Non-minimal sums of disjoint products

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Abstract

We refine a conjecture of M. O. Locks and J. M. Wilson [*Reliability Engineering and System Safety* **46** (1994), 283-286] regarding disjoint forms of the Abraham Reliability problem. We also present other examples of reliability problems with the interesting property that their minimal disjoint forms do not arise from the familiar tautology $M_1+...+M_n \equiv M_1+(M_2\cdot\bar{M}_1)+(M_3\cdot\bar{M}_2\cdot\bar{M}_1)+...+(M_n\cdot\bar{M}_{n-1}\cdot...\cdot\bar{M}_1)$.

Keywords. Sum of disjoint products, minimal disjoint form.

1. Introduction

A *coherent binary system* or *coherent reliability problem* on a finite set E is given by its *minpaths* $M_1, ..., M_n$, which are subsets of E none of which contains any other. Each of the elements of E either *operates* or *fails*, independently of the others; the system *operates* if at least one M_i is completely operational, and otherwise the system *fails*. A (minpath-based) *SDP algorithm* calculates the reliability of the system by expressing the logical disjunction $M_1 + ... + M_n$ as an equivalent sum of disjoint products. There are two families of SDP algorithms, distinguished by the kinds of products they produce. SDP algorithms which use *single-variable inversion* (SVI) produce products which are conjunctions involving variables $a, b, c, ..$ representing individual elements of E and also involving negations $\bar{a}, \bar{b}, \bar{c}, ..$ of these variables. Algorithms which use *multiple-variable inversion* (MVI) produce products which also involve negations $\overline{a_1...a_m}$ of grouped variables; $\overline{a_1...a_m} = \bar{a}_1 + ... + \bar{a}_m$ is satisfied if at least one of the grouped variables fails. A variable may not appear more than once in a single SVI or MVI product: $ab\bar{a}c, ab\overline{ac}$ and $\overline{ab}\,\overline{ac}$ are all ineligible. See [11] for an exposition of several SDP algorithms which appear in the literature; all of the algorithms follow this outline.

1. Choose the order in which $M_1, ..., M_n$ are to be considered.
2. Analyze $M_1+...+M_n$ as a sum of disjoint events $M_1+(M_2\cdot\bar{M}_1)+(M_3\cdot\bar{M}_2\cdot\bar{M}_1)+...+(M_n\cdot\bar{M}_{n-1}\cdot...\cdot\bar{M}_1)$.
3. Analyze each individual disjoint event $M_j \cdot \bar{M}_{j-1} \cdot ... \cdot \bar{M}_1$ as a sum of disjoint products.

All published SDP algorithms share step 2 of the outline, but they differ in the way they *preprocess* minpaths in step 1 (see [2, 3, 4, 7, 8, 9, 13, 14] for various preprocessing strategies) and in the way they analyze conjunctions of negations in step 3 (negation strategies are discussed in [6, 12] for SVI-SDP and [5, 10, 11] for MVI-SDP).

In this note the effectiveness of an SDP algorithm in analyzing a system is measured using the number of disjoint terms produced by the algorithm. In practice it is extremely difficult to determine all the disjoint product analyses of a system, so it is extremely difficult to verify that a particular sum of disjoint products is a minimal disjoint form (or m.d.f.) for a given system. In Section 2 we observe that it is much easier to determine the possible number of disjoint terms in a sum of disjoint products if something is known about the order in which $M_1, ..., M_n$ are listed. For instance it is often possible to determine the minimum number of terms that appear in disjoint forms in which step 1 follows the advice of [1], with smaller minpaths listed before larger ones. As an illustration of the technique we refine a conjecture stated in [9], that a disjoint form of the best-known example in the literature, a network mentioned in [1], must have at least 55 terms for SVI-SDP and 35 terms for MVI-SDP. We prove that an SVI disjoint form of this example in which smaller minpaths are listed before larger ones must have at least 54 terms, and we exhibit such a disjoint form; we also exhibit a slightly different SVI disjoint form which has only 53 terms, obtained using a preprocessing strategy discussed in detail in [2]. In addition we prove that an MVI disjoint form of this example obtained by listing smaller minpaths before larger ones must have at least 35 terms. We do not know whether a 53-term SVI disjoint form or a 35-term MVI disjoint form is minimal for this problem.

