho_{ijk} \rangle,$$

in which there are relators

$$\rho_{ijk} = [[x_i, x_j], x_k]$$

whenever $1 \leq i, j, k \leq \mu$ and

$$\rho_i = \prod_{j \neq i} [x_i, x_j]^{\ell(K_i, K_j)}$$

whenever $1 \leq i \leq \mu$. Here the generator x_i represents the meridian $m_i G_3$. (Also, any one of the relators ρ_i is a consequence of the other relators, and can be deleted from the presentation without effect.)

He also showed that the abelian group G_2/G_3, written multiplicatively, has the presentation

$$\langle x_{12}, \dots, x_{\mu-1\,\mu};\ \sigma_i,\ \sigma_{ijhk} \rangle$$

in which there is a generator x_{ij} whenever $1 \leq i < j \leq \mu$, a relator

$$\sigma_{ijhk} = [x_{ij}, x_{hk}]$$

whenever $1 \leq i < j \leq \mu$ and $1 \leq h < k \leq \mu$, and a relator

$$\sigma_i = \Big(\prod_{j<i} x_{ji}^{\ell(K_i, K_j)}\Big) \cdot \Big(\prod_{j>i} x_{ij}^{-\ell(K_i, K_j)}\Big)$$

whenever $1 \leq i \leq \mu$. Here the generator x_{ij} represents the commutator $[m_i, m_j]G_3$. (As before, any one relator σ_i is redundant, and can be deleted.) Equivalently, if we define a $\mu \times \binom{\mu}{2}$ matrix λ, with rows and columns indexed by $\{1, \dots, \mu\}$ and $\{(p, q) | 1 \leq p < q \leq \mu\}$ respectively, by

$$\lambda_{i(p,q)} = \begin{cases} \ell(K_i, K_q) & \text{if } i = p \\ -\ell(K_i, K_p) & \text{if } i = q \\ 0 & \text{if } p \neq i \neq q, \end{cases}$$

then λ is a presentation matrix for the abelian group G_2/G_3.

Using these presentations, it is a simple matter to verify

THEOREM 1. *Let* $L=K_1 \cup \cdots \cup K_\mu$ *and* $\tilde{L}=\tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu \subseteq S^3$ *be tame links with groups* G *and* $\tilde{G}$, *and suppose that either*

$$\ell(K_i, K_j) = \ell(\tilde{K}_i, \tilde{K}_j) \quad \text{for all } i \neq j \in \{1,\ldots,\mu\} \quad \text{or} \quad \ell(K_i, K_j) = -\ell(\tilde{K}_i, \tilde{K}_j)$$

for all $i \neq j \in \{1,\ldots,\mu\}$. *Then there are meridian-ptrserving isomorphisms* $G/G_3 \cong \tilde{G}/\tilde{G}_3$ *and* $G_2/G_3 \cong \tilde{G}_2/\tilde{G}_3$.

Counterexamples to the converse of THEOREM 1 are constructed in §1, using connected sums of links. Motivated by these, we make the following

DEFINITION: A link L has *inseparable linking numbers* (loosely, L is *inseparable*) iff whenever the linking numbers in a connected sum $L' \# L''$ are identical with those in L, at least one of L', L'' is null (or a knot).

This definition is discussed further in §2. Using it, we prove

THEOREM 2. *Let* $L=K_1 \cup \cdots \cup K_\mu$ *and* $\tilde{L}=\tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ *be tame links in* S^3, *with groups* G *and* $\tilde{G}$. *Then any two of the following are equivalent.*

(a) *Whenever* $K_{i_1} \cup \cdots \cup K_{i_\eta}$ *has inseparable linking numbers, either* $\ell(K_{i_j}, K_{i_k}) = \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k})$ *for all* $j \neq k \in \{1,\ldots,\eta\}$ *or* $\ell(K_{i_j}, K_{i_k}) = -\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k})$ *for all* $j \neq k \in \{1,\ldots,\eta\}$.

(b) *There is a meridian-preserving isomorphism* $G/G_3 \cong \tilde{G}/\tilde{G}_3$.

(c) *There is a meridian-preserving isomorphism* $G_2/G_3 \cong \tilde{G}_2/\tilde{G}_3$.

(d) *There are matrices* B *and* C *with* $\lambda = B\tilde{\lambda}$ *and* $\tilde{\lambda} = C\lambda$, *where* $\tilde{\lambda}$ *is the matrix defined as* λ *was above, but using the linking numbers in* $\tilde{L}$ *rather than those in* L,

In particular, then, the converse of Theorem 1 does hold for inseparable links.

Theorem 2 also yields a simple necessary condition for the existence of meridian-preserving isomorphisms, when combined with the observation that any two-component link is inseparable, namely, the following

COROLLARY. *Let* $L, \tilde{L} \subseteq S^3$ *be tame links with groups* G *and* $\tilde{G}$, *and suppose there is a meridian-preserving isomorphism* $G_2/G_3 \cong \tilde{G}_2/\tilde{G}_3$. *Then* $\ell(K_i, K_j) = \pm \ell(\tilde{K}_i, \tilde{K}_j)$ *for all* $i \neq j \in \{1,\ldots,\mu\}$.

Given a $\mu \times \mu$ integral matrix A, let $\hat{A}$ be the $\binom{\mu}{2} \times \binom{\mu}{2}$ matrix, with rows and columns indexed by $\{(p, q) | 1 \leq p < q \leq \mu\}$, whose $(i, j)(p, q)$ entry is $\hat{a}_{(i,j)(p,q)} = a_{ip}a_{jq} - a_{iq}a_{jp}$ whenever $1 \leq i < j \leq \mu$ and $1 \leq p < q \leq \mu$. As can easily be verified, if A is nonsingular, $\hat{A}$ will be too; in fact, $(\hat{A})^{-1} = (\widehat{A^{-1}})$.

