The Efron dice voting system

E. Gilson[1], C. Cooley[1], W. Ella[2], M. Follett[3] and L. Traldi[3]

[1]Northwestern University, Evanston IL 60208
[2]George Washington University, Washington DC 20052
[3]Lafayette College, Easton PA 18042

June 14, 2011

Abstract

About fifty years ago, Efron noted some counterintuitive properties of the long-term behavior of contests involving dice. For instance, consider the 6-sided dice whose sides are labeled (4,4,4,4,0,0), (3,3,3,3,3,3), (6,6,2,2,2,2), and (5,5,5,1,1,1). Each die has a 2/3 probability of rolling a higher number than the next one in the list, and the last has the same 2/3 probability of rolling a higher number than the first. The nontransitivity of games involving non-identical dice was popularized by Gardner in *Scientific American*. Although Gardner and other authors have observed that nontransitive dice serve to illustrate the complexities of the theory of voting, it does not seem that much attention has been paid to the corresponding voting system. Our purpose in this paper is to present this voting system, and compare its properties with those of other voting systems. One of the most interesting properties is the fact that cancellation with respect to the Efron dice voting system can replace cancellation with respect to pairwise preferences in Young's characterization of the social choice function associated with the Borda Count.

1 Introduction

Suppose a set of n voters is presented with the opportunity to choose among m candidates, with each voter providing a strict preference ordering of the candidates. One way to assess the relative support for candidates x_1 and x_2 would be to ask the following question: among the n^2 ways to choose voters v_i and v_j at random from the population, does it happen more often that v_i's ranking of x_1 is higher than v_j's ranking of x_2, or that v_i's ranking of x_1 is lower than v_j's ranking of x_2? If the answer is "higher" then it would be reasonable to think that support for x_1 is generally stronger than support for x_2, and if the answer is "lower" then it would be reasonable to think the opposite.

This voting system may also be described using dice. For each candidate, construct an n-sided die that lists the various rankings given to that candidate by the voters. Then assessing the likelihood that randomly chosen voters v_i and v_j will have v_i's ranking of candidate x_1 higher than v_j's ranking of candidate x_2 is the same as assessing the likelihood that rolling the dice corresponding to candidates x_1 and x_2 will result in a higher roll for x_1. We call the resulting voting system *Efron dice voting*. Using "dice" as part of the name for this voting system may be misleading, though, because "rolling the dice" usually describes a random process. The Efron dice voting system has no random character, as it does not draw conclusions from single dice rolls; rather all rolls of the dice representing two candidates are taken into account.

For example, consider a profile of five candidates and five voters. We tally votes by awarding four points for a first place vote, three points for a second place vote, and so on. Suppose candidate x_1 receives two first place votes and three third place votes; the corresponding die is $(4, 4, 2, 2, 2)$. If candidate x_2 receives three first place votes and two fourth place votes, then the corresponding die is $(4, 4, 4, 1, 1)$. There are 25 possible results when rolling the two dice; 6 are ties, 10 are wins for x_1 and 9 are wins for x_2, so the Efron dice system judges that x_1 has won, 10-9. In contrast, the Borda Count (which compares the total rankings of the candidates) judges that x_1 and x_2 are tied with 14 points apiece, and Condorcet's method (which compares the numbers of voters who prefer each candidate over the other) judges that x_2 has won, 3-2.

Even though the Efron dice voting system does not always give the same results as the Borda Count or Condorcet's method, it shares many characteristics with one or the other. Like Condorcet's method, the Efron dice system provides pairwise rankings of the candidates, which are not generally transitive. Like the Borda Count, the Efron dice system takes into account the precise placement of two candidates in the voters' preference rankings, not merely each voter's preference between the two. That is to say, the Efron dice system is a "positional" voting system in a general sense. However the Efron dice system is not a positional voting system in the sense of Saari [5], as it does not decide between candidates by adding up point totals.

In Section 2 we introduce our basic terminology regarding ballots and profiles, and in Section 3 we define the voting systems we will discuss. In Section 4 we consider profiles in which voters express only two or three levels of preference. A striking result is that in such profiles, Efron dice voting provides transitive social preferences that are sometimes different from those provided by the Borda Count. Several other properties of Efron dice voting are also discussed in Sections 4 and 5, including the unusual property that the results of a dice election may be changed by voters who are indifferent (i.e., they have precisely the same opinion of every candidate).

In some sense, the Efron dice system is "between" the Borda Count and Condorcet's method, and as we discuss in Sections $6-7$, its relationship with either one of these two familiar voting systems is rather similar to their relationship with each other. Pairwise dice results can be aggregated to yield the Borda Count, in much the same way as ordinary pairwise results can (see Chapter 4 of [6]). This aggregation process yields several other similarities among the Borda-Condorcet, Borda-dice and dice-Condorcet relationships. For instance:

if either Condorcet's method or the Efron dice voting system concludes that all the candidates in a strict election are tied, then the Borda Count must do the same; and if x_r is the unique winner of a strict election according to any one of these three systems, then it cannot be the unique loser according to either of the other two systems.

In Section 8 we prove our most striking result, that Young's characterization of the social choice function associated with the Borda Count [9] may be modified to refer to Efron dice voting rather than pairwise preferences.

Theorem 1 *The social choice function associated with the Borda Count is uniquely characterized by five properties. Three appear in Young's characterization [9]: consistency, faithfulness and neutrality. The fourth property, anonymity, is not explicitly mentioned in Young's characterization; but it appears implicitly, as it follows from the other properties given there. The fifth property is that for every strict profile in which all candidates are tied according to the Efron dice voting system, the social choice function also indicates that all the candidates are tied.*

In the last section of the paper we mention some open questions and directions for future research.

2 Preference orders and ratings

We follow [1] for much of our notation and terminology, with some modifications. Before going into specifics we should make it clear that our discussion is naïve: we do not consider the distinction between the opinions a voter expresses and the opinions the voter actually holds. This naïveté is unrealistic both because sincerity often does not serve a voter's interest, and because choosing a particular kind of ballot excludes sincere voters whose opinions are not structured in the appropriate way.

Consider a set of $n \geq 2$ voters, $V = \{v_1, ..., v_n\}$, and a set of m candidates, $X = \{x_1, ..., x_m\}$. Voter v_i expresses his or her opinions of the candidates, giving a *weak preference order* $\succsim_i$ on $\{x_1, ..., x_m\}$; the n-tuple $(\succsim_1, ..., \succsim_n)$ is a *voter preference profile.* We presume each voter's preference order $\succsim_i$ is complete (for every pair r and s, $x_r \succsim_i x_s$ or $x_s \succsim_i x_r$) and transitive (if $x_r \succsim_i x_s$ and $x_s \succsim_i x_t$, then $x_r \succsim_i x_t$). The associated *indifference* and *strict preference* relations are given by: (a) $x_r \sim_i x_s$ if $x_r \succsim_i x_s$ and $x_s \succsim_i x_r$, and (b) $x_r \succ_i x_s$ if $x_r \succsim_i x_s$ and $x_s \not\succsim_i x_r$. The transitivity of $\succsim_i$ implies that $\succ_i$ and $\sim_i$ are also transitive; moreover $\sim_i$ is an equivalence relation.

A basic feature of the Efron dice voting system is that it involves comparing the preferences of different voters. In order to perform these comparisons, we would like to re-express preference orders using numerical ratings. That is, we would like voter v_i to assign ratings $\rho_{i1}, ..., \rho_{im}$ to the candidates, in such a way that $\rho_{ir} \geq \rho_{is}$ if and only if $x_r \succsim_i x_s$. Then we will compare v_i's assessment of x_r to v_j's assessment of x_s by comparing ρ_{ir} to ρ_{js}. We distinguish two types of ratings.

Definition 2 *Suppose each voter's preference order is strict, i.e.,* $x_{r_{j_1}} \succ_i x_{r_{j_2}} \succ_i \ldots \succ_i x_{r_{j_m}}$ *for some choice of distinct indices* $r_{j_1}, \ldots, r_{j_m}$. *Then the corresponding* strict ratings *are* $\rho_{ir_{j_1}} = m-1, \rho_{ir_{j_2}} = m-2, \ldots, \rho_{ir_{j_m}} = 0$.

In a strict profile, v_i's preference order is equivalent to v_i's ratings; either determines the other.

Definition 3 *Suppose* $k \leq m$ *and for every voter* v_i, *the indifference relation* $\sim_i$ *produces no more than* k *equivalence classes. The voters' preference orders are then* weakly k-chotomous.

The adverb "weakly" may be omitted if for every voter v_i, the indifference relation $\sim_i$ produces precisely k equivalence classes. The most common special cases of Definition 3 are indicated by the terms *(weakly) dichotomous* ($k = 2$) and *(weakly) trichotomous* ($k = 3$). The version of Definition 3 that appears in [1] includes only k-chotomous profiles. We extend the definition to allow weakly k-chotomous profiles in order to highlight two interesting properties of the Efron dice voting system: weakly trichotomous profiles yield transitive societal preferences, and indifferent voters can make a difference. Details and examples are provided in Section 4.

Definition 4 *A set of ratings is* weakly k-chotomous *if (a) the associated preference orders are weakly* k*-chotomous, (b)* $\rho_{ir} \in \{0, \ldots, k-1\}$ *for every voter* v_i *and candidate* x_r, *and (c)* $\rho_{ir} \geq \rho_{is}$ *if and only if* $x_r \succsim_i x_s$.

In a weakly k-chotomous profile, v_i's preference order $\succsim_i$ is determined by v_i's ratings: $x_r \succsim_i x_s$ if and only if $\rho_{ir} \geq \rho_{is}$. If $\sim_i$ produces k different equivalence classes, then v_i's ratings are determined by $\succsim_i$. However, if $\sim_i$ produces fewer than k equivalence classes, then v_i's ratings are not determined by $\succsim_i$; for instance, an indifferent weakly dichotomous voter may choose to assign 0 to every candidate, or choose to assign 1 to every candidate. Consequently we regard a weakly k-chotomous profile as consisting of the voters' ratings, rather than their preference orders.

3 Five voting systems

Outside of Section 8 we focus on pairwise voting systems, which assess societal preferences between pairs of alternatives, rather than social choice functions, which choose winners from the entire slate of candidates. The reason for this focus is simply that we are writing about the Efron dice voting system, which is a mechanism for choosing between pairs of candidates.

Definition 5 *A* pairwise voting system *is a function that assigns to each profile and each pair of candidates a* (societal) preference, $x_r \succsim x_s$ *or* $x_s \succsim x_r$.