In Section 3 we observe that for some examples the use of $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$ in step 2 is not optimal. There are other sums of disjoint events which are equivalent to $M_1 + ... + M_n$ and produce smaller disjoint forms of these examples.

2. The Abraham Reliability problem

If steps 1 and 2 of an SDP algorithm are applied to a problem as mentioned in the introduction, then in step 3 each event $M_j \cdot \bar{M}_{j-1} \cdot ... \cdot \bar{M}_1$ is analyzed as a sum of disjoint products. Equivalently, the relative complements $M_i - M_j = \{e \in E \mid e \in M_i \text{ and } e \notin M_j\}$ with $i < j$ are analyzed; only those which are minimal with respect to inclusion must be considered.

Given in Table 1 is an SVI-SDP analysis of the Abraham Reliability problem [1], one of the most-studied examples in the literature. The minpaths of the problem are listed in the first column of the table, and to the right of each minpath M_j are disjoint terms whose conjunctions with M_j give a disjoint sum equivalent to $M_j \cdot \bar{M}_{j-1} \cdot ... \cdot \bar{M}_1$. This disjoint form involves 54 terms, so it contradicts a conjecture of [9]; moreover a term may be "saved" by listing $acfjl$ before $acdh$.

We claim that the disjoint form given in Table 1 is minimal among SVI disjoint forms which always list smaller minpaths before larger ones, and we also claim that an MVI disjoint form which follows the same kind of minpath order must involve at least 35 terms. We prove these claims by analyzing each size group of minpaths separately.

Any listing of the minpaths in increasing size order must start with jkl. The 4-element minpaths are discussed in the following chart.

minpath	min. rel. comps. (required)	min. rel. comps. (possible)	min. number of disjoint terms	number of disjoint terms if last
$acdh$	jkl	bjl, fk	3 (SVI) or 1 (MVI)	5 (SVI) or 2 (MVI)
$bcjl$	k	adh	1 (SVI) or 1 (MVI)	3 (SVI) or 1 (MVI)
$dfhk$	jl	ac	2 (SVI) or 1 (MVI)	4 (SVI) or 1 (MVI)

The chart presents information about each minpath's contributions to SVI and MVI disjoint forms. The most difficult fact to verify is that $acdh$ requires 5 disjoint terms if it is listed last; this is verified by calculating the negation of the system of minimal relative complements jkl, bjl, fk. We use Boolean multiplication and absorption as discussed in [6] and [12].

$$\begin{aligned} & \overline{(jkl + bjl + fk)} \\ = \;& (\bar{j} + \bar{k} + \bar{l})(\bar{b} + \bar{j} + \bar{l})(\bar{f} + \bar{k}) \\ = \;& \bar{j}\,\bar{f} + \bar{j}\bar{k} + \bar{l}\,\bar{f} + \bar{l}\,\bar{k} + \bar{k}\bar{b} \end{aligned}$$

There are 5 minimal negating terms, so at least 5 disjoint SVI terms are required. It happens that these 5 SVI terms can be covered with 2 disjoint MVI terms, $\bar{k}\,\overline{bjl} + k\bar{f}\,\overline{jl}$.

We see that in an SVI-SDP analysis, listing any one of these minpaths last "costs" 2 SVI terms over the minimum; hence no list can produce an SVI-SDP analysis with fewer than 8 terms, the number of terms appearing in Table 1. The list $acdh, bcjl, dfhk$ requires only 3 MVI terms. Observe that an optimal list for MVI-SDP must start with $acdh$, while an optimal list for SVI-SDP cannot start with $acdh$. This illustrates the fact that optimal preprocessing for SVI-SDP is generally different from optimal preprocessing for MVI-SDP; see [2] for a more complete discussion.

The 5-element minpaths are discussed in the following chart.

minpath	min. rel. comps. (required)	min. rel. comps. (possible)	min. number of disjoint terms	number of disjoint terms if last
$acfjl$	k, b, dh	egh, eil	2 (SVI) or 1 (MVI)	6 (SVI) or 3 (MVI)
$abdhk$	jl, f, c	gij	2 (SVI) or 1 (MVI)	3 (SVI) or 2 (MVI)
$bcdfh$	jl, k, a		2 (SVI) or 1 (MVI)	2 (SVI) or 1 (MVI)
$ghijk$	l, df, acd	abd, ace, ef	3 (SVI) or 2 (MVI)	5 (SVI) or 4 (MVI)
$acegh$	jkl, bjl, d	$fjl, ijk, il, fk, fikl$	3 (SVI) or 2 (MVI)	6 (SVI) or 4 (MVI)
$aceil$	jk, bj, dh	$fj, gh, fghk, fk$	4 (SVI) or 2 (MVI)	6 (SVI) or 4 (MVI)
$efghk$	jl, d	$ij, ac, il, acil$	2 (SVI) or 1 (MVI)	6 (SVI) or 2 (MVI)
$efikl$	j, dh	$acgh, ac, gh$	2 (SVI) or 1 (MVI)	4 (SVI) or 2 (MVI)