THEOREM 3. *Let* $L=K_1 \cup \cdots \cup K_\mu$ *and* $\tilde{L}=\tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ *be tame links in* S^3, *with groups* G *and* $\tilde{G}$. *Then* $G/G_3 \cong \tilde{G}/\tilde{G}_3$ *iff there are* $\mu \times \mu$ *matrices* A, B, *and* C, A *invertible, with* $\lambda = B\tilde{\lambda}\hat{A}^{-1}$ *and* $\tilde{\lambda} = C\lambda\hat{A}$.

Before concluding this section, we must thank Sharon Richter for her illustrations. It should also be mentioned that an earlier version of some of this work was presented to the American Mathematical Society on August 19, 1981, in Pittsburgh, Pennsylvania.

1. Some Counter-Examples to the Converse of Theorem 1.

The class of examples we consider consists of certain links which can be decomposed as "products" or "connected sums". Following Schubert [4, p. 142], this concept is defined as follows. Let $S \subseteq S^3$ be a tame embedded two-sphere; then $S^3 - S$ is the union of two open three-balls, say, B' and B''. Let $L' = K'_1 \cup \cdots \cup K'_\alpha$ and $L'' = K''_1 \cup \cdots \cup K''_\beta$ be tame links such that $L' \subseteq S \cup B'$, $L'' \subseteq S \cup B''$, and $L' \cap S = L'' \cap S$ consists of a single simple arc (that is, a homeomorphic image of the closed interval $[0, 1]$). Let I be the interior of this arc, and suppose that I receives opposite orientations from the components of L' and L'', say, K'_i and K''_j, that contain it. Then $L = L' \cup L'' - I$ is a link of $\mu = \alpha + \beta - 1$ components in S^3, each of which inherits an orientation from the component of L' or L'' that intersects it; the ordering of the components of L can follow any pattern one may choose. We say L is a *connected sum of L' and L'' obtained by splicing together K'_i and K''_j*. An example of such a connected sum is illustrated in FIGURE 1; there $L = K_1 \cup K_2 \cup K_3 \cup K_4$ is a connected sum of $L' = K'_1 \cup K'_2$ and $L'' = K''_1 \cup K''_2 \cup K''_3$ obtained by splicing together K'_2 and K''_1. (K'_2 and K''_1 have been moved off S, in the interest of clarity.)

A simple application of the Seifert-van Kampen theorem [3, p. 370] allows one to deduce that if L is a connected sum of L' and L'' obtained by splicing together K'_i and K''_j, and G, G' and G'' are the groups of L, L', and L'' (respectively), then G is isomorphic to the free product with amalgamation

$$G' \underset{C}{*} G'',$$

where C is an infinite cyclic group and the homomorphisms $C \to G'$ and $C \to G''$ which define the amalgamation map a generator of C to the ith meridian of L' and jth meridian of L''. Furthermore, the meridians of L are precisely the images in G of the meridians of L' and L''.

We may now describe the examples we have in mind. Let $L' = K'_1 \cup \cdots \cup K'_\alpha$, $\bar{L}' = \bar{K}'_1 \cup \cdots \cup \bar{K}'_\alpha$, and $L'' = K''_1 \cup \cdots \cup K''_\beta$ be tame links in S^3, in each of which some pair of components has nonzero linking number, and such that $\ell(K'_i, K'_j) = -\ell(\bar{K}'_i, \bar{K}'_j)$ for all $i \neq j \in \{1, \ldots, \alpha\}$. Let L be a connected sum of L' and L'' obtained by splicing together, say, K'_α and K''_1, and let $\bar{L}$ be a connected sum of $\bar{L}'$ and L'' obtained by splicing together $\bar{K}'_\alpha$ and K''_1. Let G, $\bar{G}$, G', $\bar{G}'$, and G'' be the groups of L, $\bar{L}$, L', $\bar{L}'$, and L'', respectively. Then G/G_3 is isomorphic to the quotient of

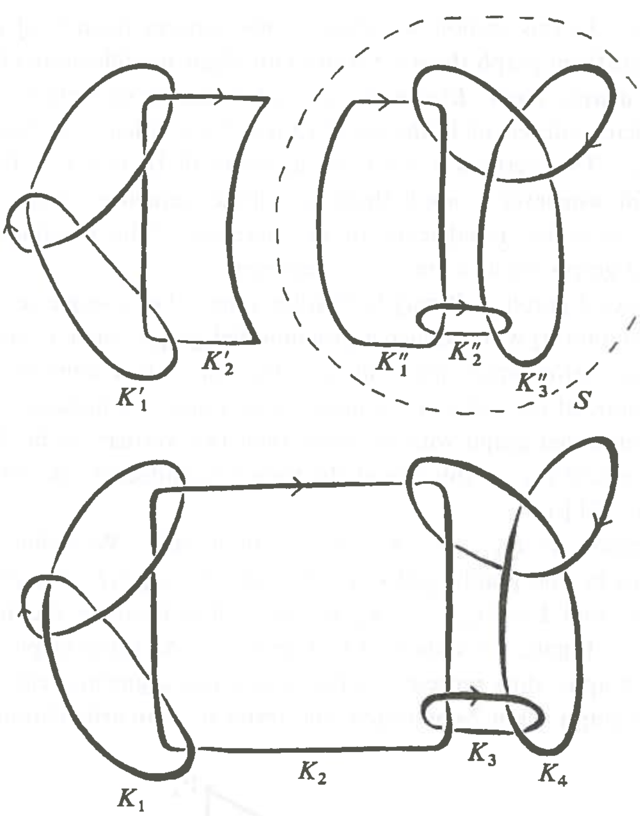

FIGURE 1

$$(G'/G'_3)\underset{C}{*}(G''/G''_3)$$

by its third lower central series subgroup, and $\bar{G}/\bar{G}_3$ is isomorphic to the quotient of

