TOPOLOGICAL REALIZATIONS OF ORTHO-PROJECTION GRAPHS

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Abstract. We discuss the production of ortho-projection graphs from alternating knot diagrams, and introduce a more general construction of such graphs from "splittings" of closed, non-orientable surfaces. As our main result, we prove that this new topological construction generates *all* ortho-projection graphs. We present a minimal example of an ortho-projection graph that does not arise from a knot diagram, and provide a surface-splitting that realizes this graph.

1. Introduction

The aim of this article is to reveal the equivalence of certain seemingly unrelated objects from graph theory, linear algebra over $F = \mathbb{Z}/2\mathbb{Z}$ and low-dimensional topology. This equivalence is given in the following theorem, which is an amalgamation of the results found below.

Theorem 1.1. *Let A be a symmetric $n \times n$ matrix over F. Then the following are equivalent:*

1. *The (looped) graph G with adjacency matrix A has the property that two vertices are adjacent if and only if they have an odd number of common neighbors.*
2. *A is idempotent over F.*
3. *Multiplication by A defines an orthogonal projection $F^n \to F^n$, with respect to the standard dot product (mod 2).*
4. *There exists a splitting of a (marked) closed, non-orientable surface Σ of genus n whose associated projection $H_1(\Sigma; F) \to H_1(\Sigma; F)$ has matrix A.*

We call graphs that have the property specified in (1) *ortho-projection graphs*, for the reason given in (3). Thus, ortho-projection graphs "encode" the various orthogonal decompositions of finite-dimensional vector spaces over F. These graphs, which arise naturally in knot theory and low-dimensional topology, are closely related to circle graphs, which have been studied extensively; for example, see [1, 2] and their references. Specifically, ortho-projection graphs provide a succinct answer to the following question, which stems from work of Gauss: Which circle graphs are realizable by generic closed curves in the plane? The answer, which appears as Theorem 2 b) in [4], is that an arbitrary circle graph Λ is realizable by a plane curve if and only if it "extends" to an ortho-projection graph, i.e., if and only if there exists an ortho-projection graph that, when stripped of its loops, becomes Λ.

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Just as some circle graphs arise from chord diagrams produced by immersed circles in the plane, some ortho-projection graphs arise from signed chord diagrams generated by alternating knot projections. Indeed, a computer-assisted search has shown that all ortho-projection graphs with eight or fewer vertices are realized in this way [5]. However, there exist ortho-projection graphs that do not arise in this manner; we discuss a minimal example in Section 3.

In this paper, we review the knot-theoretic production of ortho-projection graphs, and we present a more general topological construction using splittings of closed, non-orientable surfaces. (A *splitting* of a surface Σ is simply a decomposition with $\Sigma = \Sigma_A \cup \Sigma_B$ and $S = \Sigma_A \cap \Sigma_B$, where S is a separating simple closed curve in Σ, and Σ_A and Σ_B are the closures in Σ of the two components of $\Sigma - S$.) Our main result is Theorem 6.2, which shows that our new construction suffices to realize *all* ortho-projection graphs. The construction itself is iterative in nature—a suitable splitting is built inductively starting from a trivial decomposition of an appropriate surface. The creation of the desired surface-splitting is analogous to the creation of a specific orthogonal splitting of a vector space by successively shifting particular orthonormal basis vectors from a subspace to its orthogonal complement. The reader who so wishes can interpret our main result as a topological characterization of certain distinguished subspaces of vector spaces over F: a subspace is the image of an ortho-projection $F^n \to F^n$ (i.e., it has trivial radical) if and only if it arises from a topological splitting of a closed, non-orientable surface of genus n.

We end this introduction by summarizing the organization of our article. In Section 2, we discuss ortho-projection graphs and their connections to orthogonal splittings of vector spaces over F. We review the production of such graphs from alternating knot diagrams in Section 3, where we also present an ortho-projection graph Γ that does not arise in this way. In Section 4, we describe our construction of ortho-projection graphs from splittings of closed, non-orientable surfaces, using homology with coefficients in F as a primary tool. Section 5 contains a review of useful results on orthogonalization of vector spaces over F. Finally, in Section 6, we state and prove our main result (Theorem 6.2), we illustrate the non-uniqueness of surface-splittings realizing a given ortho-projection graph, and we finish by presenting a splitting that yields the graph Γ.

2. Ortho-Projection Graphs

Let $n \geq 1$ and let G be an undirected, looped graph (without multiple loops and without multiple edges) on the vertex set $\{v_1, \ldots, v_n\}$. As introduced above, G is an *ortho-projection graph* (or, *op-graph*) if it satisfies the following adjacency condition: arbitrary (not necessarily distinct) vertices are neighbors in G if and only if they have an odd number of common neighbors. (We say that a vertex is a neighbor of itself if and only if it is looped.) See Figure 1 for an example.

Since the number of common neighbors of v_i and v_j equals the number of (possibly non-simple) paths of length two joining v_i and v_j, the definition above has the following algebraic equivalent: G is an op-graph if and only if the mod-2 reduction of its adjacency matrix is idempotent over $F = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Thus if G is an op-graph, and we define an $n \times n$ matrix A_G over F by $A_G(i, j) = \bar{1}$ if and only if v_i and v_j are neighbors, we have $A_G = (A_G)^{\mathrm{T}}$ and $A_G = (A_G)^2$. Multiplication by A_G defines an endomorphism of F^n, which by the equations above is self-adjoint

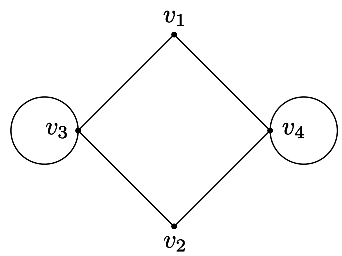

FIGURE 1. An ortho-projection graph.

(with respect to the standard mod-2 dot product) and idempotent. Such an endomorphism is an orthogonal projection (or, *ortho-projection*); see Proposition 2.1. It is for this reason that we call G an op-graph.

Proposition 2.1. *Let A be an arbitrary endomorphism of F^n. Let $\ker A$ and $\operatorname{fix} A$ denote the (possibly trivial) eigenspaces of A, and let $\operatorname{im} A$ denote the image of A. If A is self-adjoint (with respect to the standard dot product $\langle\ ,\ \rangle\colon F^n \times F^n \to F$) and idempotent then*

(1) $\operatorname{fix} A = \operatorname{im} A$,
(2) $\ker A \perp \operatorname{fix} A$, *and*
(3) $F^n = \ker A \oplus \operatorname{fix} A$.

Thus A represents orthogonal projection onto $\operatorname{im} A \subseteq F^n$.

Proof. Clearly $\operatorname{fix} A \subseteq \operatorname{im} A$. Since $A(Av) = A^2v = Av$ for all $v \in F^n$, $\operatorname{im} A \subseteq \operatorname{fix} A$, proving (1). Let $v \in \ker A$ and $w \in \operatorname{fix} A$ be arbitrary. Then $\langle v, w\rangle = \langle v, Aw\rangle = \langle Av, w\rangle = \langle 0, w\rangle = \bar{0}$, proving (2). For arbitrary $v \in F^n$, we have $v = (v+Av)+Av$, where $v + Av \in \ker A$ and $Av \in \operatorname{fix} A$. Thus $F^n = \ker A + \operatorname{fix} A$. If $v \in \ker A \cap \operatorname{fix} A$ then $0 = Av = v$. So $F^n = \ker A \oplus \operatorname{fix} A$, proving (3). $\square$

Example 2.2. The graph G in Figure 1 has adjacency matrix

$$A_G = \begin{bmatrix} \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}.$$

Since this matrix is symmetric and idempotent over F, we see that G is an op-graph.

