# Generalized activities and reliability

Lorenzo Traldi
Department of Mathematics, Lafayette College
Easton, Pennsylvania 18042

**Abstract**

The analysis of the Tutte polynomial of a matroid using activities is associated with a shelling of the family of spanning sets. We introduce an activities analysis of the reliability of a system specified by an arbitrary clutter, associated with an $\mathcal{S}$-partition rather than a shelling. These activities are related to a method of constructing Boolean interval partitions developed by Dawson in the early 1980s.

## 1. Introduction

Let $\mathfrak{C}$ be a clutter on a finite set $E$, i.e., a family of subsets of $E$, none of which contains any other. If $X \subseteq E$ then the *deletion* $\mathfrak{C} - X$ is $\{C \in \mathfrak{C} \,|\, C \cap X = \emptyset\}$ and the *contraction* $\mathfrak{C}/X$ consists of the minimal elements of $\{S \subseteq E - X \,|\, S \cup X \in \mathfrak{F}(\mathfrak{C})\}$; both are clutters on $E - X$. If $X = \{e\}$ then $\mathfrak{C} - X$ and $\mathfrak{C}/X$ are denoted $\mathfrak{C} - e$ and $\mathfrak{C}/e$, respectively. Deletions and contractions are clearly associative and commutative in the sense that $(((\mathfrak{C}-W)/X)-Y)/Z = (\mathfrak{C}-(W\cup Y))/(X\cup Z) = (\mathfrak{C}/(X\cup Z))-(W\cup Y) = \{\text{minimal } S \subseteq E - (W \cup X \cup Y \cup Z) \,|\, S \cup X \cup Z \in \mathfrak{F}(\mathfrak{C})\}$ for any pairwise disjoint $W, X, Y, Z \subseteq E$. A *loop* is an element of $E$ which appears in no element of $\mathfrak{C}$ and an *isthmus* is an element of $E$ which appears in every element of $\mathfrak{C}$; note that when $\mathfrak{C} = \emptyset$ every element of $E$ is both a loop and an isthmus. We use a term from reliability theory and refer to the elements of $\mathfrak{C}$ as *minpaths*.

If $\mathfrak{F}(\mathfrak{C})$ is the associated *filter* $\mathfrak{F}(\mathfrak{C}) = \{S \subseteq E | S \text{ contains an element of } \mathfrak{C}\}$ then the polynomial

$$\mathrm{rel}(\mathfrak{C}) = \sum_{S \in \mathfrak{F}(\mathfrak{C})} p^{|S|}(1-p)^{|E-S|}$$

represents the *reliability* or *percolation probability* of a system whose minimal states are the elements of $\mathfrak{C}$; the indeterminate $p$ represents the probability of successful operation of an individual element of $E$. Observe that

$\mathrm{rel}(\mathfrak{C})$ may also be defined recursively: if $e$ is a loop of $\mathfrak{C}$ then $\mathrm{rel}(\mathfrak{C}) = \mathrm{rel}(\mathfrak{C}-e)$, if $e$ is an isthmus of $\mathfrak{C}$ then $\mathrm{rel}(\mathfrak{C}) = p \cdot \mathrm{rel}(\mathfrak{C}/e)$, if $e$ is neither a loop nor an isthmus of $\mathfrak{C}$ then $\mathrm{rel}(\mathfrak{C}) = (1-p) \cdot \mathrm{rel}(\mathfrak{C}-e) + p \cdot \mathrm{rel}(\mathfrak{C}/e)$, $\mathrm{rel}(\emptyset) = 0$ and $\mathrm{rel}(\{\emptyset\}) = 1$.

If $\mathfrak{C}$ is the clutter of bases of a matroid $M$ then $\mathrm{rel}(\mathfrak{C})$ is a specialization of the Tutte polynomial $t(M)$:

$$\mathrm{rel}(\mathfrak{C}) = p^{r(E)}(1-p)^{|E|-r(E)} \cdot t(M)(1, \frac{1}{1-p}).$$

We refer to the literature (e.g., [20] or [21]) for an introduction to the Tutte polynomial. The property of particular interest to us is the fact that recursive calculations of $t(M)$ give rise to interval partitions of the power-set $2^E$ using *activities* [1, 3, 7, 12, 14]. (Recall that an *interval* of sets is of the form $[X,Y] = \{S \mid X \subseteq S \subseteq Y\}$). An activities partition of $2^E$ has the property that each interval intersects $\mathfrak{F}(\mathfrak{C})$ in a subinterval; moreover these subintervals constitute a *shelling* of $\mathfrak{F}(\mathfrak{C})$. (See [21] for an introduction to shellings, but note that shellings are usually defined for complexes rather than filters, so the definition must be "turned upside down" for our purposes). Such a shelling of $\mathfrak{F}(\mathfrak{C})$ is associated with a recursive calculation of $\mathrm{rel}(\mathfrak{C})$ in much the same way that the original activities partition of $2^E$ is associated with a calculation of $t(M)$.