$$(\bar{G}'/\bar{G}'_3)\underset{C}{*}(G''/G''_3)$$

by its third lower central series subgroup. By Theorem 1, there is a meridian-preserving isomorphism between G'/G'_3 and $\bar{G}'/\bar{G}'_3$; hence there is also a meridian-preserving isomorphism between G/G_3 and $\bar{G}/\bar{G}_3$, even though L and $\bar{L}$ do not satisfy the hypothesis of Theorem 1.

2. Links with Inseparable Linking Numbers

Restating a definition given earlier, a link $L=K_1 \cup \cdots \cup K_\mu$ has inseparable linking numbers iff whenever a connected sum $\bar{L}'\#\bar{L}''=\bar{K}_1 \cup \cdots \cup \bar{K}_\mu$ has $\ell(\bar{K}_i, \bar{K}_j)=\ell(K_i, K_j)$ for all $i\neq j \in \{1,\ldots, \mu\}$, at least one of $\bar{L}'$ and $\bar{L}''$ is either null or a knot. In this section we discuss this concept further, after recalling some definitions from graph theory, taken (with slight modifications) from [5].

A *simple graph* $\mathscr{G}=(V, E)$ consists of a set V (the set of *vertices* of $\mathscr{G}$) and a set E of doubleton subsets of V (the set of *edges of* $\mathscr{G}$); either E, or both E and V, may be empty. Two vertices $v, w \in V$ are *adjacent* iff $\{v, w\} \in E$. The graph $\mathscr{G}$ is *connected* iff whenever $v, w \in V$ there is a finite sequence $v=v_0, \dots, v_m=w$ of vertices, each of which is adjacent to its successor. The maximal connected subgraphs of a graph are its *connected components.*

In a connected graph $\mathscr{G}$, it may be that the removal of a single vertex (and the edges which contain it) will produce a disconnected graph; such a vertex is sometimes called an *articulation point of* $\mathscr{G}$. We will call a connected graph *2-connected* iff none of its vertices is an articulation point; for instance, we consider any simple, connected graph with no more than two vertices to be 2-connected. (Though convenient for us, this use of the term "2-connected" is not completely standard [5, p. 131].)

Now, suppose $L=K_1 \cup \cdots \cup K_\mu \subseteq S^3$ is a tame link. We define the *linking graph of* L to be the graph $\ell\mathscr{G}(L)=(V, E)$ with $V=\{v_i | \ell(K_i, K_j)\neq 0$ for some $j\neq i \in \{1,\dots, \mu\}\}$ and $E=\{\{v_i, v_j\} | \ell(K_i, K_j)\neq 0\}$. For instance, the linking graph of the link L of Figure 1 is indicated in Figure 2. (As usual in pictorial representations of graphs, dots represent vertices, and line segments represent edges.) Note that this graph is not 2-connected; the vertex v_2 is an articulation point.

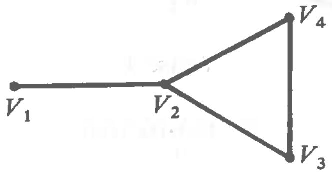

FIGURE 2

This is no mere coincidence, as we see in

PROPOSITION 2.1. *The link L has inseparable linking numbers iff its linking graph $\ell\mathscr{G}(L)$ is 2-connected.*

PROOF. If the linking numbers in L are not inseparable, then there is a connected sum $L' \# L''$, whose components have the same linking numbers as those of L, and in which neither L' nor L'' is either null or a knot; then $\ell\mathscr{G}(L)=\ell\mathscr{G}(L' \# L'')$. The graph $\ell\mathscr{G}(L' \# L'')$ cannot be 2-connected, though, for it is connected, the vertex corresponding to the component along which L' and L'' are spliced together will, clearly, be an articulation point.

On the other hand, suppose $\ell\mathscr{G}(L)$ is not 2-connected. If it is not connected, let L' be the sublink of L consisting of those components with the same indices as the vertices of some single connected component of $\mathscr{lg}(L)$, and L'' the link consisting of the remaining components of L together with a single unknot, unlinked from them. Any connected sum of links equivalent to L' and L'', in which this unknot in L'' is spliced together with some component of L', will, clearly, have the same linking numbers as L.

If $\mathscr{lg}(L)$ is connected but not 2-connected, it must have an articulation point v_i. Let L' be the sublink of L consisting of K_i and those components of L with the same indices as the vertices of some single connected component of the graph obtained from $\mathscr{lg}(L)$ by deleting v_i (and all edges containing v_i). Let L'' be the sublink of L containing K_i and, in addition, all the components of L not contained in L'. Then the connected sum of links equivalent to L' and L'', in which the two copies of K_i are spliced together, will have the same linking numbers as L.

Either way, we conclude that if $\mathscr{lg}(L)$ is not 2-connected L cannot have inseparable linking numbers. Q. E. D.