At this point we have established the equivalence of (1), (2) and (3) in Theorem 1.1. The equivalence of these to (4) in that theorem is established by Proposition 4.2 and Theorem 6.2 below.

3. Knots and Ortho-Projections

Let $n \geq 1$ and let K be a classical knot diagram with n crossings. Label the crossings $1, \ldots, n$ in any manner. Choose a non-crossing point $* \in K$, and give K either of its two possible orientations. Start at $*$ and trace along K in the positive direction, recording the crossing labels sequentially as you encounter them. Continue until you return to $*$.

This process yields a *double-occurrence word* (in the symbols $1, \ldots, n$) called the *Gauss code* of K; equivalently, it gives a *chord diagram* for K with n labeled chords. Each crossing in K has a sign $+$ or $-$, according to the convention shown in Figure 2. Prefix each symbol in the Gauss code with the sign of its crossing to obtain the *signed Gauss code* of K. Similarly, tag each chord in the chord diagram with the sign of its crossing to produce the *signed chord diagram* for K.

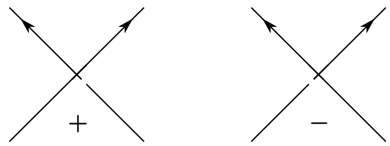

FIGURE 2. Crossing signs.

The *trip matrix* T_K is the symmetric $n \times n$ matrix over F gotten from the signed chord diagram for K as follows: for $j \neq k$, $T_K(j,k) = \bar{1}$ if and only if chords j and k intersect; for $j = k$, $T_K(j,k) = \bar{1}$ if and only if the sign of crossing $j = k$ is $-$.

Remark 3.1. The second author introduced the trip matrix in [9], and showed that the Jones polynomial of the associated knot can be calculated from this matrix using elementary linear algebra over F. In [10], he proved that the trip matrix of an *alternating* classical knot diagram is an ortho-projection matrix with respect to the standard mod-2 dot product, a fact first noted by Richard Stong [6].

Remark 3.2. Reversing the orientation of K has no effect on the crossing signs, and thus no effect on T_K. T_K is also unaffected by relocation of the basepoint $*$. Relabeling the crossings of K changes T_K via conjugation by a permutation matrix.

Remark 3.3. The graph G_K whose adjacency matrix is T_K is called the *looped interlacement graph* of K in [8], following the usage in [3]. In light of Remark 3.1, if K is an alternating diagram then G_K is an op-graph. If K is a positive knot diagram then, by definition, G_K is a circle graph. More generally, the graph obtained by stripping G_K of its loops is a circle graph, for any diagram K.

Example 3.4. The labeled knot diagram K in Figure 3 yields the signed Gauss code $-6-2+1+3+4+5-2-6+3+4+5+1$ and the signed chord diagram shown in Figure 4. The trip matrix of K is

$$T_K = \begin{bmatrix} \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}.$$

The corresponding looped interlacement graph G_K appears in Figure 5. Since K is an alternating diagram, we know that T_K is idempotent and G_K is an op-graph.

A computer-assisted analysis shows that each op-graph with eight or fewer vertices is the looped interlacement graph of an alternating classical knot diagram [5]. However, the 9-vertex op-graph Γ in Figure 6 is not such a graph. To confirm this, it suffices by Remark 3.3 to show that Γ is not a circle graph. So suppose it is, and let $\mathcal{D}$ denote a corresponding chord diagram. From Γ, we see that chords 1, 2 and 3 are pairwise-disjoint in $\mathcal{D}$. We also see that chord 4 intersects chords 2 and 3 but

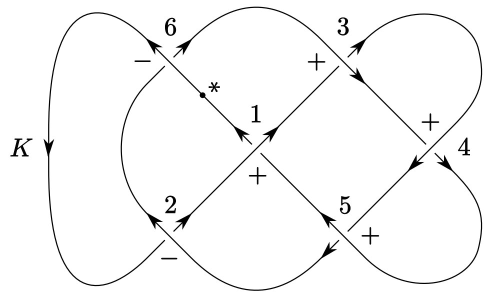

FIGURE 3. A labeled knot diagram.

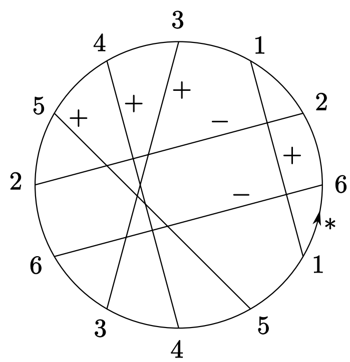

FIGURE 4. A signed chord diagram.

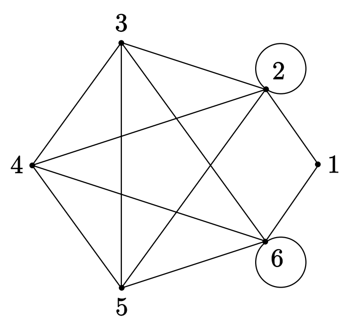

FIGURE 5. A looped interlacement graph.

not chord 1, chord 5 intersects chords 1 and 3 but not 2, and chord 6 intersects chords 1 and 2 but not 3. Thus none of the chords 1, 2 or 3 separates the other two, which forces chords 1 through 6 in $\mathcal{D}$ to form a "truncated Star of David." Since chord 7 intersects chords 1, 2, 4 and 5 but not 3 or 6, chords 1 through 7 in $\mathcal{D}$ *must* appear as in Figure 7, up to homeomorphism. Since chord 7 separates chords 3 and 6, it is impossible to place chord 8 in a manner consistent with Γ. Thus no such $\mathcal{D}$ exists, and Γ is not a circle graph.

Remark 3.5. Note that we have actually shown that the induced subgraph on vertices 1 through 8 of Γ is not a circle graph. In fact, the induced subgraph on the vertex set $\{1,3,4,5,7,8\}$ is not a circle graph. This can be seen via an argument similar to the one given above, or by noting that local complementation of this subgraph at vertex 3 yields the wheel W_5, which is a circle graph obstruction according to [1].

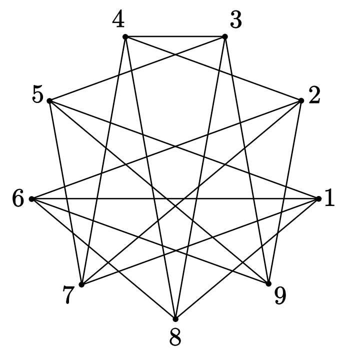

FIGURE 6. An op-graph Γ that does not arise from a knot.

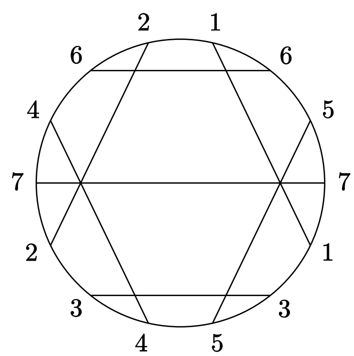

FIGURE 7. A partial chord diagram for Γ.