The societal preferences given by a pairwise voting system yield an indifference relation and a strict preference relation in the same way as the voters' preference orders do: $x_r \sim x_s$ if $x_r \succsim x_s$ and $x_s \succsim x_r$; and $x_r \succ x_s$ if $x_r \succsim x_s$ and $x_s \not\succsim x_r$. The societal preferences need not be transitive in any sense; that is, $x_r \sim x_s$ and $x_s \sim x_t$ need not imply $x_r \sim x_t$, $x_r \succ x_s$ and $x_s \succ x_t$ need not imply $x_r \succ x_t$, $x_r \succ x_s$ and $x_s \sim x_t$ need not imply $x_r \succ x_t$, etc.

Definition 6 *The* Efron dice voting system (DV) *is the pairwise voting system in which $x_r \succsim x_s$ if and only if*

$$|\{(i,j)|\rho_{ir} \geq \rho_{js}\}| \geq |\{(i,j)|\rho_{is} \geq \rho_{jr}\}|.$$

Note that we use $|S|$ to denote the number of elements in a set S.

Definition 7 Condorcet's method (CM) *is the pairwise voting system in which $x_r \succsim x_s$ if and only if*

$$|\{i|\rho_{ir} \geq \rho_{is}\}| \geq |\{i|\rho_{is} \geq \rho_{ir}\}|, \text{ or equivalently } |\{i|x_r \succsim_i x_s\}| \geq |\{i|x_s \succsim_i x_r\}|.$$

When a pairwise voting system is intended only for a certain kind of profile, we refer to it as a *strict* or *(weakly) k-chotomous* voting system.

Definition 8 *The* approval voting system (AV) *is the weakly dichotomous pairwise voting system in which $x_r \succsim x_s$ if and only if*

$$|\{i|\rho_{ir} \geq \rho_{is}\}| \geq |\{i|\rho_{is} \geq \rho_{ir}\}|, \text{ or equivalently } \sum_{i=1}^{n} \rho_{ir} \geq \sum_{i=1}^{n} \rho_{is}.$$

Definition 9 *The* combined approval voting system (CAV) *is the weakly trichotomous pairwise voting system in which $x_r \succsim x_s$ if and only if*

$$\sum_{i=1}^{n} \rho_{ir} \geq \sum_{i=1}^{n} \rho_{is}.$$

Definition 10 *The* Borda Count (BC) *is the strict pairwise voting system in which $x_r \succsim x_s$ if and only if*

$$\sum_{i=1}^{n} \rho_{ir} \geq \sum_{i=1}^{n} \rho_{is}.$$

Observe that under AV, CAV and BC each candidate x_r is given a score $\sum_{i=1}^{n} \rho_{ir}$, and societal preferences are determined by comparing scores. As numerical inequalities are transitive, societal preferences are transitive in these systems. DV and CM do not work in the same way, and often give intransitive societal preferences.

4 Weakly dichotomous and weakly trichotomous systems

Theorem 11 *On a weakly dichotomous profile DV, CM and AV are all the same. Moreover, if a strict profile has only two candidates then each of these systems is the same as the Borda Count.*

Proof. For each candidate x_r, let α_{r1} be the number of voters who assign x_r the rating 1; then $\alpha_{r0} = n - \alpha_{r1}$ is the number of voters who give x_r the rating 0.

In approval voting, $x_r \succsim x_s$ if and only if $\alpha_{r1} \geq \alpha_{s1}$.

In dice voting, $x_r \succsim x_s$ if and only if $\alpha_{r1}(n - \alpha_{s1}) \geq \alpha_{s1}(n - \alpha_{r1})$; as this is equivalent to $\alpha_{r1} n \geq \alpha_{s1} n$, it is true if and only if $\alpha_{r1} \geq \alpha_{s1}$.

For $i, j \in \{0, 1\}$ let n_{ij} denote the number of voters who give x_r the rating i and x_s the rating j; then $x_r \succsim x_s$ according to Condorcet's method if and only if $v_{10} \geq v_{01}$. This occurs if and only if $v_{10} + v_{11} \geq v_{01} + v_{11}$, i.e., $\alpha_{r1} \geq \alpha_{s1}$.

If there are only two candidates in a strict profile then for each candidate x_r, the sum

$$\sum_{i=1}^{n} \rho_{ir}$$

that appears in the definition of BC is α_{r1}. ■

When three or more ratings are used, the situation is quite different: the voting systems of Section 3 are all distinct. As a first step in verifying this, we recall the following surprising result of [8].

Proposition 12 *The Efron dice voting system is transitive on weakly trichotomous profiles.*

Proof. For $\rho \in \{0, 1, 2\}$ and $r \in \{1, ..., m\}$ let $\alpha_{r\rho}$ denote the number of voters who assign candidate x_r the rating ρ. Then $x_r \succsim x_s$ according to DV if and only if

$$2 \cdot |\{(i,j)|\rho_{ir} > \rho_{js}\}| \geq 2 \cdot |\{(i,j)|\rho_{ir} < \rho_{js}\}|,$$

or equivalently

$$2\alpha_{r2} \cdot (\alpha_{s1} + \alpha_{s0}) + 2\alpha_{r1} \cdot \alpha_{s0} \geq 2\alpha_{s2} \cdot (\alpha_{r1} + \alpha_{r0}) + 2\alpha_{s1} \cdot \alpha_{r0}.$$

With a little algebra we see that this is equivalent to

$$2\alpha_{r2}\alpha_{s1} + \alpha_{r2}\alpha_{s0} - \alpha_{s2}\alpha_{r0} - 2\alpha_{s1} \cdot \alpha_{r0} \geq 2\alpha_{s2}\alpha_{r1} + \alpha_{s2}\alpha_{r0} - \alpha_{r2}\alpha_{s0} - 2\alpha_{r1} \cdot \alpha_{s0}.$$

Adding $\alpha_{r2}\alpha_{s2} - \alpha_{r0}\alpha_{s0}$ to both sides, we see that this is equivalent to

$$(\alpha_{r2} - \alpha_{r0}) \cdot (\alpha_{s2} + 2\alpha_{s1} + \alpha_{s0}) \geq (\alpha_{s2} - \alpha_{s0}) \cdot (\alpha_{r2} + 2\alpha_{r1} + \alpha_{r0}).$$

Consequently x_r is preferred over x_s in the Efron dice voting system if and only if

$$\frac{\alpha_{r2} - \alpha_{r0}}{\alpha_{r2} + 2\alpha_{r1} + \alpha_{r0}} \geq \frac{\alpha_{s2} - \alpha_{s0}}{\alpha_{s2} + 2\alpha_{s1} + \alpha_{s0}}.$$

Note that the two sides of the inequality differ only in that r appears on the left where s appears on the right. We conclude that the Efron dice voting system assesses societal preferences by assigning each candidate a numerical score, and then comparing the scores. As inequalities of real numbers are transitive, the proposition follows. ∎

We refer to the fraction

$$\frac{\alpha_{r2} - \alpha_{r0}}{\alpha_{r2} + 2\alpha_{r1} + \alpha_{r0}}$$

as the *dice strength* of candidate x_r, and denote it $str(x_r)$. Observe that the dice strength gives a connection between the Efron dice system and combined approval voting: when there are only three ratings, DV prefers x_r over x_s if and only if $str(x_r) > str(x_s)$, and CAV prefers x_r over x_s if and only if the numerator of $str(x_r)$ is greater than the numerator of $str(x_s)$. As the denominator varies from candidate to candidate, the two systems sometimes disagree. In fact:

Theorem 13 *On weakly trichotomous profiles, CM, DV and CAV are distinct. On strict profiles with three or more candidates, CM, DV and BC are distinct.*

Proof. As CAV coincides with BC on strict profiles with precisely three candidates, we may verify the two statements simultaneously by providing examples of strict three-candidate profiles for which the voting systems come to different conclusions. The ballots are indicated by listing the candidates in order of increasing preference; for instance $x_3x_2x_1$ indicates a ballot with $x_1 \succ_i x_2 \succ_i x_3$ and (hence) $\rho_{i3} = 0$, $\rho_{i2} = 1$ and $\rho_{i1} = 2$.

Profile 1:

$x_3x_2x_1$	$x_2x_3x_1$	$x_3x_1x_2$	$x_1x_3x_2$	$x_2x_1x_3$	$x_1x_2x_3$
45	4	10	5	0	36

Here $\alpha_{12} = 49$, $\alpha_{11} = 10$, $\alpha_{10} = 41$, $\alpha_{22} = 15$, $\alpha_{21} = 81$, $\alpha_{20} = 4$, $\alpha_{32} = 36$, $\alpha_{31} = 9$ and $\alpha_{30} = 55$. In this situation the Efron dice system asserts that $x_1 \succ x_2 \succ x_3$, because

$$str(x_1) = \frac{8}{110} \approx 0.073 > str(x_2) = \frac{11}{181} \approx 0.061 > str(x_3) = \frac{-19}{109} \approx -0.174.$$

The Borda Count and Condorcet's method, instead, assert that $x_2 \succ x_1 \succ x_3$.

Profile 2:

$x_3x_2x_1$	$x_2x_3x_1$	$x_3x_1x_2$	$x_1x_3x_2$	$x_2x_1x_3$	$x_1x_2x_3$
49	0	6	9	4	32

Once again $\alpha_{12} = 49$, $\alpha_{11} = 10$, $\alpha_{10} = 41$, $\alpha_{22} = 15$, $\alpha_{21} = 81$, $\alpha_{20} = 4$, $\alpha_{32} = 36$, $\alpha_{31} = 9$ and $\alpha_{30} = 55$. As these are the same values obtained from Profile 1, the Efron dice system again asserts that $x_1 \succ x_2 \succ x_3$, and the Borda Count again yields $x_2 \succ x_1 \succ x_3$. This time, though, Condorcet's method agrees with the Efron dice system.