Dawson [8, 10, 11] investigated generalizations of these activities partitions of $2^E$. He observed that if $\mathfrak{B}$ is any nonempty family of subsets of $E = \{e_1, ..., e_m\}$ then for each $S \in 2^E$ there is a unique $B = \beta_{\mathfrak{B}}(S) \in \mathfrak{B}$ which minimizes

$$\sum_{e_i \in B \Delta S} 2^i.$$

(Here $B \Delta S$ is the symmetric difference $(B-S) \cup (S-B)$.) $\beta_{\mathfrak{B}}(S)$ is constructed recursively: if $\beta_{\mathfrak{B}}(S) \cap \{e_{i+1}, ..., e_m\}$ is determined, then (1) $e_i \in \beta_{\mathfrak{B}}(S)$ if every $B \in \mathfrak{B}$ with $B \cap \{e_{i+1}, ..., e_m\} = \beta_{\mathfrak{B}}(S) \cap \{e_{i+1}, ..., e_m\}$ has $e_i \in B$, (2) $e_i \notin \beta_{\mathfrak{B}}(S)$ if every $B \in \mathfrak{B}$ with $B \cap \{e_{i+1}, ..., e_m\} = \beta_{\mathfrak{B}}(S) \cap \{e_{i+1}, ..., e_m\}$ has $e_i \notin B$, and otherwise (3) $e_i \in \beta_{\mathfrak{B}}(S)$ or $e_i \notin \beta_{\mathfrak{B}}(S)$ according to whether $e_i \in S$ or $e_i \notin S$. For each $B \in \mathfrak{B}$, $\beta_{\mathfrak{B}}^{-1}(\{B\})$ is an interval in $2^E$, so $\beta_{\mathfrak{B}}$ defines a partition of $2^E$ into intervals each of which contains precisely one element of $\mathfrak{B}$. Dawson gave several conditions on $\beta_{\mathfrak{B}}$ which are equivalent to $\mathfrak{B}$ being a matroid basis clutter; we give another in Corollary 2.7 below.

If $\mathfrak{C}$ is not a matroid basis clutter then Dawson's $\beta_{\mathfrak{C}}$ might not give an interval partition of $\mathfrak{F}(\mathfrak{C})$, because the intervals of Dawson's partition of $2^E$ might not intersect $\mathfrak{F}(\mathfrak{C})$ in subintervals. Consequently this partition might not correspond to a recursive calculation of $\mathrm{rel}(\mathfrak{C})$. The purpose of the present paper is to introduce activities partitions for arbitrary clutters which do correspond to recursive calculations of $\mathrm{rel}(\mathfrak{C})$.

We complete our introduction by assuring the skeptical reader that the reliability of arbitrary clutters is not an instance of "abstract nonsense." See [17, 18] for two types of practical network reliability problems which are *universal* in the sense that they provide concrete realizations of arbitrary clutters.

## 2. Results

If $\mathfrak{C}$ is a clutter on $E = \{e_1, ..., e_m\}$ then by a *resolution* of $\mathfrak{C}$ we mean a sequence $\mathfrak{C} = \mathfrak{C}_m, \mathfrak{C}_{m-1}, ..., \mathfrak{C}_0$ of clutters obtained from $\mathfrak{C}$ by removing the elements of $E$ in reverse order, first $e_m$, then $e_{m-1}$, and so on; each element is to be removed either by contraction or by deletion. We associate to a resolution the set $S$ of elements contracted during that resolution, and we denote the resolution $\mathrm{Res}(\mathfrak{C}, S)$. Observe that if $C \in \mathfrak{C}$ and $\mathrm{Res}(\mathfrak{C}, C)$ is $\mathfrak{C}_m, \mathfrak{C}_{m-1}, ..., \mathfrak{C}_0$ then for $i < m$, $C - \{e_{i+1}, ..., e_m\} \in \mathfrak{C}_i$; consequently $\mathrm{Res}(\mathfrak{C}, C)$ never involves either the contraction of a loop or the deletion of an isthmus. This observation actually characterizes the minpaths of matroid basis clutters [12]. However, if $\mathfrak{C}$ is non-matroidal there may be other subsets of $E$ whose associated resolutions have neither contracted loops nor deleted isthmuses.

**Definition 2.1**. *If $B \in \mathfrak{F}(\mathfrak{C})$, $B \notin \mathfrak{C}$ and $\mathrm{Res}(\mathfrak{C}, B)$ involves neither the deletion of any isthmus nor the contraction of any loop then $B$ is a fake minpath of $\mathfrak{C}$. We denote $\mathfrak{B}(\mathfrak{C}) = \mathfrak{C} \cup \{$fake minpaths of $\mathfrak{C}\}$. For the sake of contrast we sometimes refer to the elements of $\mathfrak{C}$ as true minpaths of $\mathfrak{C}$.*

The requirement that a fake minpath must be an element of $\mathfrak{F}(\mathfrak{C})$ is generally unnecessary: except when $\mathfrak{C} = \emptyset$, it follows from the requirement that the corresponding resolution not involve the deletion of any isthmus. See Proposition 3.1.

For an example of the definition, consider $\mathfrak{C} = \{\{1\}, \{2, 3\}\}$. The resolution of $\mathfrak{C}$ corresponding to $B = \{1, 3\}$ is the sequence $\mathfrak{C}/3 = \{\{1\}, \{2\}\}$, $(\mathfrak{C}/3) - 2 = \{\{1\}\}$, $((\mathfrak{C}/3) - 2)/1 = \{\emptyset\}$. No element is deleted as an isthmus or contracted as a loop during the resolution, so $B$ is a fake minpath of $\mathfrak{C}$.