To prove the claim that the 24 terms in Table 1 constitute a minimal SVI-SDP analysis of these minpaths we must prove that every SVI-SDP analysis exceeds the total of the listed minimum numbers of disjoint terms by at least 4.

If $aceil$ is listed after $efikl$ then no matter what else is listed before $aceil$, $aceil$ requires 6 SVI terms, because

$$\begin{aligned}
& \overline{(jk + bj + dh + fk)} \\
= \quad & (\bar{j} + \bar{k})(\bar{b} + \bar{j})(\bar{d} + \bar{h})(\bar{f} + \bar{k}) \\
= \quad & (\bar{j} + \bar{b}\bar{k})(\bar{d}\bar{f} + \bar{d}\,\bar{k} + \bar{h}\bar{f} + \bar{h}\bar{k}) \\
= \quad & \bar{j}\bar{d}\bar{f} + \bar{j}\bar{d}\,\bar{k} + \bar{j}\bar{h}\bar{f} + \bar{j}\bar{h}\bar{k} + \bar{b}\bar{d}\,\bar{k} + \bar{b}\bar{h}\bar{k}
\end{aligned}$$

and taking one or both of fk, gh into account does not reduce the number of required terms. Similarly, if $aceil$ is listed before $efikl$ then no matter what else is listed before $efikl$, $efikl$ requires 4 SVI terms. That is, it "costs" 2 terms over the minimum to list either of $aceil, efikl$ before the other.

If $acegh$ is listed after $efghk$ then $acegh$ requires at least 5 SVI terms, because

$$\begin{aligned} & \overline{(jkl+bjl+fk)} \\ = \; & (\bar{j}+\bar{k}+\bar{l})(\bar{b}+\bar{j}+\bar{l})(\bar{f}+\bar{k}) \\ = \; & \bar{j}\bar{f}+\bar{j}\bar{k}+\bar{l}\bar{f}+\bar{l}\,\bar{k}+\bar{b}\bar{k} \end{aligned}$$

and taking one, two or all of fjl, ijk, il into account does not reduce the number of required terms. Similarly, if $acegh$ is listed before $efghk$ then no matter what else is listed before $efghk$, $efghk$ requires at least 4 SVI terms. Hence listing either of $acegh, efghk$ before the other "costs" 2 terms over the minimum. This verifies our claim that every listing of the 5-element minpaths produces an SVI-SDP analysis with at least 24 disjoint terms, 4 more than the minimum indicated in the chart.

We claim also that an MVI-SDP analysis of these minpaths must exceed the indicated minimum number of disjoint terms by at least 3. This claim is verified by observing that there are three list decisions which "cost" at least one term each: which of $abdhk$ and $ghijk$ comes first, which of $acegh$ and $aceil$ comes first, and which of $efghk$ and $efikl$ comes first. We see that the 5-element minpaths require at least 14 MVI terms.

The 6-element minpaths are discussed in the following chart.

minpath	min. rel. comps. (required)	min. rel. comp. (possible)	min. no. of disjoint terms	no. of disjoint terms if last
$abeghk$	jl, d, ij, c, f	il	2 (SVI) or 2 (MVI)	3 (SVI) or 2 (MVI)
$bcefgh$	jl, d, k, a	il, ij	2 (SVI) or 1 (MVI)	3 (SVI) or 2 (MVI)
$dehijk$	l, f, ac, ab, g		2 (SVI) or 2 (MVI)	2 (SVI) or 2 (MVI)
$abeikl$	j, f, c, dh	gh	2 (SVI) or 1 (MVI)	2 (SVI) or 2 (MVI)
$acdgil$	jk, bj, fj, h, e	fk	2 (SVI) or 2 (MVI)	3 (SVI) or 2 (MVI)
$bcefil$	j, dh, a, k	gh	2 (SVI) or 1 (MVI)	2 (SVI) or 2 (MVI)
$bcghij$	l, df, ad, k, ae	ef	3 (SVI) or 2 (MVI)	3 (SVI) or 3 (MVI)
$dfgikl$	j, h, e	ac	1 (SVI) or 1 (MVI)	2 (SVI) or 1 (MVI)

