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SHORTER NOTES

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LINKS WITH FREE GROUPS ARE TRIVIAL

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ABSTRACT. In this note we extend a well-known consequence of Dehn's Lemma to include tame links in the three-sphere.

Recall that a link in S^3 is a disjoint union of finitely many embedded circles (the *components* of the link). Two links are *equivalent* iff there is a homeomorphism of S^3 with itself which carries one onto the other, and a link is *tame* iff it is equivalent to a polygonal link. A link is *trivial* iff it is equivalent to a link lying in a plane in R^3.

THEOREM. *Let* $L \subseteq S^3$ *be a tame link whose group* $G = \pi_1(S^3 - L)$ *is free. Then* L *is a trivial link.*

PROOF. Let μ be the number of components of L; note that G is free of rank μ, for its abelianization $H_1(S^3 - L; \mathbf{Z}) \cong H^1(L; \mathbf{Z})$ is free abelian of rank μ, by Alexander duality. If $\mu = 1$ we obtain the standard unknotting theorem for tame knots in S^3 [**2**, p. 103].

Proceeding inductively, suppose that $\mu > 1$. Then L must be a split union $L = L_1 \cup L_2$ of two nonempty sublinks L_1 and L_2 (i.e., $L = L_1 \cup L_2$ and there is an embedded 2-sphere $S \subseteq S^3 - L$ such that L_1 and L_2 are contained in different components of $S^3 - S$); for if not then $S^3 - L$ would be an Eilenberg-Mac Lane space $K(G, 1)$ [**1**, p. 19], from which it would follow that $H_2(S^3 - L; \mathbf{Z}) \cong H_2(G) = 0$ ($H_2(G) = 0$ because G is free). By Alexander duality, though, $H_2(S^3 - L; \mathbf{Z}) \cong \tilde{H}^0(L; \mathbf{Z})$ is free abelian of rank $\mu - 1$.

(The fact that L is splittable can also be deduced directly from Theorem (27.1) of [**1**].)

Thus L is a split union $L = L_1 \cup L_2$, and hence $G \cong G_1 * G_2$, where $G_i = \pi_1(S^3 - L_i)$. In particular, G_1 and G_2, being subgroups of the free group G, are themselves free. By inductive hypothesis, then, L_1 and L_2 are both trivial links, and hence so is their split union L. Q.E.D.

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