If $\mathfrak{C}$ is a clutter on a set $E$ and $S \subseteq E$ then we use the notation $EO(S) = \{e \notin S \mid e$ is not deleted as a loop or an isthmus in $\mathrm{Res}(\mathfrak{C}, S)\}$; the elements of $EO(S)$ are *externally ordinary with respect to $S$*.

**Theorem 2.2**. *If $\mathfrak{C}$ is a clutter on a finite set $E$ then $\mathfrak{F}(\mathfrak{C})$ is partitioned by the intervals $[B, E - EO(B)]$, $B \in \mathfrak{B}(\mathfrak{C})$.*

Theorem 2.2 immediately yields an activities formula for $\mathrm{rel}(\mathfrak{C})$, obtained by grouping terms in the definition of $\mathrm{rel}(\mathfrak{C})$.

**Corollary 2.3**.

$$\mathrm{rel}(\mathfrak{C}) = \sum_{B \in \mathfrak{B}(\mathfrak{C})} p^{|B|} \cdot (1-p)^{|EO(B)|}$$

If $\mathfrak{C}$ is the basis clutter of a matroid $M$ then the formula of Corollary 2.3 is the same as the formula for $\mathrm{rel}(\mathfrak{C}) = p^{r(E)}(1-p)^{|E|-r(E)} \cdot t(M)(1, \frac{1}{1-p})$ which arises from the activities description of $t(M)$.

It may seem that Dawson's partition of $2^E$, which is defined without reference to contractions or deletions, is fundamentally different from our activities partition, but it turns out that there is a surprisingly strong relationship between the two.

**Theorem 2.4**. *If $\mathfrak{C} \neq \emptyset$ then the activities partition of $\mathfrak{F}(\mathfrak{C})$ is induced by the Dawson partition of $2^E$ corresponding to $\mathfrak{B}(\mathfrak{C})$, i.e., $[B, E - EO(B)] = \mathfrak{F}(\mathfrak{C}) \cap \beta^{-1}_{\mathfrak{B}(\mathfrak{C})}(\{B\})$ for $B \in \mathfrak{B}(\mathfrak{C})$.*

If $S \in \mathfrak{F}(\mathfrak{C})$ then Theorem 2.4 and the definition of $\beta_{\mathfrak{B}(\mathfrak{C})}$ make it clear that $\beta_{\mathfrak{B}(\mathfrak{C})}(S)$ is simply the lexicographically greatest subset of $S$ which is an element of $\mathfrak{B}(\mathfrak{C})$. It follows that if the elements of $\mathfrak{B}(\mathfrak{C})$ are listed in decreasing lexicographic order as $B_1, ..., B_p$ then for each $i$, every $S \supseteq B_i$ is in some interval $[B_j, E - EO(B_j)]$ with $j \leq i$. Consequently activities partitions of $\mathfrak{F}(\mathfrak{C})$ are $\mathcal{S}$*-partitions* in the sense of Chari [6]. (Chari actually defined $\mathcal{S}$-partitions for complexes, not filters; the two notions are clearly complementary.) In particular, if $\mathfrak{C}$ has no fake minpath then an activities partition of $\mathfrak{F}(\mathfrak{C})$ is a (possibly non-pure) *shelling* [2, 4, 5].

The focus of this paper is on $\mathrm{rel}(\mathfrak{C})$, and consequently we focus on partitions of $\mathfrak{F}(\mathfrak{C})$ rather than partitions of $2^E$. There is however a natural way to extend the activities partition from $\mathfrak{F}(\mathfrak{C})$ to all of $2^E$. For $S \subseteq E$ let $IO(S) = \{s \in S \,|\, e$ is neither a loop nor an isthmus when contracted in $\mathrm{Res}(\mathfrak{C}, S)\}$; the elements of $IO(S)$ are *internally ordinary with respect to* $S$.

**Theorem 2.5**. *If $\mathfrak{C}$ is a nonempty clutter on a finite set $E$ then the intervals $[IO(B), E - EO(B)]$, $B \in \mathfrak{B}(\mathfrak{C})$, partition $2^E$. Moreover $[IO(B), E - EO(B)] = \beta^{-1}_{\mathfrak{B}(\mathfrak{C})}(\{B\})$ for every $B \in \mathfrak{B}(\mathfrak{C})$.*

The fake minpaths provide a characterization of matroid basis clutters.

**Theorem 2.6**. *A nonempty clutter $\mathfrak{C}$ on a finite set $E$ is a matroid basis clutter if and only if there is no ordering of $E$ which produces a fake minpath for $\mathfrak{C}$.*

We deduce a characterization of matroids using $\beta_{\mathfrak{C}}$; related characterizations were given by Dawson [10].

**Corollary 2.7.** *A nonempty clutter $\mathfrak{C}$ on a finite set $E$ is a matroid basis clutter if and only if every ordering of $E$ produces a function $\beta_{\mathfrak{C}}$ with the property that $\beta_{\mathfrak{C}}(S) \subseteq S$ for all $S \in \mathfrak{F}(\mathfrak{C})$.*

It is possible for a non-matroid basis clutter to be without fake minpaths in *some* orderings. For example, $\mathfrak{C} = \{\{1,2\},\{3\}\}$ has no fake minpath even though the re-ordered version $\{\{2,3\},\{1\}\}$ has the fake minpath $\{1,3\}$. An indication of the difficulty of characterizing these clutters is given in our last theorem.