Table 1 lists 17 disjoint SVI terms for these minpaths, one more than the indicated minimum. Whichever of $acdgil, dfgikl$ comes after the other in a list cannot achieve the minimum number of SVI terms, so this portion of Table 1 is indeed optimal. Similarly, it is not possible to achieve the indicated minimum number of MVI terms because $bcefgh$ and $bcghij$ cannot both achieve the minimum, so these minpaths require at least 13 MVI terms.

Taking into account the fact that there are four 7-element minpaths, we see that a minpath list in which smaller minpaths precede larger ones cannot give rise to disjoint forms with fewer than 54 SVI terms or 35 MVI terms. ■

3. Non-optimality of $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$

In this section we give examples of reliability problems for which it is possible to improve the performance of an SDP algorithm by changing step 2: $M_1 + ... + M_n$ is most efficiently analyzed as a sum of disjoint events other than $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$. The first example uses the logical equivalence between $M_1 + M_2 + M_3$ and $M_1M_2M_3 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2) + (M_1 \cdot \bar{M}_3)$.

Example 3.1. Let n be a positive integer, and consider a reliability problem with 3 disjoint minpaths $A = a_1...a_n$, $B = b_1...b_n$ and $C = c_1...c_n$. An SVI-SDP algorithm structured as described in the introduction produces $1 + n + n^2$ disjoint terms. However the problem has an SVI-SDP analysis with only $3n + 1$ disjoint terms: $\bar{a}_1B$, $a_1\bar{a}_2B$, ..., $a_1...a_{n-1}\bar{a}_nB$, $\bar{b}_1C$, ..., $b_1...b_{n-1}\bar{b}_nC$, $\bar{c}_1A$, ..., $c_1...c_{n-1}\bar{c}_nA$, and ABC. ■

There are similar examples with more than three disjoint minpaths. For instance, to discuss an example with 7 disjoint minpaths we use the following lemma, whose proof we leave to the reader.

Lemma 3.2. The disjunction $M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7$ of seven statements is logically equivalent to the disjunction

$$\begin{aligned}
&M_1M_2M_3M_4M_5M_6M_7 \\
&+ \sum_{a=1}^{7} M_aM_{a+1}M_{a+2}M_{a+3}M_{a+5}\bar{M}_{a+6} \\
&+ \sum_{a=1}^{7} M_a\bar{M}_{a+1}M_{a+3}\bar{M}_{a+4}M_{a+5} \\
&+ \sum_{a=1}^{7} M_a\bar{M}_{a+1}\bar{M}_{a+2}M_{a+3}M_{a+4} \\
&+ \sum_{a=1}^{7} M_a\bar{M}_{a+1}\bar{M}_{a+2}\bar{M}_{a+3}
\end{aligned}$$

of 29 disjoint statements, where all indices involving a are considered modulo 7. That is, every combination of truth values of $M_1, ..., M_7$ which involves at least one "true" will satisfy exactly one of the 29 given statements.

Example 3.3. Let n be a positive integer, and consider a reliability problem with 7 disjoint minpaths

$M_1 = e_{11}...e_{1n}$, $M_2 = e_{21}...e_{2n}$, ..., $M_7 = e_{71}...e_{7n}$. An SVI-SDP algorithm structured as described in the introduction produces $1 + n + n^2 + n^3 + n^4 + n^5 + n^6$ disjoint terms. However the problem has an SVI-SDP analysis with only $1+7n+14n^2+7n^3$ disjoint terms, given by interpreting each M_a in Lemma 3.2 as $e_{a1}...e_{an}$ and each $\bar{M}_a$ as $\bar{e}_{a1} + (e_{a1}\bar{e}_{a2}) + (e_{a1}e_{a2}\bar{e}_{a3}) + ... + (e_{a1}...e_{an-1}\bar{e}_{an})$. ■

The following lemma will be useful in discussing some reliability problems whose minimal MVI disjoint forms do not follow the pattern $M_1+(M_2\cdot\bar{M}_1)+(M_3\cdot\bar{M}_2\cdot\bar{M}_1)+...+(M_n\cdot\bar{M}_{n-1}\cdot...\cdot\bar{M}_1)$. Let $n \geq 1$, and let b_n be the number of partitions of an n-element set into subsets of cardinality ≤ 2. The first few of these numbers are $b_1 = 1$, $b_2 = 2$, $b_3 = 4$, $b_4 = 10$, $b_5 = 26$, $b_6 = 76$, and $b_7 = 232$; in general $b_{n+2} = b_{n+1} + (n+1)b_n$.