4. Splittings and Ortho-Projections

Let $n \geq 1$ and let Σ be a closed, non-orientable surface of genus n, i.e., $\Sigma \approx \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (n summands). Let $\pi \colon \mathbb{Z} \to F = \mathbb{Z}/2\mathbb{Z}$ be the canonical projection. We use F-coefficients for all our homological calculations, though we omit the symbol F from our notation.

$H_1(\Sigma)$ is an n-dimensional vector space over F. Let $C_1, \ldots, C_n$ be pairwise-disjoint, one-sided circles in Σ. (A *circle* is just a simple, closed curve.) Because each C_j is one-sided, $\langle [C_j], [C_j] \rangle = \bar{1}$ for $1 \leq j \leq n$, where $\langle\ ,\ \rangle \colon H_1(\Sigma) \times H_1(\Sigma) \to F$ is the mod-2 intersection form. Since these circles are pairwise-disjoint, $\langle [C_j], [C_k] \rangle = \bar{0}$ for $j \neq k$. Thus $\langle [C_j], [C_k] \rangle = \overline{\delta_{j,k}} = \pi(\delta_{j,k})$ for $1 \leq j, k \leq n$, which implies that $\mathcal{C} = \{[C_1], \ldots, [C_n]\}$ is a basis for $H_1(\Sigma)$. We let $C = C_1 \cup \cdots \cup C_n$, and refer to the pair (Σ, C) as a *marked surface*.

Remark 4.1. The curve system C allows us to identify $H_1(\Sigma)$ with F^n, by having $\mathcal{C}$ correspond to the standard basis. Note that the mod-2 intersection product corresponds to the mod-2 dot product under this identification.

An open regular neighborhood of C in Σ is the disjoint union of n open Möbius bands. Removing such a neighborhood from Σ leaves a compact surface $\Sigma|C$ homeomorphic to a sphere with n holes. Choose an orientation for $\Sigma|C$, which induces an orientation on $\partial(\Sigma|C)$. From this orientation on $\partial(\Sigma|C)$, we obtain coherent orientations for the circles $C_1, \ldots, C_n$.

Let S be a separating circle in Σ that is in general position with respect to C. S "splits" Σ into a pair of compact, connected sub-surfaces Σ_A and Σ_B, with $\Sigma_A \cup \Sigma_B = \Sigma$ and $\Sigma_A \cap \Sigma_B = S$. Since S is separating, $[S] = 0 \in H_1(\Sigma)$, and so $\langle [S], [C_j] \rangle = \bar{0}$ for $1 \le j \le n$. Thus $|S \cap C_j|$ is even for all j; let $c(j) = \frac{1}{2}|S \cap C_j|$. Choose an orientation for S, and select a basepoint $* \in S - C$. For $1 \le j \le n$, let $*_j$ be the first successor of $*$ in $S \cap C_j$, as determined by the chosen orientation for S. The points in $S \cap C_j$ split C_j into a union of $2c(j)$ closed arcs. Trace along C_j in the positive direction starting at $*_j$ and label these arcs sequentially as you encounter them, using the labels $\alpha_{j,1}, \ldots, \alpha_{j,c(j)}$ for the arcs in $\Sigma_A \cap C_j$, and $\beta_{j,1}, \ldots, \beta_{j,c(j)}$ for the arcs in $\Sigma_B \cap C_j$. Note that $S \cup \left(\bigcup_{j,k} \alpha_{j,k}\right)$ and $S \cup \left(\bigcup_{j,k} \beta_{j,k}\right)$ are "chord diagrams" embedded in Σ_A and Σ_B, respectively. See Figure 8, in which $c(j) = 2$, S is represented by the radial arcs, and C_j is obtained from the depicted circle by antipodal identification. Note that the region inside the circle in Figure 8 is not part of the surface Σ.

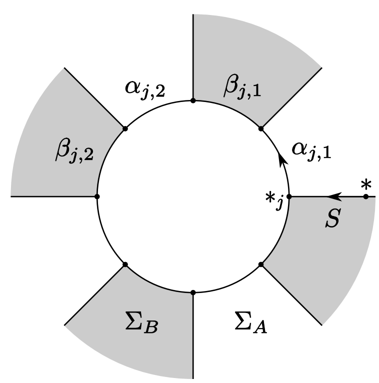

FIGURE 8. The labelings on C_j.

We refer to the triple (Σ, C, S) as a *splitting* of the (marked) surface Σ. We now consider the algebraic splitting that corresponds to this topological one. Since (Σ_A, Σ_B) is an excisive couple, we have the commutative diagram in reduced homology shown in Figure 9, in which the rungs of the horizontal ladder are induced by $(\Sigma_A, S) \hookrightarrow (\Sigma, \Sigma_B)$ and the rungs of the vertical ladder are induced by $(\Sigma_B, S) \hookrightarrow (\Sigma, \Sigma_A)$.

As indicated in Figure 9, we have endomorphisms P_A and P_B of $H_1(\Sigma)$. Specifically, $P_A = i_a \circ (j_a)^{-1} \circ (i_A)^{-1} \circ j_B$, where $j_B \colon H_1(\Sigma) \to H_1(\Sigma, \Sigma_B)$, $i_A \colon H_1(\Sigma_A, S) \to H_1(\Sigma, \Sigma_B)$, $j_a \colon H_1(\Sigma_A) \to H_1(\Sigma_A, S)$ and $i_a \colon H_1(\Sigma_A) \to H_1(\Sigma)$ are the standard maps of homology theory. Analogously, $P_B = i_b \circ (j_b)^{-1} \circ (i_B)^{-1} \circ j_A$, where

$$
\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d, "0"] & & \\
{} \arrow[r] & H_1(S) \arrow[r, "0"] \arrow[d, "0"'] & H_1(\Sigma_A) \arrow[r, "\approx", "j_a"'] \arrow[d, "i_a"] \arrow[dr, phantom, "P_A"] & H_1(\Sigma_A, S) \arrow[r] \arrow[d, "i_A"', "\approx \text{ excision}"] & {} \\
{} \arrow[r, "0"'] & H_1(\Sigma_B) \arrow[r, "i_b"'] \arrow[d, "\approx"', "j_b"] \arrow[dr, phantom, "P_B"] & H_1(\Sigma) \arrow[r, "j_B"] \arrow[d, "j_A"'] & H_1(\Sigma, \Sigma_B) \arrow[r, "0"'] & {} \\
& H_1(\Sigma_B, S) \arrow[r, "i_B", "\substack{\approx \\ \text{excision}}"'] \arrow[d] & H_1(\Sigma, \Sigma_A) \arrow[d, "0"] & & \\
& {} & {} & &
\end{tikzcd}
$$

FIGURE 9

$j_A \colon H_1(\Sigma) \to H_1(\Sigma, \Sigma_A)$, $i_B \colon H_1(\Sigma_B, S) \to H_1(\Sigma, \Sigma_A)$, $j_b \colon H_1(\Sigma_B) \to H_1(\Sigma_B, S)$ and $i_b \colon H_1(\Sigma_B) \to H_1(\Sigma)$.

A direct analysis of this commutative diagram yields the following, whose proof we omit.

