On the colored Tutte polynomial of a graph of bounded treewidth

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Abstract

We observe that a formula given by S. Negami [*Trans. Amer. Math. Soc.* **299** (1987), 601-622] for the Tutte polynomial of a k-sum of two graphs generalizes to a colored Tutte polynomial. Consequently an algorithm of A. Andrzejak [*Discrete Math.* **190** (1998), 39-54] may be directly adapted to compute the colored Tutte polynomial of a graph of bounded treewidth in polynomial time. This result has also been proven by J. A. Makowsky [*Discrete Appl. Math.* **145** (2005), 276-290], using a different algorithm based on logical techniques.

Keywords: colored Tutte polynomial, k-sum, splitting formula, treewidth

1. Introduction

The *Tutte polynomial* (or *dichromate*) is one of the most-studied invariants of graphs and matroids; it has many important applications, including vertex colorings, flows, reliability, polynomial invariants of knots, and partition functions of models of statistical mechanics. We refer the reader to [5, 21, 23] for detailed expositions. Many of these applications involve some kind of edge-weighting or edge-coloring; for instance in knot theory an edge of a plane graph represents a crossing of a knot diagram, and is colored to indicate which of the two possible crossings is present.

In our discussion we follow standard terminology of graph theory and matroid theory, as for instance in [5, 17, 22]. A graph G has an associated *cycle matroid* M, whose rank function is defined as follows. For a subset $S \subseteq E = E(G)$ let $G : S$ be the subgraph of G with $V(G : S) = V(G)$ and $E(G : S) = S$; if $c(G : S)$ is the number of connected components of $G : S$ then $r(S) = |V(G)| - c(G : S)$. Whenever we discuss matroids, we presume that graphs are included in the discussion via their cycle matroids.

The (uncolored) Tutte polynomial may be defined in several equivalent ways. One is a *deletion-contraction recursion*: if M is a matroid then $T(M)$ is an element of the polynomial ring $\mathbb{Z}[X, Y]$ such that $T(M) = XT(M/e)$ for any isthmus e of M, $T(M) = YT(M-e)$ for any loop e of M, and $T(M) = T(M-e)+T(M/e)$

for any other element e. The empty matroid $\emptyset$ has $T(\emptyset) = 1$. Another definition utilizes a *subset expansion*:

$$T(M) = \sum_{S \subseteq E} (X-1)^{r(E)-r(S)} (Y-1)^{|S|-r(S)}.$$

Here r denotes the rank function of the matroid M. A third definition of the Tutte polynomial is phrased in terms of *basis activities*; this definition will not be useful in this note, so we do not discuss it in detail.

The evident exponential character of these definitions suggests the result of [10] that computing $T(M)$ is $\#P$-hard in general.

Pathwidth and *treewidth* were introduced in [18, 19, 20] as measures of the structural complexity of graphs, and since then these notions have proven to be of great interest. Many algorithmically intractable problems become tractable when restricted to graphs of bounded treewidth; in particular, the result that computing $T(M)$ is tractable for graphs of bounded treewidth was mentioned by Welsh in [21], and proven by Andrzejak in [2]. (See also [15].) The crucial ingredients of Andrzejak's proof are algorithms of Bodlaender and Hagerup which find a binary tree decomposition of a graph G of treewidth bounded by k [3, 4] and Negami's splitting formula for the Tutte polynomial of the k-sum of two graphs [14]; the splitting formula is used repeatedly as G is constructed from simpler graphs according to a binary tree decomposition. Makowsky has recently proven a broad generalization of this result [11, 12, 13], using logical techniques; this generalization covers colored versions of the Tutte polynomial and many other kinds of generating functions definable in monadic second order logic.

In this note we observe that Negami's splitting formula can be extended to a colored version of the Tutte polynomial, and hence that for this polynomial and its evaluations (like the Kauffman bracket) a colored version of Andrzejak's algorithm may be used. As Makowsky observes in [13], the greater generality of logical techniques comes at a significant computational price; an algorithm based on Negami's splitting formula is less general but more practical.

2. Negami's splitting formula, with colors

Suppose a matroid M on a set E is *colored* by a function $c : E \to \Lambda$, and suppose that for each $\lambda \in \Lambda$ elements x_λ, y_λ of a field have been chosen. (We trust that there will be no confusion between the color $c(e)$

of an edge of G and the number of components $c(H)$ of a subgraph of G.) Zaslavsky [24] calls

$$R(M,c) = \sum_{S \subseteq E} (\prod_{e \in S} x_{c(e)})(\prod_{e \in E-S} y_{c(e)})(X-1)^{r(E)-r(S)}(Y-1)^{|S|-r(S)}$$