3. The Maximal Purely Inseparable Sublinks of a Link

We will say that the linking numbers in a link are *purely inseparable* (or, loosely, that the link itself is purely inseparable) iff they are inseparable and, in addition, every component of the link has nonzero linking number with some other component. An arbitrary tame link L in S^3 may then be expressed in an essentially unique way as $L=L_{(0)} \cup \cdots \cup L_{(\nu)}$, where $L_{(0)}$ is the (possibly empty) sublink of L consisting of those components whose linking numbers with all other components are zero, and $L_{(1)},\ldots, L_{(\nu)}$ are the maximal purely inseparable sublinks of L; in particular, $\nu=0$ iff L is null. Also, note that $L_{(0)} \cap L_{(j)}$ is empty if $j \geq 1$, and if $j > i \geq 1$, $L_{(i)} \cap L_{(j)}$ is either empty or a single component of L, for if $L_{(i)}$ and $L_{(j)}$ were to have two or more components in common $L_{(i)} \cup L_{(j)}$ would be purely inseparable, violating the maximality of $L_{(i)}$ and $L_{(j)}$.

LEMMA 3.1. *$L_{(1)},\ldots, L_{(\nu)}$ can be indexed in such a way that for $j \geq 2$, $L_{(j)} \cap (L_{(1)} \cup \cdots \cup L_{(j-1)})$ is either empty or a single component of L.*

PROOF. Observe, first, that there can be no sequence $L_{(i_1)},\ldots, L_{(i_n)}$ of pairwise distinct, maximal purely inseparable sublinks of L, $n \geq 3$, such that $L_{(i_1)} \cap L_{(i_n)} \neq \emptyset \neq L_{(i_k)} \cap L_{(i_{k+1})}$ for $1 \leq k < n$; for if there were such a sequence, $L_{(i_1)} \cup \cdots \cup L_{(i_n)}$ would be purely inseparable. We inductively index the set of all maximal purely inseparable sublinks of L as follows. First of all, let $L_{(1)}$ be any maximal purely inseparable sublink of L. Suppose that, for some integer $j (1 \leq j < \nu)$, we have already chosen $L_{(1)},\ldots, L_{(j)}$. If there is a maximal purely inseparable sublink L' of L distinct from $L_{(1)},\ldots, L_{(j)}$, such that $L' \cap \{L_{(1)} \cup \cdots \cup L_{(j)}\} \neq \emptyset$, then let $L_{(j+1)}$ be any such sublink. If there are no such sublinks, then let $L_{(j+1)}$ be any maximal purely inseparable sublink of L which is distinct from

$L_{(1)},\dots, L_{(j)}$. By this process, we obtain an indexing $L_{(1)},\dots, L_{(\nu)}$ of all maximal purely inseparable sublinks of L. From the observation of the preceding paragraph, we can see that this indexing has the required properties. Q.E.D.

The reader may have recognized that this proof is essentially graph-theoretic: in the first paragraph it is observed that a certain graph with ν vertices is a forest (that is, its connected components are trees), and the remainder of the argument is based on this observation.

Given a link L and its sublinks $L_{(0)},\dots, L_{(\nu)}$ defined above, we define a sequence $\bar{L}_{(0)},\dots, \bar{L}_{(\nu)}$ of links in the following manner: if $L_{(j)} \cap (L_{(0)} \cup \cdots \cup L_{(j-1)}) = \emptyset$ (in particular, if $j=0$), $\bar{L}_{(j)}$ consists of $L_{(j)}$ and a single unknot, unlinked from $L_{(j)}$; if $L_{(j)} \cap (L_{(0)} \cup \cdots \cup L_{(j-1)}) \neq \emptyset$, $\bar{L}_{(j)} = L_{(j)}$. A link $\hat{L}$ can be obtained from links equivalent to $\bar{L}_{(0)},\dots, \bar{L}_{(\nu)}$ by repeated use of connected sums, arranged so that in the formation of the jth connected sum either the extra unknotted component of $\bar{L}_{(j)}$ is spliced together with some component of the result of the first $j-1$ connected sums (if $\bar{L}_{(j)} \neq L_{(j)}$), or else the two copies of the component $L_{(j)}$ has in common with $L_{(0)} \cup \cdots \cup L_{(j-1)}$ are spliced together (if $\bar{L}_{(j)} = L_{(j)}$). Clearly this link $\hat{L}$ has the same linking numbers as L.

Let $G_{(j)} = \pi_1(S^3 - L_{(j)})$ for $0 \leq j \leq \nu$. Let P be the group obtained from by repeated use of free products with amalgamation, arranged so that in the formation

$$G_{(0)}/G_{(0)3},\dots, G_{(\nu)}/G_{(\nu)3}$$

of the jth free product with amalgamation the amalgamated subgroups are either both trivial (if $L_{(j)} \neq \bar{L}_{(j)}$), or are both infinite cyclic subgroups, generated by the meridians to the copies of the component $L_{(j)}$ and $L_{(0)} \cup \cdots \cup L_{(j-1)}$ have in common (if $L_{(j)} = \bar{L}_{(j)}$). Then we have

PROPOSITION 3.2. *The quotients P/P_3 and G/G_3 are isomorphic and furthermore, such an isomorphism exists, under which the meridian to K_i in G/G_3 corresponds to the image in P/P_3 of the meridian to K_i in $G_{(j)}/G_{(j)3}$ whenever $K_i \subseteq L_{(j)}$.*

PROOF. By Theorem 1, if $\hat{G}$ is the group of $\hat{L}$ then there is a meridian-preserving isomorphism $G/G_3 \cong \hat{G}/\hat{G}_3$. Thus it suffices to show that there is an isomorphism, as described, between P/P_3 and $\hat{G}/\hat{G}_3$. This follows immediately from the relationship between the group of a connected sum and the groups of the links which are spliced together to form it (see §1). Q.E.D.