Proposition 4.2. *Let (Σ, C, S) be an arbitrary splitting. Then*

(1) *P_A and P_B are idempotent,*
(2) $\ker P_A = i_b\, H_1(\Sigma_B) = \operatorname{im} P_B$ *and* $\ker P_B = i_a\, H_1(\Sigma_A) = \operatorname{im} P_A$*,*
(3) *$P_A + P_B = I$, the identity on $H_1(\Sigma)$,*
(4) *$H_1(\Sigma) = i_a\, H_1(\Sigma_A) \oplus i_b\, H_1(\Sigma_B)$, and*
(5) *The splitting in (4) is orthogonal with respect to the intersection form $\langle\ ,\ \rangle$, and thus P_A and P_B are self-adjoint with respect to that form.*

Remark 4.3. The orthogonality of $i_a\, H_1(\Sigma_A)$ and $i_b\, H_1(\Sigma_B)$ follows from that fact that each class in $i_a\, H_1(\Sigma_A)$ $\big(i_b\, H_1(\Sigma_B)\big)$ can be represented by a cycle that lies in the interior of Σ_A (Σ_B).

In light of Proposition 4.2, we see that P_A and P_B are ortho-projections. We now determine the matrices $[P_A]$ and $[P_B]$ of these endomorphisms with respect to the basis $\mathcal{C} = \{[C_1], \dots, [C_n]\}$. Let $1 \le j \le n$ be arbitrary, and focus on P_A. By definition, $(i_A)^{-1}\big(j_B([C_j])\big) = [\alpha_{j,1}] + \cdots + [\alpha_{j,c(j)}] \in H_1(\Sigma_A, S)$. For $1 \le k \le c(j)$, let $S_{j,k}$ be the arc in S that begins at the terminus of the oriented arc $\alpha_{j,k}$ and follows S in the positive direction to the initial point of $\alpha_{j,k}$. Let $\widetilde{\alpha}_{j,k} = \alpha_{j,k} \cup S_{j,k}$. Then $(j_a)^{-1}\big([\alpha_{j,1}] + \cdots + [\alpha_{j,c(j)}]\big) = [\widetilde{\alpha}_{j,1}] + \cdots + [\widetilde{\alpha}_{j,c(j)}] \in H_1(\Sigma_A)$. Thus, by definition, $P_A([C_j]) = [\widetilde{\alpha}_{j,1}] + \cdots + [\widetilde{\alpha}_{j,c(j)}]$, regarded as a class in $H_1(\Sigma)$.

Remark 4.4. There are two arcs in S that span the endpoints of $\alpha_{j,k}$. In fact, either could serve as $S_{j,k}$, since $[S] = 0 \in H_1(\Sigma_A)$.

Remark 4.5. Since $(i_A)^{-1} \circ j_B$ is surjective, $\{[\alpha_{j,k}] : 1 \le j \le n, 1 \le k \le c(j)\}$ spans $H_1(\Sigma_A, S)$. Similarly, $\{[\beta_{j,k}] : 1 \le j \le n, 1 \le k \le c(j)\}$ spans $H_1(\Sigma_B, S)$; this observation is used in the proof of Lemma 6.1.

Now that we have calculated $P_A([C_j]) \in H_1(\Sigma)$, we need to express it as a linear combination of $[C_1], \ldots, [C_n]$. This is easy, since the basis $\mathcal{C}$ is orthonormal relative to $\langle\ ,\ \rangle$. We have $P_A([C_j]) = \langle P_A([C_j]), C_1\rangle C_1 + \cdots + \langle P_A([C_j]), C_n\rangle C_n$. Thus it suffices to determine $\langle[\widetilde{\alpha}_{j,k}], [C_i]\rangle$ for arbitrary i, j, k. But this is simply the parity of the number of times that the cycle $\widetilde{\alpha}_{j,k}$ crosses the circle C_i. For $i \neq j$, this is the parity of the number of times that $S_{j,k}$ crosses C_i. For $i = j$ there is a complication—the cycle $\widetilde{\alpha}_{j,k}$ contains $\alpha_{j,k}$, and so is not transverse to $C_i = C_j$. In this case, we simply modify $\widetilde{\alpha}_{j,k}$ by isotopy along $\alpha_{j,k}$. Specifically, we move the arc $\alpha_{j,k} \subset \widetilde{\alpha}_{j,k}$ parallel to itself (in either direction) to eliminate its intersection with $C_i = C_j$. Then, as above, $\langle[\widetilde{\alpha}_{j,k}], [C_i]\rangle = \langle[\widetilde{\alpha}_{j,k}], [C_j]\rangle$ is the parity of the number of times that the modified cycle $\widetilde{\alpha}_{j,k}$ crosses the circle $C_i = C_j$. Note that this second case only affects the diagonal of $[P_A]$.

We summarize this lengthy discussion with an illustrative example.

Example 4.6. Figure 10 shows a splitting of a closed, non-orientable surface Σ of genus 6 into a pair of non-orientable sub-surfaces, Σ_A and Σ_B. The unshaded surface Σ_A has Euler characteristic $\chi = 4 - 7 = -3$ and genus 4. The shaded surface Σ_B has $\chi = 6 - 7 = -1$ and genus 2. Each geometric circle in the figure—including the large one—carries an antipodal identification. The regions inside the small circles are not part of Σ, and the labels within those regions refer to the corresponding one-sided circles obtained from the antipodal identifications. Similarly, the region outside the large circle is not part of Σ.

We now determine $[P_A]$ for the splitting depicted in Figure 10. As noted above, it is necessary to isotope each $\widetilde{\alpha}_{j,k}$ to make it transverse to C_j. We do this by "pushing" each $\alpha_{j,k}$ in the direction in which S departs from the terminus of the arc, as indicated by the arrow along S near each $\alpha_{j,k}$. Following this convention and the discussion above, we obtain:

$$\begin{aligned}
P_A([C_1]) &= [C_2] + [C_6],\\
P_A([C_2]) &= ([C_5] + [C_1] + [C_2]) + ([C_3] + [C_4]),\\
P_A([C_3]) &= [C_4] + [C_2] + [C_5] + [C_1] + [C_2] + [C_6] + [C_1] + [C_2],\\
P_A([C_4]) &= [C_2] + [C_5] + [C_6] + [C_3],\\
P_A([C_5]) &= [C_6] + [C_3] + [C_4] + [C_2],\\
P_A([C_6]) &= [C_1] + [C_2] + [C_3] + [C_4] + [C_2] + [C_5] + [C_6].
\end{aligned}$$

Thus the matrix of P_A with respect to the ordered basis $\mathcal{C}$ is

$$[P_A] = \begin{bmatrix}
\bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1}\\
\bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0}\\
\bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1}\\
\bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1}\\
\bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1}\\
\bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1}
\end{bmatrix}.$$

Note that $[P_A]$ is identical to the trip matrix in Example 3.4, so the splitting in this example gives rise to the op-graph shown in Figure 5. The first four columns of $[P_A]$ form a basis for the column space of the matrix; the corresponding classes $[C_2]+[C_6]$,

$[C_1]+[C_2]+[C_3]+[C_4]+[C_5]$, $[C_2]+[C_4]+[C_5]+[C_6]$ and $[C_2]+[C_3]+[C_5]+[C_6]$ form a basis for $i_a\, H_1(\Sigma_A)$.

By (3) in Proposition 4.2, we obtain $[P_B]$ by adding the identity matrix to $[P_A]$. The first two columns of $[P_B]$ form a basis for the column space of that matrix; the corresponding classes $[C_1]+[C_2]+[C_6]$ and $[C_1]+[C_3]+[C_4]+[C_5]$ form a basis for $i_b\, H_1(\Sigma_B)$.