the *normal function* of the colored matroid. $R(M,c)$ is probably the most natural colored version of $T(M)$, though there are many others; see [6, 9, 24] and the references mentioned there. $R(M,c)$ has a basis activities expansion analogous to that of $T(M)$, and it satisfies a colored deletion-contraction recursion: $R(M,c) = ((X-1)y_{c(e)} + x_{c(e)})R(M/e,c)$ for any isthmus e of M, $R(M,c) = ((Y-1)x_{c(e)} + y_{c(e)})R(M-e,c)$ for any loop e of M, and $R(M,c) = y_{c(e)}R(M-e,c) + x_{c(e)}R(M/e,c)$ for any other element e. The empty matroid $\emptyset$ has $R(\emptyset,c) = 1$. Zaslavsky proves that $R(M,c)$ is the most interesting field-valued invariant which satisfies such a weighted deletion-contraction recursion; any other such invariant is either *degenerate* (determined by the set E, the rank of M and the loops and coloops of M) or *non-global* (defined only for certain choices of the x_λ and y_λ). This result generalizes a universal property of $T(M)$ [8].

It may be surprising that $R(M,c)$ can be a simpler invariant to understand than $T(M)$. If we regard $T(M)$ as a subset expansion, it is a generating function for the number of subsets S of E with given corank $r(E)-r(S)$ and nullity $|S|-r(S)$. Similarly, if the various x_λ and y_λ are independent indeterminates then $R(M,c)$ is a generating function for the number of subsets S of E with given corank $r(E)-r(S)$, nullity $|S|-r(S)$, and distribution of colored elements inside and outside S. Why is this simpler? Well, if the function $c : E \to \Lambda$ is injective then the distribution of colored elements inside and outside S completely determines S. After determining $r(E)$ as the corank of $\emptyset$, we see that if c is injective and the various x_λ and y_λ are independent indeterminates then $R(M,c)$ is essentially a list which gives the rank of every $S \subseteq E$, i.e., $R(M,c)$ is essentially the rank function of M.

Theorem 2.1. Let G be a k-sum of subgraphs H and K, i.e., $V(G) = V(H) \cup V(K)$, $|V(H) \cap V(K)| = k$, $E(G) = E(H) \cup E(K)$ and $E(H) \cap E(K) = \emptyset$. Then there is a formula which depends only on k, $|V(G)|$ and $c(G)$ which gives $R(M,c)$ as a function of the R-invariants and numbers of connected components of contractions of H and K.

The proof of Theorem 2.1 follows the proof of Theorem 4.2 in [14] very closely. The key to what Negami calls his "beautiful formula" is to study not $R(M,c)$ but a related invariant $f(G,c)$ with a simple deletion-contraction property which treats loops, isthmuses and other edges in the same way.

Let G be a graph, and suppose $E = E(G)$ is colored by $c : E \to \Lambda$. Suppose that t is an element of a field and for each $\lambda \in \Lambda$, x_λ and y_λ are also elements of the field; for instance the field in question could be the field of quotients of the polynomial ring $\mathbb{Z}[\{t\} \cup \{x_\lambda, y_\lambda : \lambda \in \Lambda\}]$. Then the deletion-contraction formula $f(G,c) = x_{c(e)} f(G/e,c) + y_{c(e)} f(G-e,c)$, together with the initial condition that if $E(G) = \emptyset$ then $f(G,c) = t^{|V(G)|}$, yields a well-defined invariant $f(G,c)$.

Proposition 2.2.

$$f(G,c) = \sum_{S \subseteq E(G)} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \in E-S} y_{c(e)} \right) t^{c(G:S)}.$$

Proof. This is true by definition if $E(G) = \emptyset$. Suppose inductively that $e_0 \in E(G)$; then

$$\begin{aligned} f(G,c) &= x_{c(e_0)} f(G/e_0,c) + y_{c(e_0)} f(G-e_0,c) \\ &= x_{c(e_0)} \cdot \sum_{S \subseteq E(G/e_0)} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \in E-\{e_0\}-S} y_{c(e)} \right) t^{c((G/e_0):S)} \\ &\quad + y_{c(e_0)} \cdot \sum_{S \subseteq E(G-e_0)} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \in E-\{e_0\}-S} y_{c(e)} \right) t^{c((G-e_0):S)} \\ &= \sum_{e_0 \in S \subseteq E(G)} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \in E-S} y_{c(e)} \right) t^{c(G:S)} \\ &\quad + \sum_{e_0 \notin S \subseteq E(G)} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \in E-S} y_{c(e)} \right) t^{c(G:S)}. \blacksquare \end{aligned}$$

It follows from Proposition 2.2 that $f(G,c)$ is closely related to $R(M,c)$; the corresponding relationship between the uncolored invariants $f(G)$ and $T(M)$ is well known [14, 16].