Profile 3:

$x_3x_2x_1$	$x_2x_3x_1$	$x_3x_1x_2$	$x_1x_3x_2$	$x_2x_1x_3$	$x_1x_2x_3$
0	49	5	41	0	5

In this profile $\alpha_{12} = 49$, $\alpha_{11} = 5$, $\alpha_{10} = 46$, $\alpha_{22} = 46$, $\alpha_{21} = 5$, $\alpha_{20} = 49$, $\alpha_{32} = 5$, $\alpha_{31} = 90$ and $\alpha_{30} = 5$. The Borda Count and the Efron dice system agree that $x_1 \succ x_3 \succ x_2$, but Condorcet's method results in a cycle: $x_2 \succ x_1 \succ x_3 \succ x_2$. ■

There are many other examples that illustrate the differences among the voting systems. Here is a particularly striking example, a strict profile involving nine voters and seven candidates. According to Condorcet's method, candidate x_1 is weaker than every other candidate, and x_7 is stronger than every other candidate. But according to DV, x_1 is stronger than x_7.

ballot (in order of increasing preference)	number of ballots
$x_7x_6x_5x_4x_3x_2x_1$	2
$x_1x_6x_5x_4x_3x_7x_2$	2
$x_1x_6x_5x_4x_2x_7x_3$	2
$x_6x_5x_4x_3x_2x_7x_1$	2
$x_1x_6x_5x_3x_2x_7x_4$	1

Before discussing some familiar properties of voting systems in the next section, we take a moment to point out that the Efron dice voting system has an unfamiliar property not shared by any of the other pairwise voting systems mentioned in Section 3: the addition of indifferent voters may change the outcome of a weakly k-chotomous election. (A voter v_i is *indifferent* if $x_r \sim_i x_s$ $\forall r, s$; equivalently, the rating ρ_{ir} is the same for every candidate.) Moreover, the effect of adding indifferent voters may be altered by changing their ratings, even though each of them assigns the same rating to every candidate.

For example, suppose we add 100 indifferent voters to Profile 1 mentioned in the proof of Theorem 13. If they all assign the rating $\rho_{ir} = 2$ for every r, then the Efron dice voting system again asserts that $x_1 \succ x_2 \succ x_3$, because

$$\frac{108}{210} \approx 0.514 > \frac{111}{281} \approx 0.395 > \frac{81}{209} \approx 0.388.$$

However, if the indifferent voters all assign the rating $\rho_{ir} = 1$ for every r, then the Efron dice voting system asserts that $x_2 \succ x_1 \succ x_3$, because

$$\frac{11}{381} \approx 0.029 > \frac{8}{310} \approx 0.026 > \frac{-19}{309} \approx -0.061.$$

It can even happen that adding indifferent voters has the effect of bringing the least-preferred candidate to most-preferred status. For instance in the strict profile

$x_3x_2x_1$	$x_2x_3x_1$	$x_3x_1x_2$	$x_1x_3x_2$	$x_2x_1x_3$	$x_1x_2x_3$
3	50	2	50	0	0

DV asserts that $x_1 \succ x_2 \succ x_3$, because

$$\frac{3}{107} \approx 0.028 > \frac{2}{108} \approx 0.019 > -\frac{5}{205} \approx -0.024.$$

However, if 100 additional indifferent voters all assign the rating $\rho_{ir} = 0$ for every r, then DV asserts that $x_3 \succ x_1 \succ x_2$, because

$$\frac{-105}{305} \approx -0.344 > \frac{-97}{207} \approx -0.469 > \frac{-98}{208} \approx -0.471.$$

A related property is that DV outcomes may be changed by adding symmetric blocs of voters. For instance, suppose we add 600 new voters to Profile 1 mentioned in the proof of Theorem 13, with each of the six different strict ballots submitted by 100 of the new voters. Although the Efron dice system has $x_1 \succ x_2 \succ x_3$ in Profile 1, it has $x_2 \succ x_1 \succ x_3$ in the new profile because

$$str(x_2) = \frac{11}{981} \approx 0.011 > str(x_2) = \frac{8}{910} \approx 0.009 > str(x_3) = \frac{-19}{909} \approx -0.021.$$

5 Some properties of pairwise voting systems

In this section we briefly discuss some familiar properties of pairwise voting systems. Most of the definitions are modified directly from [1], [7] or [9].

A pairwise voting system is *anonymous* if the societal preferences cannot be changed by simply permuting the voters' ballots; that is, the societal preferences are not affected if $v_1, ..., v_n$ are re-indexed. Similarly, a *neutral* voting system has the property that the societal preferences are the same if $x_1, ..., x_m$ are re-indexed before or after applying the voting system. A voting system satisfies *non-imposition* if every candidate can be the unique winner for some profile of voters. A voting system is *faithful* if, when there is only one voter, the societal preferences precisely reflect that voter's preferences. A voting system satisfies *Pareto* if, in every profile in which every voter has $x_r \succsim_i x_s$, the voting system concludes that $x_r \succsim x_s$.

All of these properties are shared by the pairwise voting systems of Section 3. The only instance of this assertion that may not be obvious is the Pareto property of the Efron dice system, as the hypothesis that no individual voter prefers x_s over x_r certainly does not prohibit one voter's rating of x_r from being lower than another voter's rating of x_s. The justification is simple, though: if $\rho_{ir} \geq \rho_{is}$ for every i, then

$$\{(i,j)|\rho_{ir} \geq \rho_{js}\} \supseteq \{(i,j)|\rho_{ir} \geq \rho_{jr}\} \supseteq \{(i,j)|\rho_{is} \geq \rho_{jr}\},$$

so $|\{(i,j)|\rho_{ir} \geq \rho_{js}\}| \geq |\{(i,j)|\rho_{is} \geq \rho_{jr}\}|$. Observe also that if $\rho_{ir} \geq \rho_{is}$ for every i and at least one voter has $x_r \succ_i x_s$ then $(i,i) \in \{(i,j)|\rho_{ir} > \rho_{js}\}$ and $(i,i) \notin \{(i,j)|\rho_{is} > \rho_{jr}\}$, so the Efron dice system concludes that $x_r \succ x_s$. The other voting systems of Section 3 also satisfy this strict form of Pareto.

In addition, all five pairwise voting systems are *monotone:* if $x_r \succsim x_s$ in P_1 and the only difference between P_1 and P_2 is that one voter v_i has a higher rating of x_r in P_2, then $x_r \succsim x_s$ in P_2 also. (The phrase "only difference" is interpreted differently for strict profiles – where it indicates that v_i has exchanged the places of x_r and the next most-preferred candidate – and weakly k-chotomous profiles – where it indicates that v_i has moved x_r from one preference class to the next most-preferred class.) Once again, the only system for which this may not be obvious is the Efron dice system; the argument is quite similar to the Pareto proof. All the systems also satisfy the strict form of the property (in which $x_r \succ x_s$ holds in both P_1 and P_2) and the decreasing form (in which the only difference between P_1 and P_2 is that one voter v_i has a lower rating of x_s in P_2).

Some familiar properties are not shared by all of the voting systems mentioned in Section 3. An example is *strict majority*: if most voters have $x_r \succ_i x_s$, then $x_r \succ x_s$. This property is satisfied by approval voting, the Efron dice system and Condorcet's method, but not by combined approval voting or the Borda Count. A *consistent* pairwise voting system has the property that whenever $\{v_1, ..., v_w\}$ and $\{v_{w+1}, ..., v_n\}$ separately prefer x_r over x_s, so does $\{v_1, ..., v_n\}$. It is easy to see that CM, AV, CAV and BC are all consistent; the Efron dice system is not, though, as indicated by the following example.

ballot	number in A	number in B	number in $A \cup B$
$x_3 \prec x_2 \prec x_1$	22	1	23
$x_2 \prec x_3 \prec x_1$	4	0	4
$x_3 \prec x_1 \prec x_2$	10	20	30
$x_1 \prec x_3 \prec x_2$	5	0	5
$x_2 \prec x_1 \prec x_3$	0	20	20
$x_1 \prec x_2 \prec x_3$	13	0	13
Dice system results	$x_3 \prec x_2 \prec x_1$	$x_3 \prec x_2 \prec x_1$	$x_3 \prec x_1 \prec x_2$

This should not be very surprising, because the separate results within Profile A and Profile B do not reflect any cross-profile comparisons.

A pairwise voting system satisfies *cancellation* if for every strict profile in which Condorcet's method indicates that the candidates are all tied, this system also indicates that the candidates are all tied. CM, AV, CAV and BC all satisfy this property but DV does not.

$x_4x_2x_3x_1$	$x_2x_3x_1x_4$	$x_1x_3x_2x_4$	$x_4x_3x_1x_2$	$x_4x_2x_1x_3$
3	1	4	1	1

For instance, in the strict 4-candidate profile indicated above, BC and CM indicate that all the candidates are tied. Candidate x_1 has the die (0,0,0,0,2,2,2,3,3,3), x_2 has the die (0,1,1,1,1,2,2,2,2,3), x_3 has the die (1,1,1,1,1,1,2,2,2,3) and x_4 has the die (0,0,0,0,0,3,3,3,3,3); we leave it to the reader to verify that $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_1$ according to DV.

Another familiar property is *Independence of Irrelevant Alternatives* (IIA). Let P be a strict profile, and let $P - x_r$ denote the strict profile obtained from P by removing x_r from the slate of candidates, and recalculating the ratings according to Definition 2. Then a strict pairwise voting system satisfies IIA if for every choice of $s, t \in \{1, ..., m\} - \{r\}$, the societal preference between x_s and

x_t in $P - x_r$ is the same as the societal preference between x_s and x_t in P. The recalculation of ratings can affect cross-ballot inequalities, and can change the candidates' rating totals; consequently it is no surprise that of the strict pairwise voting systems of Section 3, only Condorcet's method satisfies IIA. The Efron dice system and the Borda Count both violate IIA in Profile A discussed in the paragraph before last.

6 Dice and the Borda Count

In this section we recall several theorems relating Condorcet's method to the Borda Count. These results have close analogues relating the Efron dice system to the Borda Count.

6.1 Aggregation

Suppose P is a strict profile, and x_r is one of the candidates. Saari [6] calls the sum

$$\sum_{s \neq r} |\{i | x_r \succ_i x_s\}|$$

the *aggregated pairwise tally* of x_r. It is the number of times a voter chooses x_r over some other candidate.

Theorem 14 *[6] The aggregated pairwise tally of x_r is the same as the Borda score for x_r.*

Proof. We simply reorganize the sum:

$$\sum_{s \neq r} |\{i | x_r \succ_i x_s\}| = \sum_{s \neq r} \sum_{i=1}^{n} \begin{cases} 1, \text{ if } x_r \succ_i x_s \\ 0, \text{ if } x_s \succ_i x_r \end{cases}$$

$$= \sum_{i=1}^{n} \sum_{s \neq r} \begin{cases} 1, \text{ if } x_r \succ_i x_s \\ 0, \text{ if } x_s \succ_i x_r \end{cases} = \sum_{i=1}^{n} \rho_{ir}.$$

■

It turns out that dice results can be aggregated in an analogous way to obtain the Borda score. The aggregation takes all rolls of the dice into account, including ties.