**Theorem 2.8**. *The class of clutters which possess no fake minpath with respect to some ordering is closed under deletion but not under contraction.*

The present paper grew out of our study of a two-variable invariant $R(G,K)$ of a probabilistic $K$-terminal network $G$ [15, 16]. Just as the Tutte polynomial of a matroid reflects more information than $\mathrm{rel}(\mathfrak{C})$, where $\mathfrak{C}$ is the matroid's basis clutter, $R(G,K)$ reflects more information about $G$ than the $K$-terminal reliability $\mathrm{rel}(\mathfrak{C})$, where $\mathfrak{C}$ is the clutter of edge-sets of Steiner $K$-trees in $G$. Consequently the activities analysis of $R(G,K)$ presented in [15, 16] is more complicated than the present paper's activities analysis of $\mathrm{rel}(\mathfrak{C})$; in particular the *convincing fake Steiner $K$-forests* discussed there correspond to the fake minpaths discussed here.

## 3. Proof of Theorem 2.2

**Proposition 3.1**. *Let $\mathfrak{C}$ be a nonempty clutter on $E = \{e_1, ..., e_m\}$, and suppose $S \subseteq E$. Then $S \in \mathfrak{F}(\mathfrak{C})$ if and only if $\mathrm{Res}(\mathfrak{C}, S)$ does not involve the deletion of any isthmus.*

**Proof**. If $S \in \mathfrak{F}(\mathfrak{C})$ then the last clutter appearing in $\mathrm{Res}(\mathfrak{C}, S)$ is $(\mathfrak{C} - (E - S))/S = \{\emptyset\}$. On the other hand, if any isthmus is deleted in $\mathrm{Res}(\mathfrak{C}, S)$ then the last clutter appearing in $\mathrm{Res}(\mathfrak{C}, S)$ is $\emptyset$. Consequently, if $S \in \mathfrak{F}(\mathfrak{C})$ then no isthmus is deleted in $\mathrm{Res}(\mathfrak{C}, S)$.

Suppose conversely that no isthmus is deleted in $\mathrm{Res}(\mathfrak{C}, S)$. The last clutter appearing in $\mathrm{Res}(\mathfrak{C}, S)$ is a clutter on the empty set, so it must be either $\emptyset$ or $\{\emptyset\}$. If it is $\{\emptyset\}$ then $\{\emptyset\} = (\mathfrak{C} - (E - S))/S$, so $S$ must contain an element of $\mathfrak{C} - (E - S)$ and hence an element of $\mathfrak{C}$, by the definitions of clutter deletion and contraction. Suppose instead that this last clutter is $\emptyset$, and let $\mathrm{Res}(\mathfrak{C}, S)$ be $\mathfrak{C} = \mathfrak{C}_m, \mathfrak{C}_{m-1}, ..., \mathfrak{C}_0$. As $\mathfrak{C}$ itself is nonempty, there must be an $i$ such that $\mathfrak{C}_i \neq \emptyset = \mathfrak{C}_{i-1}$; necessarily then $e_i$ is an isthmus in $\mathfrak{C}_i$ and $\mathfrak{C}_{i-1} = \mathfrak{C}_i - e_i$. ■

Note that in addition to proving the proposition, we have proven that $S \in \mathfrak{F}(\mathfrak{C})$ if and only if $\{\emptyset\} = (\mathfrak{C} - (E - S))/S$, and $S \notin \mathfrak{F}(\mathfrak{C})$ if and only if $\emptyset = (\mathfrak{C} - (E - S))/S$.

**Lemma 3.2**. *If $e \in E$ then $\mathfrak{C} - e = \mathfrak{C}/e$ if and only if $e$ is a loop of $\mathfrak{C}$.*

**Proof**. Certainly if $e$ is a loop of $\mathfrak{C}$ then $\mathfrak{C} - e = \mathfrak{C}/e$. If $e$ is not a loop of $\mathfrak{C}$ then $e \in C$ for some $C \in \mathfrak{C}$; $C - \{e\} \in \mathfrak{C}/e$ and $C - \{e\} \notin \mathfrak{C} - e$, so $\mathfrak{C} - e \neq \mathfrak{C}/e$. ■

If $S \subseteq E$ then let $IL(S) = \{e \in S \,|\, e$ is contracted as a loop in $\mathrm{Res}(\mathfrak{C}, S)\}$; we call the elements of $IL(S)$ *internal loops* of $S$. Similarly, the *external loops* of $S$ are the elements of $EL(S) = \{e \notin S \,|\, e$ is deleted as a loop in $\mathrm{Res}(\mathfrak{C}, S)\}$. Using this terminology we have an inductive generalization of Lemma 3.2.