4. Proof of Theorem 2

Suppose $L = K_1 \cup \cdots \cup K_\mu$ and $\tilde{L} = \tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ are tame links in S^3

satisfying condition (a) of the statement of Theorem 2. Since any two-component link is inseparable, it follows from (a) that $\ell(K_i, K_j) = \pm\ell(\tilde{K}_i, \tilde{K}_j)$ for all $i \neq j \in \{1, \dots, \mu\}$, and so if $i_1, \dots, i_\eta \in \{1, \dots, \mu\}$ then $K_{i_1} \cup \cdots \cup K_{i_\eta}$ has (purely) inseparable linking numbers iff $\tilde{K}_{i_1} \cup \cdots \cup \tilde{K}_{i_\eta}$ does.

In particular, then, if $L = L_{(0)} \cup \cdots \cup L_{(\nu)}$ is the decomposition of L defined in §3, and for $0 \leq j \leq \nu$ we let $\tilde{L}_{(j)}$ be the sublink of $\tilde{L}$ consisting of those components $\tilde{K}_i$ such that $K_i \subseteq L_{(j)}$, then $\tilde{L} = \tilde{L}_{(0)} \cup \cdots \cup \tilde{L}_{(\nu)}$ is the decomposition of $\tilde{L}$ defined in §3. That is, $\tilde{L}_{(0)}$ consists of those components of $\tilde{L}$ whose linking numbers with all other components of $\tilde{L}$ are zero, and $\tilde{L}_{(1)}, \dots, \tilde{L}_{(\nu)}$ are the maximal purely inseparable sublinks of $\tilde{L}$. For $0 \leq j \leq \nu$ let $G_{(j)} = \pi_1(S^3 - L_{(j)})$ and $\tilde{G}_{(j)} = \pi_1(S^3 - \tilde{L}_{(j)})$. By (a) and Theorem 1, for each j there is a meridian-preserving isomorphism between $G_{(j)}/G_{(j)3}$ and $\tilde{G}_{(j)}/\tilde{G}_{(j)3}$. The existence of a meridian-preserving isomorphism between G/G_3 and $\tilde{G}/\tilde{G}_3$ now follows from Proposition 3.2.

Thus if L and $\tilde{L}$ satisfy (a), they also satisfy (b). Obviously, if L and $\tilde{L}$ satisfy (b) they must satisfy (c).

If L and $\tilde{L}$ satisfy (c), then the elements $\sigma_1, \dots, \sigma_\mu$ of the (multiplicative) free abelian group on $\{x_{ij} | 1 \leq i < j \leq \mu\}$ (given by the formula appearing in the presentation of G_2/G_3 mentioned in the introduction) must be elements of the subgroup generated by the elements $\tilde{\sigma}_1, \dots, \tilde{\sigma}_\mu$ (given by the analogous formula); that is, there must be a $\mu \times \mu$ integral matrix B with $\lambda = B\tilde{\lambda}$. Similarly, there must be a $\mu \times \mu$ integral matrix C with $\tilde{\lambda} = C\lambda$. Thus if L and $\tilde{L}$ satisfy (c), they also satisfy (d).

To complete the proof of Theorem 2, then, it suffices to show that if L and $\tilde{L}$ satisfy (d), they must satisfy (a) too. Note that since the sum of the rows of λ is zero, for any $j \in \{1, \dots, \mu\}$ there is a matrix C, with $\tilde{\lambda} = C\lambda$, whose jth column is without nonzero entries. Similarly, for any $j \in \{1, \dots, \mu\}$ there is a matrix B with $\lambda = B\tilde{\lambda}$, whose jth column consists entirely of zeroes.

Suppose $i_1, \dots, i_\eta \in \{1, \dots, \mu\}$ and $K_{i_1} \cup \cdots \cup K_{i_\eta}$ has inseparable linking numbers. We distinguish three cases: this link may be null, or purely inseparable, or neither.

Suppose, first, that $K_{i_1} \cup \cdots \cup K_{i_\eta}$ is null. For convenience' sake, we may assume that $i_1 < \cdots < i_\eta$. If $B = (b_{ij})$ and $C = (c_{ij})$ are matrices with $\lambda = B\tilde{\lambda}$ and $\tilde{\lambda} = C\lambda$ then, in particular, for $1 \leq j < k \leq \eta$ we have

$$\begin{aligned} \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) &= \tilde{\lambda}_{i_j(i_j,i_k)} = \sum_{i=1}^{\mu} c_{i_j i}\lambda_{i(i_j,i_k)} \\ &= c_{i_j i_j}\ell(K_{i_j}, K_{i_k}) - c_{i_j i_k}\ell(K_{i_j}, K_{i_k}) = 0, \end{aligned}$$

since $\ell(K_{i_j}, K_{i_k}) = 0$. Thus $\tilde{K}_{i_1} \cup \cdots \cup \tilde{K}_{i_\eta}$ is also null, so

$$\ell(K_{i_j}, K_{i_k}) = \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = -\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) \qquad \text{for all} \quad j \neq k \in \{1, \dots, \eta\}.$$