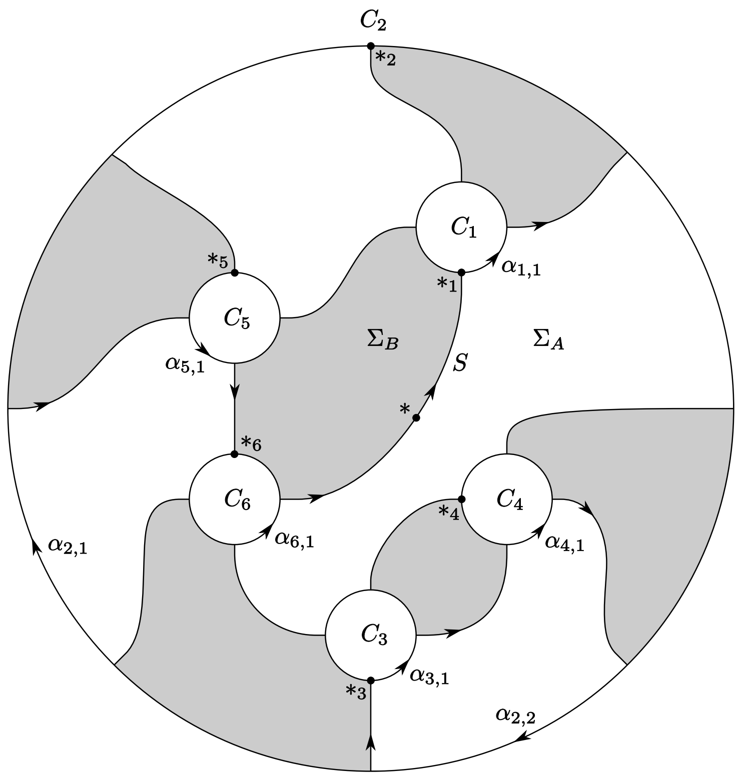

FIGURE 10. A labeled splitting.

Remark 4.7. For any (Σ, C, S), the embedded chord diagram $S \cup \left(\bigcup_{j,k} \alpha_{j,k}\right)$ splits Σ_A into "sectors," each of which yields a relation in $H_1(\Sigma_A, S)$. For example, in Figure 10 there are four unshaded sectors, which yield the relations $[\alpha_{1,1}]+[\alpha_{2,1}]+[\alpha_{3,1}]+[\alpha_{4,1}]+[\alpha_{6,1}]=0$, $[\alpha_{2,2}]+[\alpha_{3,1}]+[\alpha_{4,1}]=0$, $[\alpha_{2,1}]+[\alpha_{5,1}]+[\alpha_{6,1}]=0$ and $[\alpha_{1,1}]+[\alpha_{2,2}]+[\alpha_{5,1}]=0$.

In practice, these relations can be used to simplify (and check) the calculation of $[P_A]$. To illustrate, we calculate $P_A([C_2])$ in Example 4.6 without tracing the arcs $S_{2,1}$ and $S_{2,2}$, as follows. From the final two relations above, $[\alpha_{2,1}]+[\alpha_{2,2}]=([\alpha_{5,1}]+[\alpha_{6,1}])+([\alpha_{1,1}]+[\alpha_{5,1}])=[\alpha_{1,1}]+[\alpha_{6,1}]$. Applying $i_a \circ (j_a)^{-1}$ to this equation gives $P_A([C_2])=P_A([C_1])+P_A([C_6])$. Thus the second column of $[P_A]$ is the sum of the first and last columns of that matrix.

As noted earlier, the off-diagonal elements of $[P_A]$ are determined by the intersections of various arcs $S_{j,k}$ and circles C_i, with $i \neq j$. Consequently, we can use an abstract chord diagram rather than the embedded chord diagram $S \cup \left(\bigcup_{j,k} \alpha_{j,k}\right)$ to calculate these elements, as in the following example.

Example 4.8. Figure 11 depicts an abstract chord diagram that corresponds to the embedded chord diagram for the splitting in Example 4.6. The oriented circle represents the curve S, while the oriented "chords" correspond to the arcs $\alpha_{j,k}$ shown in Figure 10. The numerical labels at the endpoints of each chord indicate which one-sided circle contains the corresponding arc $\alpha_{j,k}$—the label is the first subscript of that $\alpha_{j,k}$. For each chord in Figure 11, the positively oriented circular arc from the terminus of the chord to its origin represents the corresponding arc $S_{j,k} \subset S$.

We can use the abstract chord diagram in Figure 11 to find the off-diagonal elements of the matrix $[P_A]$ we determined earlier. For example, the sequence of labels $2, 6$ bounded by chord 1 indicates that $[P_A]_{2,1} = [P_A]_{6,1} = \bar{1}$, and that all other off-diagonal elements in column 1 of $[P_A]$ are $\bar{0}$. Similarly, the sequence $1, 2, 3, 4, 2, 5$ bounded by chord 6 gives $[P_A]_{1,6} = [P_A]_{3,6} = [P_A]_{4,6} = [P_A]_{5,6} = \bar{1}$ and $[P_A]_{2,6} = \bar{0}$ (since we count appearances of labels mod 2). Since there are two chords labeled 2, we "add" (via symmetric difference) the sequences $3, 4$ and $5, 1$ to obtain $[P_A]_{1,2} = [P_A]_{3,2} = [P_A]_{4,2} = [P_A]_{5,2} = \bar{1}$ and $[P_A]_{6,2} = \bar{0}$. The reader is invited to consider the remaining columns.

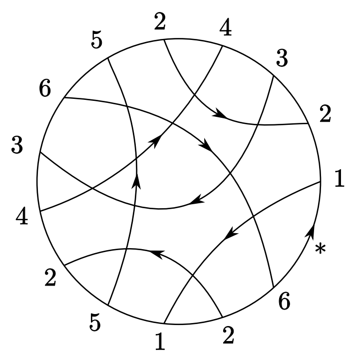

FIGURE 11. An abstract chord diagram.

Remark 4.9. In fact, the off-diagonal elements of $[P_A]$ are completely determined by the circular sequence of numbers in Figure 11; the chords themselves are irrelevant. As an illustration of this, suppose we modify the chords labeled 2 in Figure 11 to obtain the diagram shown in Figure 12. When we add the sequences $5, 6, 3, 4$ and $6, 1$ bounded by these new chords, we find that $[P_A]_{1,2} = [P_A]_{3,2} = [P_A]_{4,2} = [P_A]_{5,2} = \bar{1}$ and $[P_A]_{6,2} = \bar{0}$, just as before. (We can even use a diagram in which the chords labeled 2 cross, in which case we might add 5,6,3,4,2,5,1 and 5,1,2,6,1 to obtain 1,3,4,5, which yields the same second column of $[P_A]$ as above.) We leave a further discussion of this "independence of chording" for a future article.

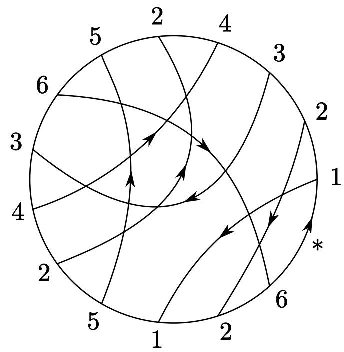

FIGURE 12. A modified chord diagram.