Definition 15 *The* aggregated dice tally *of x_r is the number of rolls won by x_r against other candidates, plus half the number of rolls tied by x_r against other candidates:*

$$\sum_{s \neq r} |\{(i,j) | \rho_{ir} > \rho_{js}\}| + \frac{1}{2} \sum_{s \neq r} |\{(i,j) | \rho_{ir} = \rho_{js}\}|.$$

Theorem 16 *If P is a strict profile then the aggregated dice tally of x_r is the product of the number of voters and the Borda score of x_r.*

Proof. Again, we simply reorganize the sum:

$$\sum_{s\neq r}|\{(i,j)|\rho_{ir}>\rho_{js}\}|+\frac{1}{2}\sum_{s\neq r}|\{(i,j)|\rho_{ir}=\rho_{js}\}|=\sum_{s\neq r}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\begin{array}{l}1,\text{ if }\rho_{ir}>\rho_{js}\\ \frac{1}{2},\text{ if }\rho_{ir}=\rho_{js}\\ 0,\text{ if }\rho_{ir}<\rho_{js}\end{array}\right.$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s\neq r}\left\{\begin{array}{l}1,\text{ if }\rho_{ir}>\rho_{js}\\ \frac{1}{2},\text{ if }\rho_{ir}=\rho_{js}\\ 0,\text{ if }\rho_{ir}<\rho_{js}\end{array}\right.=\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\begin{array}{c}\rho_{ir}+\frac{1}{2},\text{ if }\rho_{ir}<\rho_{jr}\\ \rho_{ir},\text{ if }\rho_{ir}=\rho_{jr}\\ \rho_{ir}-\frac{1}{2},\text{ if }\rho_{ir}>\rho_{js}\end{array}\right.$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ir}+\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\begin{array}{c}\frac{1}{2},\text{ if }\rho_{ir}<\rho_{jr}\\ 0,\text{ if }\rho_{ir}=\rho_{jr}\\ -\frac{1}{2},\text{ if }\rho_{ir}>\rho_{js}\end{array}\right.$$

$$=n\sum_{i=1}^{n}\rho_{ir}+\sum_{i=1}^{n}\sum_{j=i}^{n}\left\{\begin{array}{c}\pm\frac{1}{2}\mp\frac{1}{2},\text{ if }\rho_{ir}\neq\rho_{jr}\\ 0\text{ or }0+0,\text{ if }\rho_{ir}=\rho_{jr}\end{array}\right.=n\sum_{i=1}^{n}\rho_{ir}+0.$$

■

Theorem 17 *[6] Suppose P is a strict profile and candidate x_r has $x_r \succsim x_s$ $\forall s$ according to CM, i.e., no other candidate is preferred over x_r by a majority of the voters. Then one of these conditions must hold: (a) there is at least one s with $x_r \succ x_s$ according to BC, and at least one t with $x_r \succ x_t$ according to CM; or (b) all the candidates are tied according to BC, and $x_r \sim x_s$ $\forall s$ according to CM.*

Proof. The hypothesis $x_r \succsim x_s$ $\forall s$ implies that the aggregated pairwise tally of x_r satisfies

$$\begin{aligned}\sum_{s\neq r}|\{i|x_r\succ_i x_s\}| &\geq \sum_{s\neq r}|\{i|x_s\succ_i x_r\}|\\ &=\sum_{s\neq r}(n-|\{i|x_r\succ_i x_s\}|)=n(m-1)-\sum_{s\neq r}|\{i|x_r\succ_i x_s\}|\end{aligned}$$

and hence

$$2\cdot\sum_{s\neq r}|\{i|x_r\succ_i x_s\}|\geq n(m-1).$$

Consequently the Borda score for x_r is at least $\frac{n(m-1)}{2}$. Observe also that if $x_r \succ x_t$ for some t according to CM then the inequalities are strict and the Borda score of x_r is strictly greater than $\frac{n(m-1)}{2}$.

For n voters and m candidates there are $n\sum_{i=0}^{m-1} i = n\frac{m(m-1)}{2}$ points available in the Borda process; consequently if it is not the case that the candidates are all tied according to BC, then there is at least one candidate x_s whose Borda score is strictly less than the average, $n\frac{(m-1)}{2}$. Then $x_r \succ x_s$ according to BC.

If instead the candidates are all tied according to BC, then each of them must have Borda score $n\frac{(m-1)}{2}$. As observed above, the Borda score of x_r is strictly greater than $n\frac{(m-1)}{2}$ unless there is no s with $x_r \succ x_s$ according to CM. ■

In case (b) it is not necessarily true that all the candidates are tied according to CM, because there may be Condorcet cycles involving candidates other than x_r.

Here is the analogous result involving the Efron dice system.

Theorem 18 *Suppose P is a strict profile and x_r has $x_r \succsim x_s$ $\forall s$ according to DV. Then either (a) there is at least one s with $x_r \succ x_s$ according to BC, and at least one t with $x_r \succ x_t$ according to DV; or (b) all the candidates are tied according to BC, and $x_r \sim x_s$ $\forall s$ according to DV.*

Proof. The proof is essentially the same as the proof of Theorem 17. As $x_r \succsim x_s$ $\forall s$, the aggregated dice tally of x_r cannot be below the average, and will be strictly greater than the average if x_r wins any pairwise dice contest. Consequently Theorem 16 tells us that the Borda score of x_r cannot be below the average, and will be strictly greater than the average if x_r wins any pairwise dice contest. If there is one x_s whose Borda score is strictly less than the average, then $x_r \succ x_s$ according to BC. If there is no such x_s, then the candidates must all be tied according to BC. This cannot happen if x_r wins any pairwise dice contest. ■

Reversing the inequalities in the proofs, we have the “opposite” results.

Theorem 19 *[6] Suppose P is a strict profile and x_r has $x_s \succsim x_r$ $\forall s$ according to CM. Then either (a) there is at least one s with $x_s \succ x_r$ according to BC, and there is at least one t with $x_t \succ x_r$ according to CM; or (b) all the candidates are tied according to BC, and $x_r \sim x_s$ $\forall s$ according to CM.*

Theorem 20 *Suppose P is a strict profile and x_r has $x_s \succsim x_r$ $\forall s$ according to DV. Then either (a) there is at least one s with $x_s \succ x_r$ according to BC, and there is at least one t with $x_t \succ x_r$ according to DV; or (b) all the candidates are tied according to BC, and $x_r \sim x_s$ $\forall s$ according to DV.*

It is not possible to strengthen “at least one” in these results. For instance, in the following profile $x_1 \succ x_s$ $\forall s > 1$ according to both CM and DV, and $x_2 \succ x_4 \succ x_1 \succ x_3$ according to BC.

ballot (in order of increasing preference)	number of ballots
$x_4x_3x_2x_1$	2
$x_3x_4x_2x_1$	1
$x_3x_2x_4x_1$	2
$x_1x_3x_4x_2$	1
$x_1x_3x_2x_4$	3

The agreement between CM and DV in this profile is coincidental. In general, CM and DV may almost always disagree with each other; examples are given in Section 7.

It appears from computer testing that if BC and DV give opposite societal preferences for two candidates, then the greatest possible discrepancy in the Borda scores will occur when one candidate receives only second-place votes, and the other receives a bare majority of first-place votes, supplemented by last-place votes. This example indicates that the difference between the Borda scores of two dice in an m-candidate, n-voter profile, in which the die with the smaller Borda score is favored by DV, can be as large as $n(m-2)-\left\lfloor \frac{n+1}{2} \right\rfloor (m-1)$.

6.2 Subset relations

Subset relations for the Borda Count have been explored by Saari in [5] and other papers. These relations are suggested by the voting system property IIA, but are enough weaker that they do not give rise to impossibility results. We first present a subset relation for the Borda Count, which is somewhat more general than those of [5]; then we show how analogous subset relations for the Efron dice voting system are derived from the Borda relations.

If P is a strict profile and $S \subseteq \{x_1, ..., x_m\}$ then we use $P|S$ to denote the strict profile obtained in the natural way from P by restricting the voters' preference orders to the candidates in S.

Theorem 21 *Let P be a strict profile, and suppose $\mathcal{S}$ is a family of subsets of $\{x_1, ..., x_m\}$ such that x_1 is contained in every $S \in \mathcal{S}$ and every other candidate is contained in exactly k of the $S \in \mathcal{S}$, $k > 0$. Suppose further that for each $S \in \mathcal{S}$, the Borda score of x_1 in $P|S$ is at least $n \cdot \frac{|S|-1}{2}$, the average Borda score in $P|S$. Then the Borda score of x_1 in P is at least the average value, $n \cdot \frac{m-1}{2}$. If in addition the Borda score of x_1 is strictly greater than $n \cdot \frac{|S|-1}{2}$ in at least one restriction $P|S$ with $S \in \mathcal{S}$, then the Borda score of x_1 is strictly greater than $n \cdot \frac{m-1}{2}$ in P.*

Proof. According to Theorem 14, for each $S \in \mathcal{S}$ the sum

$$\sum_{\substack{x_s \in S \\ s>1}} |\{i \in \{i, ..., n\} | x_1 \succ_i x_s\}|$$

is the Borda score of x_1 in S. As the Borda score of x_1 in S is at least $n \cdot \frac{|S|-1}{2}$ by hypothesis, we conclude that

$$\sum_{\substack{x_s \in S \\ s>1}} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}| \geq n \cdot \frac{|S|-1}{2}$$

for each $S \in \mathcal{S}$ and consequently

$$\sum_{S \in \mathcal{S}} \sum_{\substack{x_s \in S \\ s>1}} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}| \geq \binom{n}{2} \cdot \sum_{S \in \mathcal{S}} (|S|-1).$$

As each $x_s \neq x_1$ is contained in precisely k sets $S \in \mathcal{S}$, it follows that

$$k \cdot \sum_{s>1} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}|$$

$$= \sum_{S \in \mathcal{S}} \sum_{\substack{x_s \in S \\ s>1}} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}| \geq \binom{n}{2} \cdot (m-1) \cdot k,$$

and hence

$$\sum_{s>1} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}| \geq \binom{n}{2} \cdot (m-1).$$

The closing sentence of the statement follows from the fact that if there is at least one $S \in \mathcal{S}$ with

$$\sum_{\substack{x_s \in S \\ s>1}} |\{i \in \{1, ..., n\} | x_1 \succ_i x_s\}| > n \cdot \binom{|S|-1}{2}$$

then the inequalities obtained by summing over $\mathcal{S}$ are also strict. ■

We conclude that dice results that are consistent in certain families of restrictions cannot be completely reversed in P:

Corollary 22 *Let P be a strict profile, and suppose $\mathcal{S}$ is a family of subsets of $\{x_1, ..., x_m\}$ such that x_1 is contained in every $S \in \mathcal{S}$ and every other candidate is contained in exactly k of the $S \in \mathcal{S}$, $k > 0$. Suppose further that for each restricted profile $P|S$ with $S \in \mathcal{S}$, DV asserts that $x_1 \succsim x_s \; \forall x_s \in S$. Then there must be at least one $s > 1$ with $x_1 \succsim x_s$ in P according to DV. If in addition there is at least one restriction $P|S$ with $S \in \mathcal{S}$ such that $x_1 \succ x_s$ for at least one $x_s \in S$ according to DV, then there is at least one $s > 1$ such that $x_1 \succ x_s$ in P according to DV.*

Proof. By Theorem 16, the hypotheses of the corollary imply the hypotheses of Theorem 21. Also, the negations of the conclusions of the corollary imply the negations of the conclusions of Theorem 21. Consequently, if the hypotheses of the corollary hold, then the negations of the conclusions cannot. ■

We leave it to the reader to formulate analogous results obtained by reversing all the inequalities and preferences in Theorem 21 and Corollary 22. There is also an analogue of Theorem 21 with equalities and indifferences, but there is no such analogue of Corollary 22 as DV does not satisfy cancellation.