Lemma 3.4. If $n \geq 1$ then

$$b_{n+2} > n\sum_{i=1}^{n} b_i.$$

Proof. The inequality is clearly true for the listed values of b_n. If $n \geq 6$ then

$$b_{n+2} = b_{n+1} + (n+1)b_n = b_n + nb_{n-1} + (n+1)b_n = nb_n + nb_{n-1} + 2b_n$$

and hence by induction

$$b_{n+2} > nb_n + nb_{n-1} + 2(n-2)\sum_{i=1}^{n-2} b_i > nb_n + nb_{n-1} + n\sum_{i=1}^{n-2} b_i. \blacksquare$$

Example 3.5. Let $E = \{e_{\{i,j\}} \,|\, 1 \leq i \neq j \leq n\}$, and consider the reliability problem on E with minpaths $M_1, ..., M_n$ given by $M_j = \{e_{\{i,j\}} \,|\, i \neq j\}$; we also use M_j to denote the conjunction of variables corresponding to the elements of the set M_j.

Suppose $2 \leq p \leq q \leq n-1$. We claim that if $j_1, ..., j_q$ are distinct indices between 1 and n then an MVI m.d.f. of $M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ requires exactly b_{p-1} terms, and an MVI m.d.f. of $M_{j_q} \cdot ... \cdot M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ requires no more than b_{p-1} terms. Before verifying the claim, we discuss its significance. If an MVI-SDP algorithm which follows the outline given in the introduction is applied, then according to the claim the MVI analysis of $M_{n-1} \cdot \bar{M}_{n-2} \cdot ... \cdot \bar{M}_1$ requires b_{n-2} terms. Consider instead an MVI disjoint form obtained by replacing the last 7 portions of step 2, $M_{n-6} \cdot \bar{M}_{n-7} \cdot ... \cdot \bar{M}_1$ through $M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1$, with 29 conjunctions of the form $S \cdot \bar{M}_{n-7} \cdot ... \cdot \bar{M}_1$, where S is obtained from one of the 29 statements of Lemma 3.2 by using M_{n-7+a} in place of M_a. The claim implies that each portion of the analysis has an MVI disjoint form with no more that b_ν terms, where ν is the number of negations appearing in the logical statement that describes that portion of the analysis. That is, this MVI disjoint form has no more than

$$\begin{aligned} & 1+b_1+b_2+...+b_{n-9}+b_{n-8}+b_{n-7}+7b_{n-6}+7b_{n-5}+7b_{n-5}+7b_{n-4} \\ < \quad & 14\sum_{i=1}^{n-4} b_i \end{aligned}$$

terms. Lemma 3.4 tells us that $\sum_{i=1}^{n-4} b_i < \frac{b_{n-2}}{n-4}$. We conclude that any MVI-SDP algorithm which follows the pattern $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$ in step 2 of the outline given in the introduction produces an MVI disjoint form which has at least $\frac{n-4}{14}$ times as many terms as a minimal MVI disjoint form has.

It remains to prove the claim that if $2 \leq p \leq q \leq n-1$ and $j_1, ..., j_q$ are distinct indices between 1 and n then an MVI m.d.f. of $M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ requires exactly b_{p-1} terms, and an MVI m.d.f. of $M_{j_q} \cdot ... \cdot M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ requires no more than b_{p-1} terms. Observe that it is enough to prove that an MVI m.d.f. of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$ requires exactly b_{p-1} terms. The corresponding claim about $M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ follows directly, because the M_j are symmetric with respect to permutations of the indices. Also, we can construct a disjoint MVI form of $M_{j_q} \cdot ... \cdot M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ with at most b_{p-1} MVI terms, by modifying an m.d.f. of $M_{j_p} \cdot \bar{M}_{j_{p-1}} \cdot ... \cdot \bar{M}_{j_1}$ in obvious ways: removing all the variables corresponding to elements of $M_{j_{p+1}} \cup ... \cup M_{j_q}$ from any grouped negations, conjoining all these variables to every MVI term, and eliminating any terms which have become unsatisfiable.