5. ORTHOGONALIZATION OVER $F = \mathbb{Z}/2\mathbb{Z}$

In this section, we discuss the "orthogonalization" of a finite-dimensional vector space V over F with respect to an arbitrary non-degenerate, symmetric bilinear form $\langle\ ,\ \rangle\colon V \times V \to F$. (Recall that $\langle\ ,\ \rangle$ is *non-degenerate* if $\langle v, v'\rangle = \bar{0}$ for all $v' \in V$ only if $v = 0$, i.e., $\mathrm{rad}\,(V) = V \cap V^{\perp} = \{0\}$.) Specifically, we address the existence of an "orthonormal" basis for V relative to $\langle\ ,\ \rangle$. As we show in Proposition 5.6, such a basis exists if (and only if) V possesses a non-self-orthogonal vector. Since the lack of such a vector implies that V is even-dimensional (Proposition 5.1), an orthonormal basis surely exists if V has odd dimension, as is noted in Corollary 5.7.

Although the results in this section are standard (for example, see [7]), we include them here as a service to the reader.

Proposition 5.1. *Let* $\dim(V) = n \geq 1$. *If* $\langle v, v\rangle = \bar{0}$ *for all* $v \in V$ *then* $n = 2m$ *for an integer* $m \geq 1$, *and there exists a basis* $\mathcal{B} = \{f_1, g_1, \ldots, f_m, g_m\}$ *for* V *with* $\langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ *and* $\langle f_j, g_k\rangle = \overline{\delta_{j,k}}$.

Remark 5.2. A basis like $\mathcal{B}$ is a *symplectic basis* relative to $\langle\ ,\ \rangle$.

Proof. The proof is by induction on n. For $n = 1$, $V = \{0, v\}$. Since $\langle\ ,\ \rangle$ is non-degenerate, $\langle v, v\rangle = \bar{1}$. Thus the antecedent in the implication is false, so the statement is true. For $n = 2$, let $\{f_1, g_1\}$ be an arbitrary basis for V, and assume $\langle f_1, f_1\rangle = \langle g_1, g_1\rangle = \bar{0}$. Since $\langle\ ,\ \rangle$ is non-degenerate, $\langle f_1, g_1\rangle = \bar{1}$, and thus the statement holds. For $n \geq 3$, let $\{v_1, \ldots, v_n\}$ be an arbitrary basis for V, and assume $\langle v_j, v_j\rangle = \bar{0}$ for $1 \leq j \leq n$. Since $\langle\ ,\ \rangle$ is non-degenerate, $\langle v_1, v_j\rangle = \bar{1}$ for some $j > 1$. Without loss of generality, assume $j = 2$. Let $f_1 = v_1$ and $g_1 = v_2$. For each $k > 2$, let $v'_k = v_k + \langle g_1, v_k\rangle f_1 + \langle f_1, v_k\rangle g_1$. Then $\langle f_1, v'_k\rangle = \langle g_1, v'_k\rangle = \bar{0}$ for $k > 2$. Let $H = \mathrm{span}\{f_1, g_1\}$ and $V' = \mathrm{span}\{v'_3, \ldots, v'_n\}$, so that V is the orthogonal direct sum of H and V'. Note that $\dim(V') = n - 2 \geq 1$, that the restriction of $\langle\ ,\ \rangle$ to V' is non-degenerate, and that $\langle v', v'\rangle = \bar{0}$ for all $v' \in V'$. Thus, by induction, $n-2 = 2(m-1)$ for some integer $m \geq 2$, and there exists a basis $\mathcal{B}' = \{f_2, g_2, \ldots, f_m, g_m\}$ for V' such that $\langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ and $\langle f_j, g_k\rangle = \overline{\delta_{j,k}}$. So $n = 2m$ for some integer $m \geq 2$, and there exists a basis $\mathcal{B} = \{f_1, g_1\} \cup \mathcal{B}'$ for V such that $\langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ and $\langle f_j, g_k\rangle = \overline{\delta_{j,k}}$.

$\square$

Proposition 5.3. *Let* $\dim(V) = n = l + 2m$, *for integers* $l \geq 1$ *and* $m \geq 0$. *If there exists a basis* $\mathcal{B} = \{e_1, \ldots, e_l, f_1, g_1, \ldots, f_m, g_m\}$ *for* V *such that* $\langle e_j, f_k\rangle = \langle e_j, g_k\rangle = \langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ *and* $\langle e_j, e_k\rangle = \langle f_j, g_k\rangle = \overline{\delta_{j,k}}$ *then there exists a basis* $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ *for* V *with* $\langle e'_j, e'_k\rangle = \overline{\delta_{j,k}}$.

Remark 5.4. A symmetric bilinear form $\langle\ ,\ \rangle$ that satisfies the conditions in the antecedent is necessarily non-degenerate, since its matrix relative to $\mathcal{B}$ is non-singular.

Remark 5.5. A basis like $\mathcal{B}'$ is an *orthonormal basis* relative to $\langle\ ,\ \rangle$.

Proof. The proof is by induction on m. The result is trivial for $m = 0$; simply let $e'_j = e_j$ for $1 \leq j \leq l = n$. For $m \geq 1$, let $\mathcal{B} = \{e_1, \ldots, e_l, f_1, g_1, \ldots, f_m, g_m\}$ be an arbitrary basis for V, and assume $\langle e_j, f_k\rangle = \langle e_j, g_k\rangle = \langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ and $\langle e_j, e_k\rangle = \langle f_j, g_k\rangle = \overline{\delta_{j,k}}$. Then $\langle e_l, f_1\rangle = \langle e_l, g_1\rangle = \langle f_1, f_1\rangle = \langle g_1, g_1\rangle = \bar{0}$ and $\langle e_l, e_l\rangle = \langle f_1, g_1\rangle = \bar{1}$. Let $e'_j = e_j$ for $j < l$. Let $e'_l = e_l + f_1 + g_1$, $e'_{l+1} = f_1 + e_l$ and $e'_{l+2} = g_1 + e_l$. It is easy to verify that $\{e'_1, \ldots, e'_l, e'_{l+1}, e'_{l+2}, f_2, g_2, \ldots, f_m, g_m\}$ is a basis for V such that $\langle e'_j, f_k\rangle = \langle e'_j, g_k\rangle = \langle f_j, f_k\rangle = \langle g_j, g_k\rangle = \bar{0}$ and $\langle e'_j, e'_k\rangle = \langle f_j, g_k\rangle = \overline{\delta_{j,k}}$. Thus, by induction, there exists a basis $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ for V with $\langle e'_j, e'_k\rangle = \overline{\delta_{j,k}}$.

$\square$

Proposition 5.6. *Let* $\dim(V) = n \geq 1$. *If* $\langle v, v\rangle = \bar{1}$ *for some* $v \in V$ *then there exists a basis* $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ *for* V *with* $\langle e'_j, e'_k\rangle = \overline{\delta_{j,k}}$.