7 Dice and Condorcet's method

In this section we explore the relationship between the Efron dice system and Condorcet's method. There is no analogue of the aggregation result of Theorem 16, but there are the following analogues of Theorem 17 and Theorem 18.

Theorem 23 *Suppose P is a strict profile and x_r is a candidate with $x_r \succsim x_s$ $\forall s$ according to CM. Then either (a) there is at least one s with $x_r \succ x_s$ according to DV; or (b) $x_r \sim x_s$ $\forall s$ according to both CM and DV.*

Proof. According to Theorem 14 the hypothesis that $x_r \succsim x_s$ $\forall s$ according to CM tells us that the Borda score of x_r is no less than the average, and equals the average only if $x_r \sim x_s$ $\forall s$ according to CM.

Suppose condition (a) of the statement fails; then $x_s \succsim x_r$ $\forall s$ according to DV. Theorem 16 tells us that the Borda score of x_r is no more than the average, and equals the average only if $x_r \sim x_s$ $\forall s$ according to DV. Considering the first sentence of the proof, we see that the failure of (a) implies that the Borda score of x_r must equal the average; under the circumstances this implies that $x_r \sim x_s$ $\forall s$ according to both CM and DV. ∎

Observe that Theorem 23 is not completely analogous to Theorem 17. As DV does not satisfy cancellation, it is possible for a strict profile to yield $x_r \sim x_s$ $\forall s$ according to CM, but not according to DV; an example appears in Section 5.

A direct consequence of Theorem 23 is the following.

Corollary 24 *If P is a strict profile and $x_r \succ x_s$ $\forall s \neq r$ according to CM, then there is at least one s with $x_r \succ x_s$ according to DV.*

The property of DV given in Corollary 24 is mentioned by Saari [4, 5], as part of a characterization of BC among *positional voting methods*, which are pairwise voting systems defined by assigning each candidate a score according to the number of voters who rank that candidate first, second, third, etc. The fact that DV satisfies the property, even though it can disagree with BC, does not contradict Saari's result because DV is not a positional voting method.

Theorem 25 *Suppose P is a strict profile and x_r is a candidate with $x_r \succsim x_s$ $\forall s$ according to DV. Then either (a) there is at least one s with $x_r \succ x_s$ according to CM or (b) $x_r \sim x_s$ $\forall s$ according to both CM and DV.*

Corollary 26 *If P is a strict profile and $x_r \succ x_s$ $\forall s \neq r$ according to DV, then there is at least one s with $x_r \succ x_s$ according to CM.*

Proof. The arguments are essentially the same as the proofs of Theorem 23 and Corollary 24. Alternatively, we may deduce these results from Corollary 22. ∎

As in Section 6, there are "opposite" results involving a candidate x_r that is not preferred over any other; we leave it to the reader to formulate them. The examples described in Theorems 27 and 29 show that it is not generally possible to improve the "at least one" in part (a) of Theorem 23 or 25.

Theorem 27 *If $m \geq 3$ then for all $n \geq 8m - 15$ there are strict profiles with n voters and m candidates, such that $x_1 \succ x_r$ $\forall r > 1$ according to CM and x_1 loses all but one of its pairwise DV contests.*

Proof. Consider a profile in which $\lfloor \frac{n}{2} \rfloor + 1$ voters - that is, slightly more than half of the voters in the profile - rank the candidates in the following order: $x_1 \succ x_2 \succ \cdots \succ x_m$. We construct the rest of the profile as follows. First, all of the remaining $n - 1 - \lfloor \frac{n}{2} \rfloor$ voters rank x_1 last and x_m second-to-last. Second, the first-place votes among these voters are distributed as evenly as possible among $x_2, ..., x_{m-1}$. As $n \geq 8m - 15$, $n - 1 - \lfloor \frac{n}{2} \rfloor \geq 4m - 8$ and consequently each of $x_2, ..., x_{m-1}$ receives at least four first-place votes.

Because x_1 is ranked first by more than half of the voters, $x_1 \succ x_r \; \forall r > 1$ according to CM. We claim that whenever $1 < r < m$, $x_r \succ x_1$ according to DV. The claim is verified by counting rolls of the dice. The die corresponding to x_1 has $\lfloor \frac{n}{2} \rfloor + 1$ labels equal to $m - 1$ and $n - 1 - \lfloor \frac{n}{2} \rfloor$ labels equal to 0, while the die corresponding to x_r has at least four labels equal to $m - 1$, and all labels greater than 0. Consequently no more than

$$(n-4)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$$

rolls of the dice favor x_1, and precisely

$$n(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor)$$

rolls of the dice favor x_r. The claim follows from the fact that the difference is negative:

$$\begin{aligned}(n-4)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - n\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) &= (2n-4)\left\lfloor \frac{n}{2} \right\rfloor + 2n - 4 - n^2 \\ \leq (2n-4)\left(\frac{n}{2}\right) + 2n - 4 - n^2 &= n^2 - 2n + 2n - 4 - n^2 = -4.\end{aligned}$$

■

The inequality $n \geq 8m - 15$ is not a tight bound. For instance, if n is odd then the construction used in the proof of Theorem 27 requires only two first-place votes for each of $x_2, ..., x_{m-1}$, because

$$\begin{aligned}(n-2)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - n\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) &= (2n-2)\left\lfloor \frac{n}{2} \right\rfloor + 2n - 2 - n^2 \\ = (2n-2)\left(\frac{n-1}{2}\right) + 2n - 2 - n^2 &= n^2 - 2n + 1 + 2n - 2 - n^2 = -1.\end{aligned}$$

Consequently if n is odd and $n \geq 4m - 7$ then there are strict profiles with n voters and m candidates, such that $x_1 \succ x_r \; \forall r > 1$ according to CM and x_1 loses all but one of its pairwise DV contests.

Even though the bound $n \geq 8m - 15$ is not tight, examples like those of Theorem 27 cannot have arbitrarily small numbers of voters; indeed $n < m - 1$ is impossible in such a profile.

Proposition 28 *Let P be a strict profile with n voters and m candidates, such that $x_1 \succ x_r \; \forall r > 1$ according to CM. Then x_1 wins at least $\frac{m-1}{n}$ of its pairwise DV contests.*

Proof. Let P be a strict profile, and let b denote the Borda score of x_1. According to Theorem 14,

$$b = \sum_{s>1} |\{i|x_1 \succ_i x_s\}| \,.$$

As $x_1 \succ x_s$ $\forall s > 1$ according to CM, $|\{i|x_s \succ_i x_1\}| \leq |\{i|x_1 \succ_i x_s\}| - 1$ $\forall s > 1$ and hence

$$\begin{aligned}(m-1)n &= \sum_{s>1} |\{i|x_1 \succ_i x_s\}| + \sum_{s>1} |\{i|x_s \succ_i x_1\}| \\ &\leq \sum_{s>1} |\{i|x_1 \succ_i x_s\}| + \sum_{s>1} (|\{i|x_1 \succ_i x_s\}| - 1) \\ &= 2 \cdot \sum_{s>1} |\{i|x_1 \succ_i x_s\}| - (m-1)\end{aligned}$$

and consequently

$$(m-1)(n+1) \leq 2 \cdot \sum_{s>1} |\{i|x_1 \succ_i x_s\}| = 2b.$$

By Theorem 16, nb equals the aggregated dice tally of x_1. It follows that the aggregated dice tally of x_1 cannot be less than

$$\frac{(m-1)n(n+1)}{2}.$$

The total number of rolls of x_1's die against the die corresponding to an x_s with $s > 1$ is n^2, so if x_1 loses or ties the dice contest with x_s then the aggregated dice tally of x_1 cannot include more than $\frac{n^2}{2}$ winning rolls against x_s. Consequently if x_1 wins precisely k pairwise DV contests then it must be that

$$\frac{(m-1)n(n+1)}{2} \leq kn^2 + (m-1-k)\left(\frac{n^2}{2}\right)$$

and hence

$$\frac{(m-1)(n+1)}{n} \leq 2k + (m-1-k) = m+k-1.$$

This requires

$$k \geq \frac{(m-1)(n+1)}{n} - (m-1) = \frac{m-1}{n}.$$

■

Theorem 29 *If $m \geq 3$ then there are strict profiles with m candidates, such that $x_1 \succ x_r$ $\forall r > 1$ according to DV and x_1 loses all but one of its pairwise CM contests.*

Proof. If $m = 3$ an example is provided by Profile 1 mentioned in the proof of Theorem 13.

Suppose $m \geq 4$. We begin with a profile P_0 consisting of $(m-1)!$ voters, in which every possible strict ballot involving $x_1, ..., x_{m-1}$ occurs once. A new profile P is constructed by modifying P_0 in two ways. First, we adjoin one new candidate x_m, with the proviso that every voter's preference order lists x_m directly after x_1. Second, we adjoin one new voter, with preference order $x_{m-1} \succ x_{m-2} \succ ... \succ x_1 \succ x_m$. Clearly then every candidate x_r with $2 \leq r \leq m-1$ has $x_r \succ x_1$ according to CM, and also $x_1 \succ x_m$ according to CM.