Corollary 2.3. $(X-1)^{c(G)}(Y-1)^{|V(G)|}R(M,c)$ is obtained from $f(G,c)$ by evaluating $t \mapsto (X-1)(Y-1)$, each $x_{c(e)} \mapsto x_{c(e)}(Y-1)$ and each $y_{c(e)} \mapsto y_{c(e)}$. Also, $t^{-c(G)} f(G,c)$ is obtained from $R(M,c)$ by evaluating $X \mapsto t+1$, $Y \mapsto 2$, each $x_{c(e)} \mapsto x_{c(e)}$ and each $y_{c(e)} \mapsto y_{c(e)}$.

Suppose G is a k-sum of subgraphs H and K, i.e., $V(G) = V(H) \cup V(K)$, $|V(H) \cap V(K)| = k$, $E(G) = E(H) \cup E(K)$ and $E(H) \cap E(K) = \emptyset$. Let $U = V(H) \cap V(K)$, and let $\Gamma(U)$ be the set of all partitions of U; $\Gamma(U)$ is a lattice when ordered by refinement. For $\gamma, \gamma' \in \Gamma(U)$ let $\gamma \wedge \gamma'$ be the *meet* in this lattice. That is, $\gamma \wedge \gamma'$ is the partition defined by: u and u' are in the same element of $\gamma \wedge \gamma'$ if there is a sequence $u = u_1, u_2, ..., u_a = u'$ such that when $1 \le i < a$, u_i and u_{i+1} are in the same element of either γ or γ'. Using an arbitrary ordering of $\Gamma(U)$ as $\{\gamma_1, \gamma_2, ...\}$ Negami [14] defines T_k to be the square matrix whose (i,j) entry is $t^{|\gamma_i \wedge \gamma_j|}$ and observes that if t is an indeterminate then T_k is nonsingular: the product of terms on the diagonal is clearly of higher degree than any other product of terms that contributes to the determinant, so the determinant must be nonzero.

For $\gamma \in \Gamma(U)$ a contraction G/γ may be defined in the obvious way: each element of γ gives rise to a single vertex of G/γ and each edge e of G gives rise to an edge e of G/γ which connects the vertices of G/γ containing the end-vertices of e. Clearly non-loop edges of G may become loops in G/γ, and isthmuses of G may become non-isthmuses in G/γ.

Proposition 2.4 below gives a splitting formula for $f(G, c)$; Theorem 2.1 follows immediately from this formula and Corollary 2.3. Observe that $|V(G)|$ and the various connected component counts which appear in Theorem 2.1 come from Corollary 2.3, not Proposition 2.4; in the terminology of [14] $f(G, c)$ *splits nicely* but $R(M, c)$ does not. We do not give the proof of Proposition 2.4 because it is no different from Negami's, but it is worth noting that the argument depends on the fact that the deletion-contraction property of f is insensitive to whether an edge is a loop, an isthmus, or neither; this insensitivity makes it easy to compare computations of $f(G, c)$ and $f(H, c)$.

Proposition 2.4. Let ${}^t\mathbf{f}(K/\Gamma(U), c)$ be the row vector $(f(K/\gamma_1, c), f(K/\gamma_2, c), ...)$ and $\mathbf{f}(H/\Gamma(U), c)$ be the column vector

$$\begin{pmatrix} f(H/\gamma_1, c) \\ f(H/\gamma_2, c) \\ \vdots \end{pmatrix}.$$

Then

$${}^t\mathbf{f}(K/\Gamma(U), c) \cdot T_k^{-1} \cdot \mathbf{f}(H/\Gamma(U), c) = f(G, c).$$

Corollary 2.5. If G is a colored graph of bounded treewidth, $R(M, c)$ may be calculated in polynomial time.

Andrzejak [2] gives a detailed discussion of an algorithm which calculates $T(M)$ by determining a binary, rooted tree decomposition of G and repeatedly using Negami's splitting formula; the algorithm runs in time $O(n^{2+7\log_2 c})$, where $n = |V(G)|$ and c is twice the number of partitions of a set with $3k + 3$ elements, presuming that arithmetic operations are performed at constant cost. The same discussion applies here, with the caveat that the cost of arithmetic operations may depend on the field that contains the x_λ and y_λ.

Similarly, if we consider the relationship between $R(M, c)$ and the rank function of M we conclude

Corollary 2.6. For a graph G of bounded treewidth the rank function of the cycle matroid M may be determined in polynomial time.

We do not know how these results may generalize to matroids which are not cycle matroids of graphs; Negami's argument requires both the simple deletion-contraction recursion of f and the presence of vertices in $H \cap K$, so it does not apply to arbitrary matroids. There are results in the literature regarding Tutte polynomials of generalized parallel connections of general matroids [1, 7], but those generalized parallel connections are not sufficiently general to include all k-sums of graphs.

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