Proof. The proof is by induction on n. For $n = 1$, $V = \{0, v\}$ with $\langle v, v\rangle = \bar{1}$. Thus the statement holds with $e'_1 = v$. For $n \geq 2$, let $\{v_1, \ldots, v_n\}$ be an arbitrary basis for V, and assume $\langle v_j, v_j\rangle = \bar{1}$ for some j. Without loss of generality, $j = 1$. Let $e_1 = v_1$, and let $v'_k = v_k + \langle e_1, v_k\rangle e_1$ for $k > 1$, so that $\langle e_1, v'_k\rangle = \bar{0}$ for $k > 1$. Let $V' = \operatorname{span}\{v'_2, \ldots, v'_n\}$. Note that the restriction of $\langle\ ,\ \rangle$ to V' is non-degenerate. There are two cases:

(1) If $\langle v', v'\rangle = \bar{1}$ for some $v' \in V'$, induction provides an orthonormal basis $\{e'_2, \ldots, e'_n\}$ for V'. With $e'_1 = e_1$, $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ is then an orthonormal basis for V.
(2) If $\langle v', v'\rangle = \bar{0}$ for all $v' \in V'$, $\dim(V') = 2m$ with $m \geq 1$ and V' has a symplectic basis $\{f_1, g_1, \ldots, f_m, g_m\}$, by Proposition 5.1. Thus $\mathcal{B} = \{e_1, f_1, g_1, \ldots, f_m, g_m\}$ satisfies the hypotheses of Proposition 5.3, and so there is an orthonormal basis $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ for V.

$\square$

Corollary 5.7. *If* $\dim(V) = n$ *is odd then there exists a basis* $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ *for* V *with* $\langle e'_j, e'_k\rangle = \overline{\delta_{j,k}}$.

Remark 5.8. The constructions from the proofs above can be assembled to give an orthogonalization (with respect to a non-degenerate, symmetric bilinear form) algorithm for finite-dimensional vector spaces V over F, akin to the standard Gram-Schmidt process for finite-dimensional inner product spaces over $\mathbb{R}$. This algorithm converts an arbitrary basis for V into a basis that is either symplectic or orthonormal with respect to the given form.

6. Realization via surface-splitting

In this section we prove our main result, that each op-graph arises from a splitting of a closed, non-orientable surface. For those graphs that come from alternating knot diagrams, the construction of one such surface-splitting is simple—just trade the crossings in the diagram for cross-caps. This is suggested in Figure 13, which depicts a splitting that generates the op-graph in Figure 5. (The unshaded surface in Figure 13 is compactified by a point at infinity.) A proof of the correctness of this construction is essentially contained in the proof of Theorem 3 in [10].

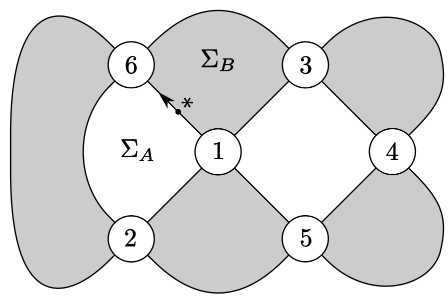

Figure 13. A splitting from a knot diagram.

However, as we showed in Section 3, not all op-graphs arise from knot diagrams. Thus we prove Theorem 6.2, which describes the construction of a surface-splitting realizing an arbitrary op-graph. The construction relies on Proposition 5.6 and Lemma 6.1 below. As before, we use F-coefficients in our homological arguments, except for a single application of integral homology in Remark 6.4.

Lemma 6.1. *Let (Σ, C, S) be an arbitrary splitting of a closed, non-orientable surface. Then each class in $H_1(\Sigma_B, S)$ can be represented by an arc properly embedded in (Σ_B, S).*

Proof. By Remark 4.5, an arbitrary class $b \in H_1(\Sigma_B, S)$ can be represented by a disjoint union of properly-embedded arcs. Label these arcs $\beta_1, \dots, \beta_m$ and let $\mathcal{E}$ be the set of their endpoints. If $m \geq 2$, focus on one endpoint of β_m and its two neighbors (along S) in $\mathcal{E}$. At least one of these neighbors is an endpoint of an arc β_i, with $i \neq m$. Modify the arcs β_m and β_i near S, as indicated in Figure 14. The result is a disjoint union of $m-1$ arcs properly embedded in (Σ_B, S) that represents the same class b. Iterate this reduction process until b is represented by a single properly-embedded arc.

□

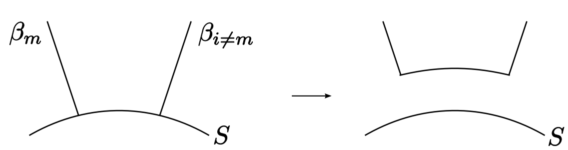

Figure 14. Arc reduction.

Theorem 6.2. *Let G be an arbitrary op-graph with $n \geq 1$ labeled vertices. Then there exists a splitting (Σ, C, S) of a (marked) closed, non-orientable surface Σ of genus n such that $[P_A] = A_G$, where $[P_A]$ is the ortho-projection matrix discussed following Proposition 4.2 and A_G is the adjacency matrix of G over F.*

Remark 6.3. An arbitrary subspace is the image of an ortho-projection $F^n \to F^n$ if and only if its radical is trivial. What Theorem 6.2 asserts is that such subspaces are precisely those that correspond to (the homologies of) embedded sub-surfaces of Σ that have a *single* boundary component.

Proof. Since G is an op-graph, F^n is the orthogonal direct sum of N_G and C_G, where N_G is the nullspace and C_G is the column (and row) space of A_G. At least one of these subspaces contains a non-self-orthogonal vector, since F^n does. Assume for the moment that C_G contains such a vector. Then, by Proposition 5.6, C_G has an orthonormal basis $\mathcal{B}' = \{e'_1, \ldots, e'_l\}$.

Let (Σ, C) be a marked, closed, non-orientable surface of genus n. Identify $H_1(\Sigma)$ with F^n by having $[C_j]$ correspond to the standard basis vector e_j. Note that the intersection product corresponds to the mod-2 dot product under this identification. For $1 \leq j \leq l$, let $b'_j \in H_1(\Sigma)$ be the class that corresponds to $e'_j \in C_G$, so that $\{b'_1, \ldots, b'_l\}$ is an orthonormal basis for the l-dimensional subspace of $H_1(\Sigma)$ corresponding to C_G.

To begin the construction, let $\Sigma_{A,0}$ be a closed disk embedded in $\Sigma - C$, with $S_0 = \partial(\Sigma_{A,0})$ and $\Sigma_{B,0} = \overline{\Sigma - \Sigma_{A,0}}$. Thus, by Proposition 4.2, $H_1(\Sigma) = i_a\, H_1(\Sigma_{A,0}) \oplus i_b\, H_1(\Sigma_{B,0}) = 0 \oplus H_1(\Sigma)$. Let b_1 denote the class in $H_1(\Sigma_{B,0}, S_0)$ that corresponds to b'_1 under the standard isomorphism j_b. By Lemma 6.1, $b_1 = [\gamma_1]$ for some arc γ_1 properly embedded in $(\Sigma_{B,0}, S_0)$ and transverse to C.

Create $\Sigma_{A,1}$ by attaching to $\Sigma_{A,0}$ a 1-handle whose core is γ_1. Because $\langle b'_1, b'_1 \rangle = \bar{1}$, this 1-handle includes an odd number of half-twists, which ensures that $\partial(\Sigma_{A,1})$ is a *single* circle S_1 transverse to C. With $\Sigma_{B,1} = \overline{\Sigma - \Sigma_{A,1}}$, we now have $H_1(\Sigma) = i_a\, H_1(\Sigma_{A,1}) \oplus i_b\, H_1(\Sigma_{B,1}) = W_1 \oplus {W_1}^{\perp}$, where $W_1 = \operatorname{span}\{b'_1\}$. Let b_2 denote the class in $H_1(\Sigma_{B,1}, S_1)$ that corresponds to b'_2 under the standard isomorphism j_b. By Lemma 6.1, $b_2 = [\gamma_2]$ for some arc γ_2 properly embedded in $(\Sigma_{B,1}, S_1)$ and transverse to C.