We claim that $x_1 \succ x_{m-1}$ according to DV. Observe that the die corresponding to x_1 has $(m-2)!$ labels equal to k for each $k \in \{2, ..., m-1\}$, and $1 + (m-2)!$ labels equal to 1. The die corresponding to x_{m-1} has $(m-2)!$ labels equal to 0, one for each ballot of P_0 on which x_{m-1} is least preferred. However there are only $(m-3) \cdot (m-3)!$ labels equal to 1, corresponding to the ballots of P_0 on which x_{m-1} is next-to-last and x_1 is not last. There are also $(m-3) \cdot (m-3)!$ labels equal to 2; $(m-4) \cdot (m-3)!$ of them correspond to the ballots of P_0 on which x_{m-1} is next-to-next-to-last and $x_1 \succ x_{m-1}$, and another $(m-3)!$ of them correspond to ballots of P_0 on which x_{m-1} is next-to-last and x_1 is last. Similarly, there are $(m-3) \cdot (m-3)!$ labels equal to k whenever $2 \leq k \leq m-2$. The remaining $1 + (m-2)!$ labels of the die corresponding to x_{m-1} all equal $m-1$.

Consequently the die corresponding to x_{m-1} may be obtained from the die corresponding to x_1 by making the following changes: a single label is changed from $m-2$ to $m-1$; $(m-3)! - 1$ labels are changed from $m-2$ to 0; for $2 \leq k \leq m-3$, $(m-3)!$ labels are changed from k to 0; and $(m-3)! + 1$ labels are changed from 1 to 0.

To justify the claim, observe first that in the set of all possible rolls of the die corresponding to x_1 against an identical replica of itself, each die must have the same number of wins and losses. (This observation follows immediately from the fact that the two are identical.) Now consider what happens if we change the second of these identical dice into x_{m-1}'s die. The single change of one label from $m-2$ to $m-1$ introduces $(m-2)!$ new winning rolls for the second die, one for each roll in which this label is matched against a label of $m-2$ on x_1's die. Every other label change introduces at least $(m-2)!$ new losing rolls for the second die, though. There are at least two of these label-reducing changes, as $m \geq 4$ implies that $(m-3)! + 1 \geq 2$. Consequently the second die must do worse overall after the changes; that is, $x_1 \succ x_{m-1}$ according to DV.

If $2 \leq r < m-1$ then the die corresponding to x_r is obtained from the die corresponding to x_{m-1} by reducing one label, and clearly this cannot improve the die's performance in any dice contest; hence the fact that DV asserts $x_1 \succ x_{m-1}$ implies that DV also asserts $x_1 \succ x_r$.

As every voter prefers x_1 over x_m, the die corresponding to x_m is obtained from x_1's die by reducing every label; consequently it is obvious that $x_1 \succ x_m$ according to DV. ■

The examples given in Theorem 29 involve many more voters than those of Theorem 27; we do not know whether or not it is actually necessary to have so many. An argument like that of Proposition 28 tells us only that $n < \sqrt{m-1}$ is impossible in such a profile.

Proposition 30 *Let P be a strict profile with n voters and m candidates, such that $x_1 \succ x_r$ $\forall r > 1$ according to DV. Then x_1 wins at least $\frac{m-1}{n^2}$ of its pairwise CM contests.*

Proof. Let P be a strict profile, and let b denote the Borda score of x_1. According to Theorem 16,

$$nb = \sum_{s>1} |\{(i,j)|\rho_{i1} > \rho_{js}\}| + \frac{1}{2}\sum_{s>1} |\{(i,j)|\rho_{i1} = \rho_{js}\}|.$$

As $x_1 \succ x_r$ $\forall r > 1$ according to DV,

$$(m-1)n^2 =$$

$$\sum_{s>1} |\{(i,j)|\rho_{i1} > \rho_{js}\}| + \sum_{s>1} |\{(i,j)|\rho_{i1} = \rho_{js}\}| + \sum_{s>1} |\{(i,j)|\rho_{is} > \rho_{j1}\}|$$

$$\leq \sum_{s>1} |\{(i,j)|\rho_{i1} > \rho_{js}\}| + \sum_{s>1} |\{(i,j)|\rho_{i1} = \rho_{js}\}|$$

$$+ \sum_{s>1} |\{(i,j)|\rho_{i1} > \rho_{js}\}| - (m-1),$$

so

$$(m-1)(n^2+1) \leq 2nb.$$

Theorem 14 tells us that the pairwise tally of x_1 is b, and the inequality above tells us that it cannot be less than

$$\frac{(m-1)(n^2+1)}{2n}.$$

If x_1 wins precisely k pairwise CM contests then the pairwise tally of x_1 is at most

$$kn + (m-1-k)\left(\frac{n}{2}\right).$$

Consequently

$$\frac{(m-1)(n^2+1)}{2n} \leq kn + (m-1-k)\left(\frac{n}{2}\right),$$

so

$$\frac{(m-1)(n^2+1)}{n^2} \leq 2k + (m-1-k) = k + (m-1).$$

This requires

$$\frac{(m-1)(n^2+1)}{n^2} - (m-1) = \frac{m-1}{n^2} \leq k.$$

■

8 Social choice functions and the Borda Count

Young [9] showed that if a social choice function defined on all strict profiles satisfies cancellation, consistency, faithfulness and neutrality, then it is equivalent to the social choice function associated with the Borda Count. Lemma 1 of [9] tells us that cancellation and consistency together imply anonymity, so Young's characterization includes anonymity implicitly. In Corollary 3.2 of [5], Saari observed that Young's characterization still holds if anonymity is included as an explicit requirement, and cancellation is replaced by some other (possibly much weaker) kind of compatibility between a social choice function and pairwise preferences. In this section we prove that cancellation may also be replaced in Saari's version of Young's characterization by requiring compatibility with the Efron dice voting system, rather than pairwise preferences.

Definition 31 *A* social choice function *is a function that assigns to each strict profile a non-empty subset W of $\{x_1, ..., x_m\}$, called the* choice set *or the* winning set*.*

The *trivial* social choice function declares all candidates to be winners; it has $W = \{x_1, ..., x_m\}$ for every profile.

Any pairwise voting system gives rise to a social choice function in a natural way: the choice set W is defined to be the intersection of all the nonempty subsets $X \subseteq \{x_1, ..., x_m\}$ such that no candidate $x_r \notin X$ has $x_r \succsim x_s$ for any $x_s \in X$. (The fact that the intersection must be nonempty is readily deduced from the fact that for any two such sets X and X', $X \subseteq X'$ or $X' \subseteq X$.) Observe that we lose a great deal of information when we construct the social choice function associated with a non-transitive pairwise voting system; for instance, if CM has $x_1 \succsim x_2 \succsim \cdots \succsim x_m \succsim x_1$ then the choice set is $\{x_1, ..., x_m\}$ no matter how many of the $\succsim$ are actually $\sim$.

Every pairwise voting system has an *opposite*, obtained by reversing all societal preferences. For each strict profile P, the choice set of the social choice function associated with the opposite is the *losing set* of candidates of P according to the original voting system.

8.1 Cancellation

Definition 32 *A social choice function satisfies* cancellation *if for every strict profile in which CM indicates that all the candidates are tied pairwise, the choice set includes all the candidates.*

Definition 33 *A social choice function satisfies* Borda cancellation *if for every strict profile in which BC indicates that all the candidates are tied pairwise, the choice set includes all the candidates.*

Definition 34 *A social choice function satisfies* dice cancellation *if for every strict profile in which DV indicates that all the candidates are tied pairwise, the choice set includes all the candidates.*

These definitions are not equivalent to the analogous definitions for pairwise voting systems, because the social choice functions associated with pairwise voting systems are so much less informative. An example given in Section 5 shows that the Efron dice voting system does not satisfy cancellation, as the candidates are all tied under CM but form a cycle under DV; the same example shows that DV does not satisfy Borda cancellation. Similarly, in the standard Condorcet cycle (i.e., the strict profile with $m = n$, in which each ballot $x_r \succ \cdots \succ x_m \succ x_1 \succ \cdots \succ x_{r-1}$ occurs once) the candidates all have the same die, so they are all tied under DV; as $x_1 \succ \cdots \succ x_m \succ x_1$ under CM, we conclude that CM does not satisfy dice cancellation. The same example shows that CM does not satisfy Borda cancellation. In contrast, we have the following.

Theorem 35 *The social choice functions associated with BC, CM and DV satisfy all three types of cancellation.*

Proof. Clearly each of the three social choice functions satisfies cancellation with respect to the pairwise voting system from which it is derived. It is also easy to see that the social choice function associated with the Borda Count satisfies Definitions 32 and 34: if all candidates are tied under CM (or DV), then they all have the same aggregated (dice) tally, so they all have the same Borda score.

Let P be a strict profile in which the choice set of the social choice function associated with CM is a proper subset $W \subset \{x_1, ..., x_m\}$. Say $1 \leq w < m$ and $W = \{x_1, ..., x_w\}$. We claim that then $x_1, ..., x_m$ do not all have the same Borda score. Recall that the Borda score of a candidate is the same as the aggregated pairwise tally. For each candidate among $x_1, ..., x_w$, we split the aggregated tally into two parts: the first part tallies the number of times that candidate is preferred over another one of $x_1, ..., x_w$, and the second part tallies the number of times that candidate is preferred over one of $x_{w+1}, ..., x_m$. Suppose the first part of the aggregated tally of x_1 is the largest; then it cannot be less than the average, which is $n \cdot \frac{w-1}{2}$. As x_1 is strictly preferred over every one of $x_{w+1}, ..., x_m$ according to CM, the second part of the aggregated tally of x_1 must be strictly greater than $(m - w) \cdot \frac{n}{2}$. Consequently the Borda score of x_1 is strictly greater than

$$n \cdot \left(\frac{w-1}{2}\right) + (m - w) \cdot \frac{n}{2} = \frac{(m-1)n}{2},$$

which is the average Borda score. This verifies the claim that $x_1, ..., x_m$ do not all have the same Borda score.

A similar argument using aggregated dice tallies shows that if P is a strict profile in which the choice set of the social choice function associated with DV does not include all the candidates, then the candidates are not all tied according to BC. We conclude that if P is a strict profile in which all candidates are tied according to BC, then the choice sets of the social choice functions associated with CM and DV must include all the candidates. That is, the social choice functions associated with CM and DV both satisfy Borda cancellation.

It follows that the social choice function associated with CM or DV satisfies cancellation with respect to the other voting system, because a strict profile P in which all candidates are tied according to CM or DV must also have all candidates tied according to BC. ■

8.2 Simple scoring functions

We recall some ideas used by Young [10].