**Corollary 3.3**. *Suppose $S, T \subseteq E$ and $\mathrm{Res}(\mathfrak{C}, S) = \mathrm{Res}(\mathfrak{C}, T)$. Then $S - T \subseteq EL(T) \cap IL(S)$.*

**Proposition 3.4**. *Suppose $S, T \subseteq E$. Then $\mathrm{Res}(\mathfrak{C}, S) = \mathrm{Res}(\mathfrak{C}, T)$ if and only if $S \Delta T \subseteq EL(S) \cup IL(S)$.*

**Proof**. If $\mathrm{Res}(\mathfrak{C}, S) = \mathrm{Res}(\mathfrak{C}, T)$ then $S \Delta T \subseteq EL(S) \cup IL(S)$ by Corollary 3.3. Suppose conversely that $S \Delta T \subseteq EL(S) \cup IL(S)$. Let $\mathrm{Res}(\mathfrak{C}, S)$ be $\mathfrak{C} = \mathfrak{C}_m, \mathfrak{C}_{m-1}, ..., \mathfrak{C}_0$ and let $\mathrm{Res}(\mathfrak{C}, T)$ be $\mathfrak{C} = \mathfrak{C}'_m, \mathfrak{C}'_{m-1}, ..., \mathfrak{C}'_0$; suppose $1 \leq i \leq m$ and $\mathfrak{C}_j = \mathfrak{C}'_j$ for $j \geq i$. If $e_i \in S \cap T$ then $\mathfrak{C}_{i-1} = \mathfrak{C}_i/e_i = \mathfrak{C}'_i/e_i = \mathfrak{C}'_{i-1}$. If $e_i \notin S \cup T$ then $\mathfrak{C}_{i-1} = \mathfrak{C}_i - e_i = \mathfrak{C}'_i - e_i = \mathfrak{C}'_{i-1}$. Otherwise $e_i \in S \Delta T \subseteq EL(S) \cup IL(S)$ and hence $e_i$ is a loop of $\mathfrak{C}_i = \mathfrak{C}'_i$. One of $\mathfrak{C}_{i-1}, \mathfrak{C}'_{i-1}$ is $\mathfrak{C}_i/e_i = \mathfrak{C}'_i/e_i$ and the other of $\mathfrak{C}_{i-1}, \mathfrak{C}'_{i-1}$ is $\mathfrak{C}_i - e_i = \mathfrak{C}'_i - e_i$; either way $\mathfrak{C}_{i-1} = \mathfrak{C}'_{i-1}$ because $e_i$ is a loop in $\mathfrak{C}_i$. As $\mathfrak{C}_m = \mathfrak{C}'_m = \mathfrak{C}$, it follows that $\mathrm{Res}(\mathfrak{C}, S) = \mathrm{Res}(\mathfrak{C}, T)$. ■

**Proposition 3.5**. *Define a relation $\sim$ on $2^E$ by: $S \sim T$ if and only if $S \Delta T \subseteq EL(S) \cup IL(S)$. Then $\sim$ is an equivalence relation, and for each*

*$S \subseteq E$ the equivalence class of $S$ is the interval $I(S) = [S - IL(S), S \cup EL(S)]$. Moreover, if $S \in \mathfrak{F}(\mathfrak{C})$ then $I(S) \subseteq \mathfrak{F}(\mathfrak{C})$.*

**Proof**. That $\sim$ is an equivalence relation follows from the fact that $S \sim T$ if and only if $\mathrm{Res}(\mathfrak{C}, S) = \mathrm{Res}(\mathfrak{C}, T)$.

Suppose $S \subseteq E$ and $T \in I(S)$. Then $T - S \subseteq (S \cup EL(S)) - S = EL(S)$ and $S - T \subseteq S - (S - IL(S)) = IL(S)$, so $S \sim T$. Conversely, if $S \sim T$ then $T \subseteq S \cup (T - S) \subseteq S \cup EL(S) \cup IL(S) = S \cup EL(S)$ and $T \supseteq S - (S - T) \supseteq S - (EL(S) \cup IL(S)) = S - IL(S)$, so $T \in I(S)$.

If $\mathfrak{C} = \emptyset$ then there is no $S \in \mathfrak{F}(\mathfrak{C})$. Otherwise, if $S \in \mathfrak{F}(\mathfrak{C})$ then $\mathrm{Res}(\mathfrak{C}, S)$ does not involve the deletion of any isthmus. If $T \in I(S)$ then $\mathrm{Res}(\mathfrak{C}, T) = \mathrm{Res}(\mathfrak{C}, S)$ also does not involve the deletion of any isthmus, so $T \in \mathfrak{F}(\mathfrak{C})$ too. ■

We turn now to the proof of Theorem 2.2; the theorem is vacuously true for $\mathfrak{C} = \emptyset$.

Suppose $\mathfrak{C} \neq \emptyset$. Proposition 3.5 implies that $\mathfrak{F}(\mathfrak{C})$ is partitioned by the intervals $I(S)$, $S \in \mathfrak{F}(\mathfrak{C})$; it follows that in order to verify Theorem 2.2 it suffices to show that for every $S \in \mathfrak{F}(\mathfrak{C})$, $S - IL(S)$ is the unique true or fake minpath of $\mathfrak{C}$ in $I(S)$. The resolutions of $\mathfrak{C}$ corresponding to $S$ and $S - IL(S)$ are identical, as they differ only in their treatment of certain loops. If $e$ were an internal loop of $S - IL(S)$ then $e$ would also be an internal loop of $S$, because $\mathrm{Res}(\mathfrak{C}, S - IL(S)) = \mathrm{Res}(\mathfrak{C}, S)$; but this would imply $e \in IL(S)$, contradicting the assumption that $e \in S - IL(S)$. Consequently $IL(S - IL(S)) = \emptyset$, so $S - IL(S)$ is a true or fake minpath of $\mathfrak{C}$. On the other hand, if $S - IL(S) \neq T \in I(S)$ then $T - (S - IL(S)) \subseteq IL(T)$ by Corollary 3.3, and consequently $IL(T) \neq \emptyset$; hence $T$ is not a true or fake minpath of $\mathfrak{C}$. ■