Create $\Sigma_{A,2}$ by attaching to $\Sigma_{A,1}$ a 1-handle whose core is γ_2. Again, $\partial(\Sigma_{A,2})$ is a single circle S_2 transverse to C. With $\Sigma_{B,2} = \overline{\Sigma - \Sigma_{A,2}}$, we now have $H_1(\Sigma) = i_a\, H_1(\Sigma_{A,2}) \oplus i_b\, H_1(\Sigma_{B,2}) = W_2 \oplus {W_2}^{\perp}$, where $W_2 = \operatorname{span}\{b'_1, b'_2\}$. Continue in this fashion until l 1-handles have been attached, giving a sub-surface $\Sigma_A = \Sigma_{A,l}$ for which $i_a\, H_1(\Sigma_A) = W_l = \operatorname{span}\{b'_1, \ldots, b'_l\}$. Then $i_a\, H_1(\Sigma_A) \subseteq H_1(\Sigma)$ corresponds to $C_G \subseteq F^n$, and so $[P_A] = A_G$ and we are done.

(If C_G contains only self-orthogonal vectors, carry out the construction above using an orthonormal basis $\mathcal{B}'$ for N_G. Then simply swap the labels "Σ_A" and "Σ_B" at the end of the process.) □

Remark 6.4. In light of the freedom present in the choice of the arcs γ_j, it should not be surprising that a given op-graph can be realized by inequivalent splittings. (In order to make this statement precise, let us say that splittings (Σ, C, S) and (Σ, C, S') are *equivalent* if there exists a self-homeomorphism h of Σ, isotopic to the identity, with $h(S) = S'$.) For example, the reader is invited to check that the splitting depicted in Figure 15 yields the op-graph from Figure 5. However, this splitting is not equivalent to the splitting in Figure 10. One way to see this is to compute the classes in *integral* homology represented by the splitting curves S and S' in the two figures. Since a homeomorphism isotopic to the identity induces the identity map on homology, if the splittings were equivalent then we would have $[S] = \pm[S'] \in H_1(\Sigma;\mathbb{Z})$. But with the given orientations, direct calculation gives $[S] = 2\big([C_1]-[C_3]-[C_4]+[C_5]\big)$ and $[S'] = 2\big([C_1]+[C_3]+[C_4]+[C_5]\big)$. Since $H_1(\Sigma;\mathbb{Z})$ is generated by $[C_1],\dots,[C_6]$ subject to the single relation $2\big([C_1]+\cdots+[C_6]\big) = 0$, $[S] \neq \pm[S'] \in H_1(\Sigma;\mathbb{Z})$.

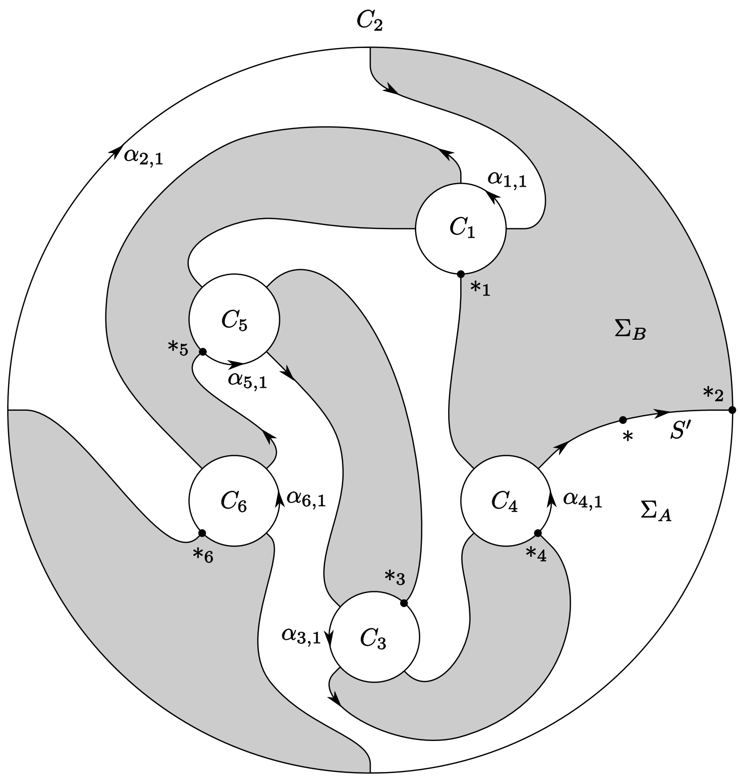

FIGURE 15. A “different” splitting realizing the op-graph in Figure 5.

We conclude with the following example, which involves the 9-vertex op-graph Γ introduced in Section 3.

Example 6.5. As noted earlier, the op-graph Γ in Figure 6 does not arise from a classical alternating knot diagram. The interested reader can check that the splitting depicted in Figure 16 realizes Γ.

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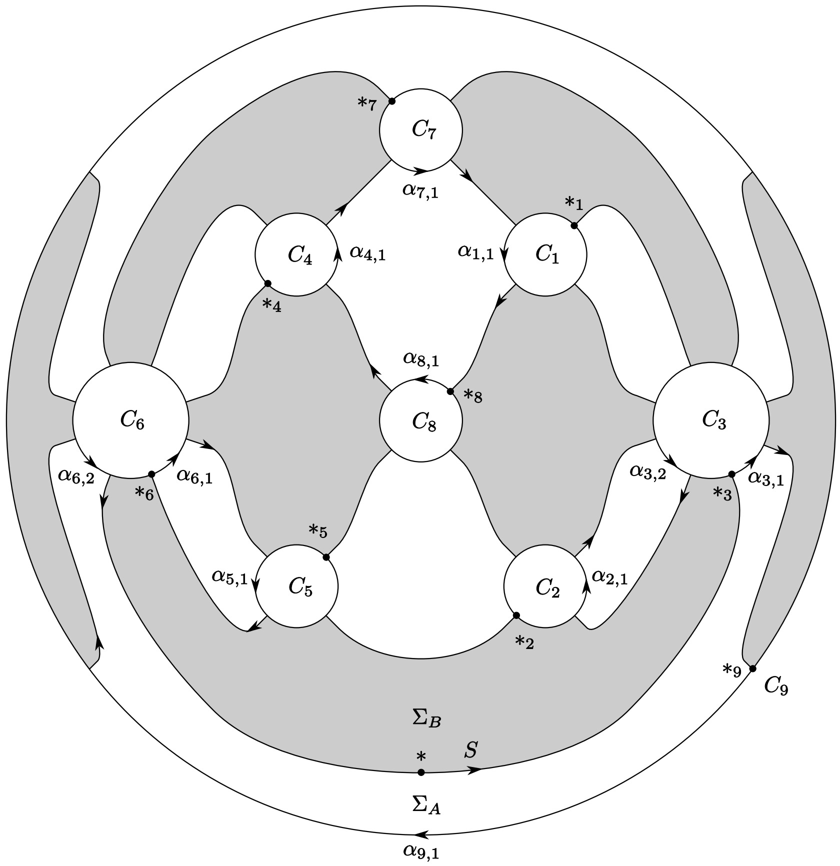

FIGURE 16. A splitting realizing the op-graph Γ.

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