We order the subsets of $\{1, ..., p-1\}$ of cardinality ≤ 2 lexicographically, i.e., if $1 \leq a < b < c \leq p-1$ then $\{a\} < \{a, b\} < \{a, c\} < \{b\} < \{b, c\} < \{c\}$. Then we order the partitions of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 lexicographically, i.e., if π and π' are partitions of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 and a is the smallest index which appears in different sets in π and π' then $\pi < \pi'$ or $\pi > \pi'$ according to the sets in π and π' which contain a. For instance, if $p = 8$ then $\{\{1, 3\}, \{2, 5\}, \{4\}, \{6, 7\}\} > \{\{1, 3\}, \{2, 4\}, \{5, 6\}, \{7\}\}$ because the smallest index which appears in different sets in the two partitions is $a = 2$, and $\{2, 5\} > \{2, 4\}$ lexicographically.

Suppose $2 \leq p \leq q \leq n-1$, $j_1, ..., j_q$ are distinct indices between 1 and n and we are given an MVI disjoint form of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$. Let π be a partition of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2, and let $Y_\pi = E - \{e_{\{i,j\}} \,|\, \{i, j\} \in \pi\} - \{e_{\{i,n\}} \,|\, \{i\} \in \pi\}$. Then Y_π contains M_p but not any other M_j with $j < p$, so the combination of truth values in which all elements of Y_π are true and all non-elements of Y_π are false satisfies $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$; hence this combination of truth values also satisfies one of the terms in the disjoint form. As an MVI term in a disjoint form of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$, this term must assert that all the variables corresponding to elements of M_p are true, and for each $i < p$ must have at least one grouped negation of variables all of which are elements of M_i. Suppose $\{i, j\} \in \pi$. The MVI term in question must have at least one grouped negation of variables all of which are elements of M_i and at least one grouped negation of variables all of which are elements of M_j; both of these grouped negations must contain $e_{\{i.j\}}$, because $e_{\{i,j\}}$ is the only element of $M_i - Y_\pi$ or $M_j - Y_\pi$. The definition of MVI products requires that no variable appear more than once in a product; consequently the grouped negations corresponding to M_i and M_j must be the same, because both mention $e_{\{i,j\}}$. The only possible grouped negation that involves elements of both M_i and M_j is the single-variable negation $\overline{e_{\{i,j\}}}$, because $e_{\{i,j\}}$ is the only common element of M_i and M_j. Hence the term in question must include $\overline{e_{\{i,j\}}}$. On the other hand, if $\{i, j\} \notin \pi$ then $e_{\{i,j\}} \in Y_\pi$ and consequently the negation $\overline{e_{\{i,j\}}}$ could not possibly appear in this MVI term. It follows that this MVI term satisfies the combination of truth values corresponding to Y_π but not the combination corresponding to any $Y_{\pi'}$ with $\pi' \neq \pi$. This shows that there must be at least b_{p-1} different terms in an MVI disjoint form of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$.

To complete the proof of the claim we construct an MVI disjoint form of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$ with b_{p-1} terms.

Suppose π is a partition of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2, and let S_π be the set of all pairs $\{a, b\}$ for which there is a partition π' of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $\pi < \pi'$, a is the smallest index which appears in different sets in π and π', and $\{a, b\} \in \pi'$. That is, if we regard π as the result of a sequence of choices made in lexicographic order then S_π is the set of pairs $\{a, b\}$ which could have been chosen but were not, and would have resulted in a lexicographically greater partition. Let $X_\pi = M_p \cup \{e_{\{a,b\}} \mid \{a, b\} \in S_\pi\}$ and $N_\pi = \{\{e_{\{a,b\}}\} \mid \{a, b\} \in \pi\} \cup \{M_a - X_\pi \mid \{a\} \in \pi\}$; obviously every combination of truth values which satisfies the MVI statement

$$P_\pi = \left(\prod_{x \in X_\pi} x \right) \left(\prod_{N \in N_\pi} \bar{N} \right)$$

also satisfies $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$. We assert that these statements P_π constitute a disjoint MVI form of $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$.