## 4. Activities and Dawson's construction

If $\mathfrak{C}$ is a clutter on $E$ then Dawson's partition of $2^E$ will be useful in analyzing $\mathrm{rel}(\mathfrak{C})$ if it provides an interval partition of $\mathfrak{F}(\mathfrak{C})$, i.e., if $\beta_{\mathfrak{C}}^{-1}(\{C\}) \cap \mathfrak{F}(\mathfrak{C})$ is an interval for each $C \in \mathfrak{C}$. In general, however, this does not happen. For instance, if $E = \{1, 2, 3, 4, 5\}$ and $\mathfrak{C} = \{\{1, 2\}, \{3, 4, 5\}\}$ then $\beta_{\mathfrak{C}}$ partitions $2^E$ into two intervals, $[\emptyset, \{1, 2, 3, 4\}]$ and $[\{5\}, \{1, 2, 3, 4, 5\}]$. The former intersects $\mathfrak{F}(\mathfrak{C})$ in the subinterval $[\{1, 2\}, \{1, 2, 3, 4\}]$ but the latter intersects $\mathfrak{F}(\mathfrak{C})$ in the more complicated set $[\{3, 4, 5\}, \{1, 2, 3, 4, 5\}] \cup \{\{1, 2, 5\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}\}$.

Evidently the failure of $\beta_{\mathfrak{C}}$ to provide an interval partition of $\mathfrak{F}(\mathfrak{C})$ is associated with the presence of sets $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}}(X) \nsubseteq X$. It is natural to try to "repair" this failure by adjoining some of these $X$ to $\mathfrak{C}$, and then applying Dawson's construction to this larger family of subsets of $E$. Adjoining all such $X$ to $\mathfrak{C}$ will result in an interval partition in which each $X$ gives rise to a singleton interval $[\{X\}, \{X\}]$; such a partition is likely to have many more intervals than are actually required. A more reasonable strategy is to adjoin to $\mathfrak{C}$ only the minimal such $X$, obtaining the family $\mathfrak{C}_1 = \mathfrak{C} \cup \{$minimal sets $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}}(X) \nsubseteq X\}$. If $\beta_{\mathfrak{C}_1}$ does not define an interval partition of $\mathfrak{F}(\mathfrak{C})$, then there must be sets $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}_1}(X) \nsubseteq X$; we may then consider $\mathfrak{C}_2 = \mathfrak{C}_1 \cup \{$minimal sets $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}_1}(X) \nsubseteq X\}$. Continuing in this way, we obtain a sequence $\mathfrak{C} = \mathfrak{C}_0 \subseteq \mathfrak{C}_1 \subseteq ...$ of families of subsets of $E$, with each $\mathfrak{C}_{i+1} = \mathfrak{C}_i \cup \{$minimal sets $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}_i}(X) \nsubseteq X\}$. The sequence must stabilize, i.e., there must be a $\mathfrak{C}_p$ such that every $X \in \mathfrak{F}(\mathfrak{C})$ has $\beta_{\mathfrak{C}_p}(X) \subseteq X$; then $\beta_{\mathfrak{C}_p}$ defines an interval partition of $\mathfrak{F}(\mathfrak{C})$. We call this $\mathfrak{C}_p$ the *Dawson completion* of $\mathfrak{C}$.

For example, consider the clutter $\mathfrak{C} = \{\{1,2\}, \{3,4,5\}\}$. It has $\mathfrak{C}_1 = \mathfrak{C} \cup \{\{1,2,5\}\}$, and $\beta_{\mathfrak{C}_1}$ provides a partition of $2^E$ into the three intervals $[\emptyset, \{1,2,3,4\}]$, $[\{5\}, \{1,2,3,5\}]$, and $[\{4,5\}, \{1,2,3,4,5\}]$. There is a unique $X \in \mathfrak{F}(\mathfrak{C})$ such that $\beta_{\mathfrak{C}_1}(X) \nsubseteq X$, namely $\{1,2,4,5\}$, so $\mathfrak{C}_2 = \mathfrak{C}_1 \cup \{\{1,2,4,5\}\}$. The corresponding function $\beta_{\mathfrak{C}_2}$ partitions $2^E$ into the intervals $[\emptyset, \{1,2,3,4\}]$, $[\{5\}, \{1,2,3,5\}]$, $[\{4,5\}, \{1,2,4,5\}]$, and $[\{3,4,5\}, \{1,2,3,4,5\}]$, which intersect $\mathfrak{F}(\mathfrak{C})$ in the subintervals $[\{1,2\}, \{1,2,3,4\}]$, $[\{1,2,5\}, \{1,2,3,5\}]$, $[\{1,2,4,5\}, \{1,2,4,5\}]$, and $[\{3,4,5\}, \{1,2,3,4,5\}]$; $\mathfrak{C}_2$ is the Dawson completion of $\mathfrak{C}$. (For a more interesting example, consider the clutter $\{\{1,4\}, \{2,3,5\}\}$.) The reader may have noticed that $\{1,2,5\}$ and $\{1,2,4,5\}$ are the fake minpaths of $\mathfrak{C}$; this is not a coincidence.