Second, suppose that $K_{i_1} \cup \cdots \cup K_{i_\eta}$ is purely inseparable; as before, we may assume that $i_1 < \cdots < i_\eta$. As noted earlier, there are matrices $B=(b_{ij})$ and $C=(c_{ij})$ such that $\lambda = B\tilde{\lambda}$, $\tilde{\lambda}=C\lambda$, and $b_{ii_\eta}=0=c_{ii_\eta}$ for all $i \in \{1,\dots,\mu\}$. Then for $1 \le j < k < \eta$ we have

$$\begin{aligned} \ell(K_{i_j}, K_{i_k}) &= \lambda_{i_j(i_j,i_k)} = \sum_{i=1}^{\mu} b_{i_j i}\tilde{\lambda}_{i(i_j,i_k)} = (b_{i_j i_j} - b_{i_j i_k})\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) \\ &= -\lambda_{i_k(i_j,i_k)} = -\sum_{i=1}^{\mu} b_{i_k i}\lambda_{i(i_j,i_k)} = (b_{i_k i_k} - b_{i_k i_j})\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) \end{aligned}$$

and, similarly,

$$\begin{aligned} \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) &= \tilde{\lambda}_{i_j(i_j,i_k)} = (c_{i_j i_j} - c_{i_j i_k})\ell(K_{i_j}, K_{i_k}) \\ &= -\tilde{\lambda}_{i_k(i_j,i_k)} = (c_{i_k i_k} - c_{i_k i_j})\ell(K_{i_j}, K_{i_k}). \end{aligned} \tag{4.1}$$

Also, for $1 \le j < \eta$ we have

$$\ell(K_{i_j}, K_{i_\eta}) = b_{i_j i_j}\ell(\tilde{K}_{i_j}, \tilde{K}_{i_\eta}) = -b_{i_\eta i_j}\ell(\tilde{K}_{i_j}, \tilde{K}_{i_\eta})$$

and

$$\ell(\tilde{K}_{i_j}, \tilde{K}_{i_\eta}) = c_{i_j i_j}\ell(K_{i_j}, K_{i_\eta}) = -c_{i_\eta i_j}\ell(K_{i_j}, K_{i_\eta}). \tag{4.2}$$

In particular, note that for $j \ne k \in \{1,\dots,\mu\}$ the integers $\ell(K_{i_j}, K_{i_k})$ and $\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k})$ are multiples of each other, and hence $\ell(K_{i_j}, K_{i_k}) = \pm\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k})$. If $\eta=2$, this is all we need prove.

We claim that necessarily $c_{i_j i_k}=0$ whenever $j \ne k \in \{1,\dots,\eta-1\}$. Since $K_{i_1} \cup \cdots \cup K_{i_\eta}$ is purely inseparable, its linking graph $\mathscr{L}\!\mathscr{G}(K_{i_1} \cup \cdots \cup K_{i_\eta})$ is connected and has no articulation points, by Proposition 2.1. In particular, if $j \ne k \in \{1,\dots,\eta-1\}$ then since v_{i_j} is not an articulation point of this graph, there must be some sequence $v_{i_k}=v_{i_{k_0}}, v_{i_{k_1}},\dots, v_{i_{k_n}}=v_{i_\eta}$ of vertices, each of which is adjacent to its successor, such that $k_p \ne j$ for $0 \le p \le n$. By replacing this sequence by an initial segment, if necessary, we may assume that $k_p \ne \eta$ for $0 \le p < n$. For $1 \le p \le n$ $v_{i_{k_{p-1}}}$ is adjacent to $v_{i_{k_p}}$, so $\ell(K_{i_{k_{p-1}}}, K_{i_{k_p}}) \ne 0$.

Now, if $1 \le p < n$ then since $k_{p-1} \ne j \ne k_p$,

$$0 = \tilde{\lambda}_{i_j(i_{k_{p-1}}, i_{k_p})} = \sum_{i=1}^{\mu} c_{i_j i}\lambda_{i(i_{k_{p-1}}, i_{k_p})} = (c_{i_j i_{k_{p-1}}} - c_{i_j i_{k_p}})\ell(K_{i_{k_{p-1}}}, K_{i_{k_p}})$$

if $k_{p-1} < k_p$, while if $k_{p-1} > k_p$

$$0 = \tilde{\lambda}_{i_j(i_{k_p}, i_{k_{p-1}})} = (c_{i_j i_{k_p}} - c_{i_j i_{k_{p-1}}})\ell(K_{i_{k_p}}, K_{i_{k_{p-1}}}).$$

In either case, since $\ell(K_{i_{k_{p-1}}}, K_{i_{k_p}}) = \ell(K_{i_{k_p}}, K_{i_{k_{p-1}}}) \ne 0$, necessarily $c_{i_j i_{k_p}} = c_{i_j i_{k_{p-1}}}$. Since this holds for $1 \le p < n$, we conclude that $c_{i_j i_{k_{n-1}}} = c_{i_j i_k}$. Also, since $k_{n-1} \ne j \ne k_n = \eta$,

$$\tilde{\lambda}_{i_j(i_{k_{n-1}}, i_\eta)} = c_{i_j i_{k_{n-1}}} \ell(K_{i_{k_{n-1}}}, K_{i_\eta}) = 0,$$

so since $\ell(K_{i_{k_{n-1}}}, K_{i_\eta}) \neq 0$, necessarily $c_{i_j i_{k_{n-1}}} = 0$. This verifies our claim that $c_{i_j i_k} = 0$ whenever $j \neq k \in \{1, \ldots, \eta - 1\}$.

A second claim we make is that $c_{i_j i_j} = c_{i_1 i_1}$ for all $j \in \{2, \ldots, \eta - 1\}$. Since $\ell g(K_{i_1} \cup \cdots \cup K_{i_\eta})$ is connected and v_{i_η} is not an articulation point, if $1 < j < \eta$ there must be some sequence $v_{i_j} = v_{i_{j_0}}, \ldots, v_{i_{j_m}} = v_{i_1}$ of vertices, each of which is adjacent to its successor, such that $j_p \neq \eta$ for $0 \leq p \leq m$. For $1 \leq p \leq m$, $\ell(K_{i_{j_{p-1}}}, K_{i_{j_p}}) \neq 0$ since $v_{i_{j_{p-1}}}$ is adjacent to $v_{i_{j_p}}$. By (4.1) and the first claim,

$$\ell(\tilde{K}_{i_{j_{p-1}}}, \tilde{K}_{i_{j_p}}) = c_{i_{j_{p-1}} i_{j_{p-1}}} \cdot \ell(K_{i_{j_{p-1}}}, K_{i_{j_p}}) = c_{i_{j_p} i_{j_p}} \cdot \ell(K_{i_{j_{p-1}}}, K_{i_{j_p}}),$$

and hence $c_{i_{j_{p-1}} i_{j_{p-1}}} = c_{i_{j_p} i_{j_p}}$, for $1 \leq p \leq m$. Thus $c_{i_j i_j} = c_{i_1 i_1}$, verifying our second claim.