Definition 36 *Let* $\mathbf{s} = \langle s_1, ..., s_m \rangle \in \mathbb{R}^m$ *be a vector. Then the* simple scoring function *associated with* $\mathbf{s}$ *is the social choice function defined as follows. Suppose* P *is a strict* m*-candidate profile, and for* $r, k \in \{1, ..., m\}$ *let* β_{rk} *denote the number of voters for whom* x_r *is the* k^{th} *most-preferred candidate. Then the sum*

$$\sum_{k=1}^{m} \beta_{rk} \cdot s_k$$

is the $\mathbf{s}$ score *of* x_r*, and the choice set includes the candidate(s) with the largest* $\mathbf{s}$ *score.*

Proposition 37 *(a) The simple scoring function associated with* $\mathbf{s}$ *is the same as the social choice function associated with the Borda Count if, and only if,* $s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m > 0$.

(b) The simple scoring function associated with $\mathbf{s}$ *is the same as the social choice function associated with the opposite of the Borda Count if, and only if,* $s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m < 0$.

(c) The simple scoring function associated with $\mathbf{s}$ *is trivial if, and only if,* $s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m = 0$.

Proof. (a) If $s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m = \delta > 0$, then a candidate whose Borda score equals b receives an $\mathbf{s}$ score equal to $n \cdot s_m + \delta \cdot b$. As n and δ are positive and do not vary from candidate to candidate, it follows that the social choice function associated with BC is the same as the simple scoring function associated with $\mathbf{s}$.

Suppose conversely that the simple scoring function associated with $\mathbf{s}$ is the same as the social choice function associated with the Borda Count. Suppose further that there is an r_0 with $s_{r_0} \leq s_{r_0+1}$. Consider the strict m-candidate profile P in which every possible strict ballot occurs precisely once. Let P' be the profile obtained from P by changing every ballot of P that has x_1 in the r_0^{th} most-preferred position and x_r in the $(r_0+1)^{st}$ most-preferred position, so that in P' the corresponding ballot has x_r in the r_0^{th} most-preferred position and x_1 in the $(r_0+1)^{st}$ most-preferred position. Then the candidates $x_2, ..., x_m$ are all equivalent to each other according to P', i.e., each of them receives the same number of first-place votes, the same number of second-place votes, and so on. Consequently $x_2, ..., x_m$ all receive both the same Borda score and the same $\mathbf{s}$ score. The Borda score of x_1 is lower than the common Borda score of $x_2, ..., x_m$, so the BC choice set is $\{x_2, ..., x_m\}$. However the $\mathbf{s}$ score of x_1 is no less than the $\mathbf{s}$ score shared by the other candidates, so the choice set of the simple scoring function associated with $\mathbf{s}$ is either $\{x_1\}$ or $\{x_1, ..., x_m\}$. Consequently the social choice function associated with BC is not the same as the simple scoring function associated with $\mathbf{s}$.

Finally, suppose $s_r > s_{r+1}$ for every r, but the differences $s_r - s_{r+1}$ are not all equal. Then there are r_0 and s_0 with $0 < s_{r_0} - s_{r_0+1} < s_{s_0} - s_{s_0+1}$. Let P again be the strict m-candidate profile in which every possible strict ballot occurs precisely once. Let P'' be the profile obtained from P by making the following changes: whenever a ballot of P has x_1 in the r_0^{th} most-preferred position, interchange x_1 with the candidate in the $(r_0+1)^{st}$ most-preferred position, and whenever a ballot of P has x_1 in the $(s_0+1)^{st}$ most-preferred position, interchange x_1 with the candidate in the s_0^{th} most-preferred position. The ballot-changes have no net effect on the Borda score of any candidate, as each candidate is moved up one place on a certain number of ballots, and moved down one place on the same number of ballots; hence the BC choice set of P'' is $\{x_1, ..., x_m\}$. On the other hand the ballot changes have a net positive effect on the $\mathbf{s}$ score of x_1 and a net negative effect on the $\mathbf{s}$ score of every other candidate, so $\{x_1\}$ is the choice set of the simple scoring function associated with $\mathbf{s}$. Once again, the social choice function associated with BC is not the same as the simple scoring function associated with $\mathbf{s}$.

This completes the proof of part (a).

The proof of part (b) is essentially the same; simply reverse every inequality. Part (c) is obvious. ∎

Theorem 38 *There are precisely three simple scoring functions that satisfy dice cancellation: the social choice function associated with the Borda Count, the social choice function associated with the opposite of the Borda Count, and the trivial social choice function.*

Proof. Let $\mathbf{s} = \langle s_1, ..., s_m \rangle \in \mathbb{R}^m$ be a scoring vector whose associated simple scoring function satisfies dice cancellation.

If $m = 2$ then the three possibilities mentioned in the statement correspond (respectively) to $s_1 > s_2$, $s_1 < s_2$ and $s_1 = s_2$.

If $m = 3$ then consider the 3-candidate profile P with two voters, one of whom has preferences $x_1 \succ x_2 \succ x_3$ and the other of whom has preferences $x_3 \succ x_2 \succ x_1$. The three candidates' dice are then $(0, 2)$, $(1, 1)$ and $(0, 2)$, so all three candidates are tied according to DV. As the simple scoring function associated with $\mathbf{s} = \langle s_1, s_2, s_3 \rangle$ satisfies dice cancellation, all three candidates must have the same $\mathbf{s}$ score. It follows that $s_1 + s_3 = 2s_2$, and hence $s_1 - s_2 = s_2 - s_3$. Proposition 37 tells us that the simple scoring function associated with $\mathbf{s}$ is one of the three mentioned in the statement.

Suppose now that $m \geq 4$.

Consider the strict profile P with two voters, one of whom has preferences $x_1 \succ x_2 \succ \cdots \succ x_m$ and the other of whom has preferences $x_m \succ x_{m-1} \succ$

$\cdots \succ x_1$. The candidates' dice are then $(0, m-1)$, $(1, m-2)$, $(2, m-3)$ and so on; clearly they are all tied according to DV. As the simple scoring function associated with $\mathbf{s} = \langle s_1, s_2, ..., s_m \rangle$ satisfies dice cancellation, the candidates must all have the same $\mathbf{s}$ score. It follows that $s_r + s_{m+1-r} = s_{r+1} + s_{m-r}$, and hence $s_r - s_{r+1} = s_{m-r} - s_{m+1-r}$, for every $r \in \{1, ..., m\}$. That is, $s_1 - s_2 = s_{m-1} - s_m$, $s_2 - s_3 = s_{m-2} - s_{m-1}$, and so on.

Suppose for the moment that the simple scoring function associated with $\mathbf{s}' = \langle s_1, ..., s_{m-1} \rangle$ satisfies dice cancellation. Considering the two-voter profile P' in which one voter has $x_1 \succ x_2 \succ \cdots \succ x_{m-1}$ and the other voter has $x_{m-1} \succ x_{m-2} \succ \cdots \succ x_1$, the argument of the previous paragraph tells us that $s_r + s_{m-r} = s_{r+1} + s_{m-r-1}$, and hence $s_r - s_{r+1} = s_{m-r-1} - s_{m-r}$, for every $r \in \{1, ..., m-1\}$. That is, $s_1 - s_2 = s_{m-2} - s_{m-1}$, $s_2 - s_3 = s_{m-3} - s_{m-2}$, and so on. Combining these equalities with those of the preceding paragraph we see that $s_{m-1} - s_m = s_1 - s_2 = s_{m-2} - s_{m-1} = s_2 - s_3 = s_{m-3} - s_{m-2} = ...$ It follows from Proposition 37 that the simple scoring function associated with $\mathbf{s}$ is one of the three mentioned in the statement.

It remains to consider the possibility that the simple scoring function associated with $\mathbf{s}' = \langle s_1, ..., s_{m-1} \rangle$ does not satisfy dice cancellation. Then there is a strict profile P with candidates $x_1, ..., x_{m-1}$ who are all tied according to DV, but do not all have the same $\mathbf{s}'$ score. Let n be the number of voters in the profile P. Theorem 16 tells us that $x_1, ..., x_{m-1}$ must all have the average Borda score in P, namely $\frac{n(m-2)}{2}$. According to Lemma 39 below, it follows that the dice corresponding to $x_1, ..., x_{m-1}$ must all be tied with any "even" die that has the same number of labels equal to each number in $\{0, ..., m-2\}$.

We construct from P a much larger strict profile Q with m candidates and $n \cdot m!$ voters, as follows.

(1) For each ballot in P, Q includes $(m-1)!$ ballots that match the original ballot, with x_m added as the least-preferred candidate.

(2) For each strict preference order of $x_1, ..., x_m$ that does not have x_m as the least-preferred candidate, Q has n ballots that match that preference order.

We claim that according to DV, the candidates $x_1, ..., x_m$ are all tied in Q. To verify the claim, we must determine the dice associated with the candidates. The die associated with x_m is simplest: each of the $n \cdot (m-1)!$ voters from (1) gives x_m a rating of 0, and for each $\rho \in \{1, ..., m-1\}$, $n \cdot (m-1)!$ of the voters from (2) give x_m a rating of ρ. Consequently x_m's die is the even die with $n \cdot (m-1)!$ instances of each label in $\{0, ..., m-1\}$. If $r < m$, then x_r's die in Q includes $(m-1)!$ copies of x_r's die from P, with each label increased by 1, from (1); $n \cdot (m-1)!$ labels of 0, from (2); and for each rating $\rho \in \{1, ..., m-1\}$, $n \cdot (m-2)! \cdot (m-2)$ labels equal to ρ, from (2).

Observe that if $r \in \{1, ..., m-1\}$ then the Borda score of x_r in Q is the sum of $(m-1)! \cdot (n + \frac{n(m-2)}{2})$, from (1), and

$$n(m-2)! \cdot (m-2) \sum_{\rho=1}^{m-1} \rho = n(m-2)! \cdot (m-2) \frac{m(m-1)}{2} = n(m-1)! \cdot \frac{m^2 - 2m}{2},$$

from (2). The sum is

$$n(m-1)! \cdot (1 + \frac{m-2}{2} + \frac{m^2-2m}{2}) = n(m-1)! \cdot \left(\frac{m^2-m}{2}\right) = \frac{n(m-1) \cdot m!}{2}.$$

As Q involves m candidates and $n \cdot m!$ voters, the average Borda score in Q is $nm! \cdot \frac{m-1}{2}$; hence x_r has the average Borda score. As x_m's die is even, Lemma 39 tells us that x_r and x_m are tied according to DV.