**Theorem 4.1**. *If $\mathfrak{C}$ is a nonempty clutter on a finite set, the Dawson completion of $\mathfrak{C}$ is $\mathfrak{C}_p = \mathfrak{B}(\mathfrak{C}) = \mathfrak{C} \cup \{$fake minpaths of $\mathfrak{C}\}$. Moreover, the activities partition of $\mathfrak{F}(\mathfrak{C})$ is identical to the partition defined by $\beta_{\mathfrak{C}_p}$.*

Theorem 4.1 follows from Propositions 4.2 - 4.4.

**Proposition 4.2**. *Let $\mathfrak{C}$ be a nonempty clutter on $E = \{e_1, ..., e_m\}$, and suppose $\mathfrak{C} \subseteq \mathfrak{D} \subseteq \mathfrak{B}(\mathfrak{C})$. Let $X$ be minimal among the elements of $\mathfrak{F}(\mathfrak{C})$ which have $\beta_{\mathfrak{D}}(X) \nsubseteq X$. Then $X$ is a fake minpath of $\mathfrak{C}$.*

**Proof**. Suppose not, and choose the greatest $j$ with $e_j \in IL(X)$. $\mathrm{Res}(\mathfrak{C}, X) = \mathrm{Res}(\mathfrak{C}, X - \{e_j\})$, so Proposition 3.1 implies that $X - \{e_j\} \in \mathfrak{F}(\mathfrak{C})$. The minimality of $X$ guarantees that $\beta_{\mathfrak{D}}(X - \{e_j\}) \subseteq X - \{e_j\}$ and hence $\beta_{\mathfrak{D}}(X-\{e_j\}) \neq \beta_{\mathfrak{D}}(X)$. If $e_j \notin \beta_{\mathfrak{D}}(X)$ the definition of $\beta_{\mathfrak{D}}(X)$ clearly implies that $\beta_{\mathfrak{D}}(X) = \beta_{\mathfrak{D}}(X-\{e_j\})$; hence $e_j \in \beta_{\mathfrak{D}}(X)$.

Recall that $\beta_{\mathfrak{D}}(X) \in \mathfrak{D} \subseteq \mathfrak{B}(\mathfrak{C})$, and hence $IL(\beta_{\mathfrak{D}}(X)) = \emptyset$. It follows that $\mathrm{Res}(\mathfrak{C}, X)$ and $\mathrm{Res}(\mathfrak{C}, \beta_{\mathfrak{D}}(X))$ differ at some point before $e_j$ is removed, for $e_j$ is removed as a loop in the former and not in the latter. Hence $X\Delta\beta_{\mathfrak{D}}(X)$ contains at least one element $e_q$ with $q > j$; choose the greatest such $q$. If $e_q \in \beta_{\mathfrak{D}}(X) - X$ then the definition of $\beta_{\mathfrak{D}}$ implies that every $B \in \mathfrak{D}$ with $B \cap \{e_{q+1}, ..., e_m\} = \beta_{\mathfrak{D}}(X) \cap \{e_{q+1}, ..., e_m\}$ has $e_q \in B$. This is impossible, though, for $\beta_{\mathfrak{D}}(X-\{e_j\}) \subseteq X$ and the definition of $\beta_{\mathfrak{D}}(X)$ clearly implies that $\beta_{\mathfrak{D}}(X-\{e_j\}) \cap \{e_{q+1}, ..., e_m\} = \beta_{\mathfrak{D}}(X) \cap \{e_{q+1}, ..., e_m\}$. (Indeed, $\beta_{\mathfrak{D}}(X-\{e_j\}) \cap \{e_{j+1}, ..., e_m\} = \beta_{\mathfrak{D}}(X) \cap \{e_{j+1}, ..., e_m\}$.) Consequently it must be that $e_q \in X - \beta_{\mathfrak{D}}(X)$; the definition of $\beta_{\mathfrak{D}}(X)$ then implies that every $B \in \mathfrak{D}$ with $B \cap \{e_{q+1}, ..., e_m\} = \beta_{\mathfrak{D}}(X) \cap \{e_{q+1}, ..., e_m\}$ has $e_q \notin B$. It follows that $\beta_{\mathfrak{D}}(X) = \beta_{\mathfrak{D}}(X-\{e_q\})$, and also that $e_q \notin \beta_{\mathfrak{D}}(X-\{e_j\})$.