By (4.1), (4.2), and our two claims,

$$\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = c_{i_1 i_1} \ell(K_{i_j}, K_{i_k})$$

for all $j \neq k \in \{1, \ldots, \eta\}$. As noted earlier, $\pm \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = \ell(K_{i_j}, K_{i_k})$ for all $j \neq k \in \{1, \ldots, \eta\}$. Since $K_{i_1} \cup \cdots \cup K_{i_\eta}$ is purely inseparable, some linking number $\ell(K_{i_j}, K_{i_k})$ is certainly nonzero; from this we may conclude that $c_{i_1 i_1} = \pm 1$. This completes our consideration of the second case.

Recall that, in the third case, $K_{i_1} \cup \cdots \cup K_{i_\eta}$ is supposed to be inseparable, but neither null nor purely inseparable. If L' is the maximal purely inseparable sublink of $K_{i_1} \cup \cdots \cup K_{i_\eta}$, then $\ell(K_{i_j}, K_{i_k}) = 0$ whenever either of K_{i_j}, K_{i_k} is not contained in L'. As in the second case above, there is a $c = \pm 1$ with $\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = c \cdot \ell(K_{i_j}, K_{i_k})$ whenever L' contains both K_{i_j} and K_{i_k}. On the other had, if $j \neq k \in \{1, \ldots, \eta\}$ and either of K_{i_j}, K_{i_k} is not contained in L', then by the first case above (applied to the null sublink $K_{i_j} \cup K_{i_k}$), $0 = \ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = c \cdot \ell(K_{i_j}, K_{i_k})$. Thus $\ell(\tilde{K}_{i_j}, \tilde{K}_{i_k}) = c \cdot \ell(K_{i_j}, K_{i_k})$ for all $j \neq k \in \{1, \ldots, \eta\}$.

Combining our three cases, we see that if links L and $\tilde{L}$ satisfy the condition (d) of the statement of Theorem 2, then they must also satisfy (a). This completes our proof of Theorem 2.

5. Proof of Theorem 3

As is well known, if K is a group and $x_1, y_1, x_2, y_2 \in K$ satisfy $x_1 K_2 = x_2 K_2$ and $y_1 K_2 = y_2 K_2$, then $[x_1, y_1] K_3 = [x_2, y_2] K_3$. (This is a simple consequence of [2, Theorem 5.3].) Thus if $x, y \in K/K_2$ there is a single element of K_2/K_3, which we will denote $[x, y] K_3$, with the property that $[x, y] K_3 = [x_1, y_1] K_3$ whenever $x = x_1 K_2$ and $y = y_1 K_2$. Also, if $K = G/G_3$ then the groups. G/G_3 and K/K_3 are canonically isomorphic, as are G_2/G_3 and K_2/K_3; we will identify these groups with each other. Using these notational conventions, we have

LEMMA 5.1. Let L and $\tilde{L}$ be tame links in S^3, with groups G and $\tilde{G}$. Then $G/G_3 \cong \tilde{G}/\tilde{G}_3$ iff there are isomorphisms $g: G/G_2 \to \tilde{G}/\tilde{G}_2$ and $h: G_2/G_3 \to \tilde{G}_2/\tilde{G}_3$ such that

$$h([x, y]G_3) = [g(x), g(y)]\tilde{G}_3$$

for all $x, y \in G/G_2$.

PROOF. If $f: G/G_3 \to \tilde{G}/\tilde{G}_3$ is an isomorphism, the isomorphisms g and h defined by f certainly satisfy the statement.

Conversely, suppose g and h are isomorphisms satisfying the statement. Let Φ be the free group on $\{x_1, \dots, x_\mu\}$. As mentioned in the introductory section of this paper, K.-T. Chen has shown [1] that there are epimorphisms $\psi: \Phi \to G/G_3$ and $\tilde{\psi}: \Phi \to \tilde{G}/\tilde{G}_3$ whose kernels are the normal subgroups of Φ generated by $\{\rho_i, \rho_{ijk}\}$ and $\{\tilde{\rho}_i, \rho_{ijk}\}$, respectively (the $\tilde{\rho}_i$ are defined in analogy with the ρ_i). Note that $\ker\psi$, $\ker\tilde{\psi} \subseteq \Phi_2$.

For $1 \le i \le \mu$ let $g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \tilde{\psi}(x_j)^{a_{ij}}\tilde{G}_2$.

There is certainly a homomorphism $F: \Phi \to \tilde{G}/\tilde{G}_3$ with

$$F(x_i) = \prod_{j=1}^{\mu} \tilde{\psi}(x_j)^{a_{ij}}$$

for each i; note that then $F(x)\tilde{G}_2 = g(\psi(x)G_2) \in \tilde{G}/\tilde{G}_2$ for all $x \in \Phi$. Consequently, $F([x, y]) = [F(x)\tilde{G}_2, F(y)\tilde{G}_2]\tilde{G}_3 = [g(\psi(x)G_2), g(\psi(y)G_2)]\tilde{G}_3 = h([\psi(x), \psi(y)])$ for all $x, y \in \Phi$, so $F(c) = h\psi(c)$ for all $c \in \Phi_2$. Since h is a monomorphism, it follows that $\Phi_2 \cap \ker F = \Phi_2 \cap \ker\psi$.