If $S \subseteq E$ contains M_p but does not contain any of $M_1, ..., M_{p-1}$ then there are certainly partitions π of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $e_{\{a,b\}} \notin S$ for all $\{a, b\} \in \pi$, for instance $\{\{1\}, \{2\}, \{3\}, ..., \{p-1\}\}$. Let $\pi(S)$ be the lexicographically greatest partition of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $e_{\{a,b\}} \notin S$ for all $\{a, b\} \in \pi(S)$, and suppose there is an $\{a, b\} \in S_{\pi(S)}$ with $e_{\{a,b\}} \notin S$. According to the definition of $S_{\pi(S)}$, there is a partition π' of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $\pi(S) < \pi'$, a is the smallest index which appears in different sets in $\pi(S)$ and π', and $\{a, b\} \in \pi'$. If π'' is obtained from π' by replacing every pair $\{c, d\} \in \pi' - \pi - \{\{a, b\}\}$ with the two singletons $\{c\}$ and $\{d\}$ then π'' is lexicographically greater than $\pi(S)$ and $\{a, b\}$ is the only pair in π'' which is not also an element of $\pi(S)$. Hence $e_{\{c,d\}} \notin S$ for all $\{c, d\} \in \pi''$, contradicting the choice of $\pi(S)$. This contradiction shows that there is no $\{a, b\} \in S_{\pi(S)}$ with $e_{\{a,b\}} \notin S$, and hence $X_{\pi(S)} \subseteq S$. Recalling that $e_{\{a,b\}} \notin S$ for all $\{a, b\} \in \pi(S)$ and that S does not contain any of $M_1, ..., M_{p-1}$, we conclude that the MVI statement $P_{\pi(S)}$ is satisfied by the combination of truth values in which all elements of S are true and all non-elements of S are false.

It follows that every combination of truth values which satisfies $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$ must satisfy at least one of the statements P_π. The reverse is obviously true, as was noted when the P_π were defined above; hence $M_p \cdot \bar{M}_{p-1} \cdot ... \cdot \bar{M}_1$ is logically equivalent to the disjunction of the statements P_π. To complete the proof of our assertion, then, we must show that if $\pi \neq \pi'$ then P_π and $P_{\pi'}$ are logically disjoint, i.e., no combination of truth values satisfies both.

Suppose $S \subseteq E$ contains M_p but does not contain any of $M_1, ..., M_{p-1}$, and suppose $\pi' \neq \pi(S)$. If $\pi' < \pi(S)$ then there must be a pair $\{a, b\} \in \pi(S) \cap P_{\pi'}$. Then $e_{\{a,b\}} \notin S$, so $X_{\pi'} \nsubseteq S$. If $\pi' > \pi(S)$ then there must be a pair $\{a, b\} \in \pi'$ such that $e_{\{a,b\}} \in S$; then $\{e_{\{a,b\}}\} \in N_{\pi'}$. Either way, $P_{\pi'}$ is not satisfied by the combination of truth values in which all elements of S are true and all non-elements of S are false. This verifies our assertion. ■

It is not difficult to rationalize the inability of SDP algorithms which use the logical pattern $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$ to find minimal disjoint forms for Examples 3.1, 3.3 and 3.5. The examples are symmetrical, and the pattern is not; consequently the pattern is not well suited to the examples, and in particular the event $M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1$ is quite complicated and requires many terms. The disjoint disjunction of Lemma 3.2 gives rise to a disjoint disjunction of n statements which is symmetric with respect to cyclic permutations of the last seven statements and has no term involving more than $n-4$ negations. It is this upper bound on the maximum number of negations per term which produces disjoint forms with fewer terms. The reader might guess that disjoint disjunctions which are more highly symmetric may have still fewer negations per term, and hence produce disjoint forms of these examples with even fewer terms than the ones discussed above. After much work we have been able to verify this guess, using the following limited generalization of Lemma 3.2.

Proposition 3.6. For every odd prime $p \leq 23$ there are disjoint disjunctions which are equivalent to M_1 +... $+M_p$, symmetric with respect to cyclic permutations of $M_1, ..., M_p$, and have no term involving more than $\frac{p-1}{2}$ negations.