Suppose $r, s \in \{1, ..., m-1\}$. We list all the rolls of x_r's die in Q against x_s's die in Q: (a) there are $((m-1)!)^2$ instances of each roll of x_r's die in P against x_s's die in P, with both labels increased by 1; (b) there are $(m-1)! \cdot n \cdot (m-2)! \cdot (m-2)$ instances of each roll of x_r's die in P against the even die with one label equal to each of $\{0, ..., m-1\}$, with both labels increased by 1; (c) there are $(m-1)! \cdot n \cdot (m-2)! \cdot (m-2)$ instances of each roll of x_s's die in P against the even die with one label equal to each of $\{0, ..., m-1\}$, with both labels increased by 1; (d) there are $n \cdot (m-1)! \cdot (n \cdot (m-1)! + (m-1) \cdot n \cdot (m-2)! \cdot (m-2))$ rolls of a 0 of x_r's die against a higher roll of x_s's die; (e) there are $n \cdot (m-1)! \cdot (n \cdot (m-1)! + (m-1) \cdot n \cdot (m-2)! \cdot (m-2))$ rolls of a 0 of x_s's die against a higher roll of x_r's die; and (f) there are $(n \cdot (m-1)!)^2$ rolls of a 0 of x_r's die against a 0 of x_s's die. Each of (a), (b), (c) and (f) results in a "net" of 0 for each of x_r and x_s, and (d) and (e) cancel each other, so x_r and x_s are tied according to DV.

This verifies the claim that according to DV, the candidates $x_1, ..., x_m$ are all tied in Q. As the simple scoring function associated with $\mathbf{s}$ satisfies dice cancellation, it follows that $x_1, ..., x_m$ all have the same $\mathbf{s}$ score in Q. This is impossible, though, as the fact that $x_1, ..., x_{m-1}$ do not have the same $\mathbf{s}'$ score in P directly implies that they do not have the same $\mathbf{s}$ score in Q. By contradiction, it cannot happen that the simple scoring function associated with $\mathbf{s}' = \langle s_1, ..., s_{m-1} \rangle$ does not satisfy dice cancellation. ■

Lemma 39 *Suppose x_1 and x_2 are two candidates in a strict profile P, and suppose the die corresponding to x_1 is* even*, i.e., x_1 has the same number of labels equal to each $\lambda \in \{0, ..., m-1\}$. Then x_1 and x_2 are tied according to DV if, and only if, they are tied according to BC.*

Proof. Suppose the die corresponding to x_1 has k labels equal to each $\lambda \in \{0, ..., m-1\}$; then P includes $n = mk$ voters. When we roll the dice corresponding to x_1 and x_2 against each other, a label equal to ρ on x_2's die wins against each of the $k\rho$ smaller labels of x_1's die, and loses against each of the $k(m-1-\rho)$ larger labels of x_1's die. Consequently a label equal to ρ on x_2's die contributes $k \cdot (\rho - (m-1-\rho)) = 2k\rho - k(m-1)$ to the net win-loss difference of x_2's die against x_1's die. The Borda score b of x_2's die is the sum of the labels on the die, so the net win-loss difference is $2kb - nk(m-1)$. The two dice tie if and only if the net win-loss difference is 0, i.e., if and only if $b = \frac{n(m-1)}{2}$, the Borda score of x_1. ■

8.3 Theorem 1

Only a little more work is required to deduce Theorem 1 from Theorem 38.

Definition 40 *[10] A* scoring function *is a social choice function obtained from a finite sequence $f_1, ..., f_k$ of simple scoring functions as follows. Given a strict profile P, first obtain the choice set according to f_1. Then eliminate every candidate whose f_2-score is suboptimal in this choice set. Then eliminate every remaining candidate whose f_3-score is suboptimal, and repeat this tie-breaking process with $f_4, ..., f_k$ in order.*

For a social choice function, *faithfulness* asserts that when there is only one voter, the choice set includes only that voter's most-preferred candidate. Consequently a vector $\mathbf{s}$ gives rise to a faithful simple scoring function if and only if s_1 is strictly larger than every one of $s_2, ..., s_m$.

Theorem 41 *The only faithful scoring function that satisfies dice cancellation is the social choice function associated with the Borda Count.*

Proof. Let f be a faithful scoring function constructed from $f_1, ..., f_k$. Suppose f satisfies dice cancellation. If P is a strict profile in which DV indicates that all candidates are tied, then the choice set under f must include all the candidates. Consequently the choice set of f_1 must include all the candidates, and for $i \geq 2$, all the candidates must have the same f_i-score. That is, every one of $f_1, ..., f_k$ must have the property that its choice set includes all the candidates in P. As this holds for every such P , every one of $f_1, ..., f_k$ must satisfy dice cancellation.

Theorem 38 tells us that every one of $f_1, ..., f_k$ is one of these three: the social choice function associated with the Borda Count, the social choice function associated with the opposite of the Borda Count, or the trivial social choice function. It cannot be that $f_1, ..., f_k$ are all trivial, for then f would be trivial, an impossibility as the trivial social choice function is not faithful. Clearly f is unchanged if we simple remove every trivial f_i from $f_1, ..., f_k$.

Then every one of $f_1, ..., f_k$ is either the social choice function associated with the Borda Count or the social choice function associated with the opposite of the Borda Count. It follows that $f_2, ..., f_k$ will never actually break any ties within a choice set resulting from f_1, so $f = f_1$.

If $f = f_1$ were the social choice function associated with the opposite of the Borda Count, then clearly f would not be faithful: in a one-voter profile the choice set would include only the voter's least-preferred candidate. By contradiction, then, $f = f_1$ must be the social choice function associated with the Borda Count. ∎

Now suppose we have a social choice function that satisfies anonymity, consistency, faithfulness, neutrality and dice cancellation. Young [10] showed that anonymity, consistency and neutrality imply that the social choice function is a scoring function. According to Theorem 41, dice cancellation and faithfulness imply that the social choice function is the same as the one associated with BC.

Even though we have not discussed the definition of consistency for social choice functions, it is worth taking a moment to observe that it is an important condition in both Young's characterization and Theorem 1. Without consistency, the characterizations are obviously incomplete: the social choice function associated with CM satisfies all the other conditions of Young's original characterization, and the social choice function associated with DV satisfies all the other conditions of Theorem 1.

9 Closing comments

We hope that the results in this paper will encourage interest in the Efron dice voting system. There are many issues that would profit from further inquiry.

1. We have not studied strategic voting in the Efron dice system. It is clear that strategic voters can affect election outcomes by casting insincere ballots that harm the strongest competitors of favorite candidates, or by changing the way weakly k-chotomous preferences are expressed in ratings; but precise, general results remain to be developed.

2. We have mentioned many properties of the Efron dice voting system. Do some of these properties provide an axiomatic characterization of DV? Are there interesting voting systems that share some properties and not others?

3. Given a strict profile P of n voters and m candidates, the Efron dice voting system produces societal preferences by constructing, for each candidate, an n-sided die that records the voters' rankings of that candidate. Consider the set containing all n^m rolls of all of these dice, and treat each roll as one voter in a new, weakly m-chotomous profile P^m; the voter's ratings of the candidates are the same as the ratings that come up in that roll of the dice. Then the result of applying the Efron dice voting system to P is the same as the result of applying Condorcet's method to P^m. To see this, consider two candidates x_r and x_s. Then CM asserts that $x_r \succ x_s$ in P^m if among the n^m voters, more favor x_r over x_s than favor x_s over x_r. But these n^m voters may be partitioned into n^2 subsets of size n^{m-2}, with each subset given by a single roll of the dice corresponding to x_r and x_s in combination with all possible rolls of the other $m-2$ dice. Consequently the number of voters of P^m who favor x_r over x_s (resp. x_s over x_r) is simply n^{m-2} times the number of rolls of the dice corresponding to x_r and x_s that are won by x_r (resp. x_s).

As Condorcet's method and the Efron dice system may disagree on P, we conclude that the map $P \mapsto P^m$ may change CM's assessment of the societal preferences. In contrast, the aggregation results of Section 6 tell us that the Borda Count comes to the same conclusions on P and P^m. (Technically, we should not apply BC to P^m as P^m is not a strict profile. This technicality may be addressed simply by broadening the definition of BC. Alternatively, P^m may be replaced by a larger strict profile, in which each each voter v of P^m is replaced by $m!$ new voters, who respect v's strict preferences and collectively break all ties in v's preference order in all possible ways, using each possible tiebreak the same number of times.) The map $P \mapsto P^m$ may be applied any number of times; each application preserves the societal preferences of the Borda Count and may alter those of Condorcet's method. Each of the resulting voting systems yields the Borda Count through aggregation, as Condorcet's method and Efron dice voting do. What properties do these voting systems have? Which other transformations of profiles yield interesting voting systems?

4. Very many voting systems and social choice functions have been proposed, in addition to the few that we have discussed. Any system that involves processing pairwise preferences may be modified to process DV preferences instead: simply apply the system to P^m (or the larger strict version of P^m) instead of applying the system directly to P. How does such "dice-based preprocessing" affect the properties of various voting systems?

Acknowledgments. The research presented here was supported by Lafayette College and the National Science Foundation, grant DMS-055282.

References

[1] S. J. Brams and P. C. Fishburn (2002), *Voting procedures*, in: Handbook of Social Choice and Welfare Volume 1 (K. J. Arrow, A. K. Sen and K. Suzumura, eds.), Elsevier, Amsterdam, 2002, pp. 173-236.

[2] D. S. Felsenthal (1989), *On combining approval with disapproval voting*, Behavioral Science 34:53-60.

[3] M. Gardner (1970), *Mathematical games: the paradox of the nontransitive dice and the elusive principle of indifference*, Scientific American 223:110-114.

[4] D. G. Saari (1989), *A dictionary for voting paradoxes*, Journal of Economic Theory 48:443-475.

[5] D. G. Saari (1990), *The Borda dictionary*, Social Choice and Welfare 7:279-317.

[6] D. G. Saari (1995), *The Basic Geometry of Voting*, Springer-Verlag, Berlin, Heidelberg and New York.

[7] A. D. Taylor (2005), *Social Choice and the Mathematics of Manipulation*, Cambridge University Press, Cambridge.

[8] L. Traldi (2006), *Dice games and Arrow's theorem*, Bulletin of the Institute of Combinatorics and its Applications 47:19-22.

[9] H. P. Young (1973), *An axiomatization of Borda's rule*, Journal of Economic Theory 9:43-52.

[10] H. P. Young (1975), *Social choice scoring functions*, SIAM Journal of Applied Mathematics 28:824-838.