If $x \in \Phi - \Phi_2$, then $\psi(x)G_2 \ne G_2$, so since g is a monomorphism, $g(\psi(x)G_2) = F(x)\tilde{G}_2 \ne \tilde{G}_2$; certainly then $x \notin \ker F$. Thus $\ker F \subseteq \Phi_2$. Also, $\ker\psi \subseteq \Phi_2$, as noted earlier. Thus $\ker F = \Phi_2 \cap \ker F = \Phi_2 \cap \ker\psi = \ker\psi$; it follows that there is a monomorphism $f: G/G_3 \to \tilde{G}/\tilde{G}_3$ with $f\psi = F$.

If $z \in \tilde{G}/\tilde{G}_3$, then since g is an epimorphism $z\tilde{G}_2 = g(\psi(x)G_2) = F(x)\tilde{G}_2$ for some $x \in \Phi$. Then $zF(x^{-1}) \in \tilde{G}_2/\tilde{G}_3$, so since h is an epimorphism $zF(x^{-1}) = h\psi(c) = F(c)$ for some $c \in \Phi_2$; then $z = F(cx) = f\psi(cx)$. Thus f is an epimorphism, and hence an isomorphism. Q.E.D.

Suppose that L, $\tilde{L} \subseteq S^3$ are tame links, and G/G_3 is isomorphic to $\tilde{G}/\tilde{G}_3$. Let g and h be isomorphisms satisfying the lemma, and let $A = (a_{ij})$ be the $\mu \times \mu$ integral matrix with

$$g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \tilde{\psi}(x_j)^{a_{ij}}\tilde{G}_2,$$

as in the proof of the lemma. Since g is an isomorphism, A must be invertible.

Recall that $[ab, cd]\tilde{G}_3 = [a, c][a, d][b, c][b, d]\tilde{G}_3$ and $[a^n, b]\tilde{G}_3 = [a, b]^n\tilde{G}_3$

$=[a, b^n]\tilde{G}_3$ for any $a, b, c, d \in \tilde{G}/\tilde{G}_2$ and any integer n [2, Corollary 5.3]. Using these facts repeatedly, if $1 \leq i < j \leq \mu$ we have

$$\begin{aligned}
h([\psi(x_i), \psi(x_j)]G_3) &= [g(\psi(x_i)G_2), g(\psi(x_j)G_2)]\tilde{G}_3 \\
&= [\prod_{p=1}^{\mu} \tilde{\psi}(x_p)^{a_{ip}}\tilde{G}_2, \quad \prod_{q=1}^{\mu} \tilde{\psi}(x_q)^{a_{jq}}\tilde{G}_2]\tilde{G}_3 \\
&= \prod_{p=1}^{\mu}\prod_{q=1}^{\mu} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}}\tilde{G}_3 \\
&= (\prod_{p=1}^{\mu}\prod_{q=p+1}^{\mu} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}}\tilde{G}_3)\cdot(\prod_{p=1}^{\mu}\prod_{q=1}^{p-1} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}}\tilde{G}_3) \\
&= (\prod_{p=1}^{\mu}\prod_{q=p+1}^{\mu} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}}\tilde{G}_3)\cdot(\prod_{q=1}^{\mu}\prod_{p=q+1}^{\mu} [\tilde{\psi}(x_q), \tilde{\psi}(x_p)]^{-a_{ip}a_{jq}}\tilde{G}_3) \\
&= \prod_{p=1}^{\mu}\prod_{q=p+1}^{\mu} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}-a_{iq}a_{jp}}\tilde{G}_3.
\end{aligned}$$

Recalling that $\hat{a}_{(i,j)(p,q)}=a_{ip}a_{jq}-a_{iq}a_{jp}$ is the $(i, j)(p, q)$ entry of $\hat{A}$, it follows from the fact that h is an isomorphism, and from the presentations of G_2/G_3 and $\tilde{G}_2/\tilde{G}_3$ mentioned in the introductory section, that there must be integral matrices B and C such that $\tilde{\lambda}=C\lambda\hat{A}$ and $\lambda=B\tilde{\lambda}\hat{A}^{-1}$.

Conversely, suppose there are matrices A, B, and C with $\tilde{\lambda}=C\lambda\hat{A}$ and $\lambda=B\tilde{\lambda}\hat{A}^{-1}$, where A is invertible. Then we may define $g\colon G/G_2\to\tilde{G}/\tilde{G}_2$ and h: $G_2/G_3\to\tilde{G}_2/\tilde{G}_3$ by the equations

$$g(\psi(x_i)G_2) = \prod_{j=1}^{\mu} \tilde{\psi}(x_j)^{a_{ij}}\tilde{G}_2$$

for $1\leq i\leq\mu$, and

$$h([\psi(x_i), \psi(x_j)]G_3) = \prod_{p=1}^{\mu}\prod_{q=p+1}^{\mu} [\tilde{\psi}(x_p), \tilde{\psi}(x_q)]^{a_{ip}a_{jq}-a_{iq}a_{jp}}\tilde{G}_3$$

for $1\leq i<j\leq\mu$. It is a simple matter to show that g and h are isomorphisms that satisfy the condition of Lemma 5.1, and so $G/G_3\cong\tilde{G}/\tilde{G}_3$.

This completes our proof of Theorem 3.

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