These cyclically symmetric disjoint disjunctions are quite large: the one mentioned in Lemma 3.2 involves only 29 terms, but the other examples we have found involve hundreds of terms ($p = 11$ or 13), thousands of terms ($p = 17$ or 19), even hundreds of thousands of terms ($p = 23$). Nevertheless for large values of n these disjunctions may be used to show that all MVI-SDP algorithms which follow the pattern $M_1 + (M_2 \cdot \bar{M}_1) + (M_3 \cdot \bar{M}_2 \cdot \bar{M}_1) + ... + (M_n \cdot \bar{M}_{n-1} \cdot ... \cdot \bar{M}_1)$ produce MVI disjoint forms of the reliability problems in Example 3.5 with up to $O(n^9)$ times as many terms as appear in minimal MVI disjoint forms. We conjecture that such disjoint disjunctions exist for all odd primes but we do not know how to try to prove this conjecture.

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jkl						
$bcjl$	$\bar{k}$					
$acdh$	$\bar{j}$	$j\bar{l}$	$jl\bar{b}\bar{k}$			
$dfhk$	$\bar{c}\bar{j}$	$\bar{c}j\bar{l}$	$c\bar{a}\bar{j}$	$c\bar{a}j\bar{l}$		
$acfjl$	$\bar{b}\bar{k}\bar{d}$	$\bar{b}\bar{k}d\bar{h}$				
$bcdfh$	$\bar{a}\bar{k}\bar{j}$	$\bar{a}\bar{k}j\bar{l}$				
$abdhk$	$\bar{c}\bar{f}\bar{j}$	$\bar{c}\bar{f}j\bar{l}$				
$ghijk$	$\bar{l}\bar{d}$	$\bar{l}d\bar{f}\bar{a}$	$\bar{l}d\bar{f}a\bar{b}\bar{c}$			
$efghk$	$\bar{d}\bar{j}$	$\bar{d}j\bar{\imath}\bar{l}$				
$acegh$	$\bar{d}\bar{j}\bar{f}$	$\bar{d}\bar{j}f\bar{k}$	$\bar{d}j\bar{l}\bar{k}$	$\bar{d}j\bar{l}k\bar{f}\bar{\imath}$	$\bar{d}jlk\bar{b}\bar{f}$	
$efikl$	$\bar{j}\bar{h}$	$\bar{j}h\bar{d}\bar{g}$				
$aceil$	$\bar{h}\bar{k}\bar{j}$	$\bar{h}\bar{k}j\bar{b}\bar{f}$	$\bar{h}k\bar{f}\bar{j}$	$h\bar{g}\bar{d}\bar{j}\bar{f}$	$h\bar{g}\bar{d}\bar{j}f\bar{k}$	$h\bar{g}\bar{d}j\bar{b}\bar{f}\bar{k}$
$dehijk$	$\bar{f}\bar{g}\bar{l}\bar{a}$	$\bar{f}\bar{g}\bar{l}a\bar{b}\bar{c}$				
$abeghk$	$\bar{c}\bar{d}\bar{f}\bar{j}$	$\bar{c}\bar{d}\bar{f}j\bar{\imath}\bar{l}$				
$bcefgh$	$\bar{a}\bar{d}\bar{k}\bar{j}$	$\bar{a}\bar{d}\bar{k}j\bar{l}$				
$bcghij$	$\bar{k}\bar{l}\bar{a}\bar{f}$	$\bar{k}\bar{l}\bar{a}f\bar{d}\bar{e}$	$\bar{k}\bar{l}a\bar{d}\bar{e}$			
$abeikl$	$\bar{c}\bar{f}\bar{j}\bar{h}$	$\bar{c}\bar{f}\bar{j}h\bar{d}\bar{g}$				
$bcefil$	$\bar{a}\bar{j}\bar{k}\bar{h}$	$\bar{a}\bar{j}\bar{k}h\bar{d}\bar{g}$				
$dfgikl$	$\bar{e}\bar{h}\bar{j}$					
$acdgil$	$\bar{e}\bar{h}\bar{j}\bar{f}$	$\bar{e}\bar{h}\bar{j}f\bar{k}$	$\bar{e}\bar{h}j\bar{k}\bar{b}\bar{f}$			
$acfghij$	$\bar{b}\bar{d}\bar{e}\bar{k}\bar{l}$					
$bcdehij$	$\bar{a}\bar{f}\bar{g}\bar{k}\bar{l}$					
$bcdfgil$	$\bar{a}\bar{e}\bar{h}\bar{j}\bar{k}$					
$abdgikl$	$\bar{c}\bar{e}\bar{f}\bar{h}\bar{j}$					

Table 1