The minimality of $X$ guarantees that $X-\{e_q\} \notin \mathfrak{F}(\mathfrak{C})$, for $\beta_{\mathfrak{D}}(X-\{e_q\}) = \beta_{\mathfrak{D}}(X) \nsubseteq X-\{e_q\}$. However, $\beta_{\mathfrak{D}}(X-\{e_j\}) \subseteq X-\{e_j\}$ and $e_q \notin \beta_{\mathfrak{D}}(X-\{e_j\})$, so $\beta_{\mathfrak{D}}(X-\{e_j\}) \subset X-\{e_q\}$; this is a contradiction because $\beta_{\mathfrak{D}}(X-\{e_j\}) \in \mathfrak{F}(\mathfrak{C})$. ■

**Proposition 4.3**. *Let $\mathfrak{C}$ be a nonempty clutter on $E = \{e_1, ..., e_m\}$, and let $\mathfrak{C} \subseteq \mathfrak{D} \subseteq \mathfrak{B}(\mathfrak{C})$. Suppose there is no $X \in \mathfrak{F}(\mathfrak{C})$ with $\beta_{\mathfrak{D}}(X) \nsubseteq X$. Then $\mathfrak{D} = \mathfrak{B}(\mathfrak{C})$.*

**Proof**. Suppose not, and let $X \notin \mathfrak{D}$ be a fake minpath of $\mathfrak{C}$. Then $\beta_{\mathfrak{D}}(X)$ is a proper subset of $X$; choose the largest $q$ so that $e_q \in X - \beta_{\mathfrak{D}}(X)$. $IL(X) = \emptyset$, so $e_q$ is not a loop of $(\mathfrak{C}-(\{e_{q+1}, ..., e_m\}-X))/(\{e_{q+1}, ..., e_m\} \cap X)$; hence there is some $T \in \mathfrak{C}' = (\mathfrak{C}-(\{e_{q+1}, ..., e_m\}-X))/(\{e_{q+1}, ..., e_m\} \cap X)$ such that $e_q \in T$.

Consider $Y = T \cup (\{e_{q+1}, ..., e_m\} \cap X)$. The definition of clutter contraction implies that $Y \in \mathfrak{F}(\mathfrak{C})$, so by hypothesis $\beta_{\mathfrak{D}}(Y) \subseteq Y$. Recall that $e_q \in X - \beta_{\mathfrak{D}}(X)$; the definition of $\beta_{\mathfrak{D}}$ implies that every $B \in \mathfrak{D}$ with $\{e_{q+1}, ..., e_m\} \cap X = \{e_{q+1}, ..., e_m\} \cap B$ has $e_q \notin B$. As $\{e_{q+1}, ..., e_m\} \cap X = \{e_{q+1}, ..., e_m\} \cap Y$, the definition of $\beta_{\mathfrak{D}}$ implies that $e_q \notin \beta_{\mathfrak{D}}(Y)$, so $\beta_{\mathfrak{D}}(Y) \subseteq Y - \{e_q\}$. This implies that $Y - \{e_q\}$ contains some element of $\mathfrak{C}$, and this in turn implies that $Y - \{e_q\} - \{e_{q+1}, ..., e_m\}$ contains some element of $(\mathfrak{C} - (\{e_{q+1}, ..., e_m\} - Y))/(\{e_{q+1}, ..., e_m\} \cap Y) = \mathfrak{C}'$. This is impossible, though, for such an element of $\mathfrak{C}'$ would be a proper subset of $T$. ■

**Proposition 4.4**. *Let $\mathfrak{C}$ be a nonempty clutter on $E = \{e_1, ..., e_m\}$, and suppose $\mathfrak{C} \subseteq \mathfrak{D} \subseteq \mathfrak{B}(\mathfrak{C})$. If $B \in \mathfrak{D}$ then $[B, B \cup EL(B)] \subseteq \beta_{\mathfrak{D}}^{-1}(\{B\})$.*

**Proof**. If $B \subseteq X \subseteq B \cup EL(B)$ and $\beta_{\mathfrak{D}}(X) \neq B$ then the definition of $\beta_{\mathfrak{D}}(X)$ is violated with respect to the element $e_j \in B \Delta \beta_{\mathfrak{D}}(X)$ which has the largest index $j$. We leave the details to the reader. ■

This completes the proof of Theorem 4.1, which implies Theorem 2.4. The next proposition is useful in the proof of Theorem 2.5.

**Proposition 4.5.** *Let $\mathfrak{C}$ be a nonempty clutter on $E = \{e_1, ..., e_m\}$, and let $\mathfrak{B} = \mathfrak{B}(\mathfrak{C})$. Suppose $S \subseteq E$ and $\beta_{\mathfrak{B}}(S) \not\subseteq S$; let $j$ be the largest index with $e_j \in \beta_{\mathfrak{B}}(S) - S$. Then $\beta_{\mathfrak{B}}(S) = \beta_{\mathfrak{B}}(S \cup \{e_j\})$, $e_j$ is deleted as an isthmus in* $\operatorname{Res}(\mathfrak{C}, S)$*, and $e_j$ is contracted as an isthmus in* $\operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(S))$*; moreover these two resolutions of $\mathfrak{C}$ coincide until the removal of $e_j$.*

**Proof.** The definition of $\beta_{\mathfrak{B}}$ implies that $\beta_{\mathfrak{B}}(S) = \beta_{\mathfrak{B}}(S \cup \{e_j\})$.

Let $B = \beta_{\mathfrak{B}}(S)$. Observe that $B \cap \{e_{j+1}, ..., e_m\} \subseteq S \cap \{e_{j+1}, ..., e_m\}$ and hence $(B \cup S) \cap \{e_{j+1}, ..., e_m\} = S \cap \{e_{j+1}, ..., e_m\}$; consequently $\operatorname{Res}(\mathfrak{C}, S)$ and $\operatorname{Res}(\mathfrak{C}, B \cup S)$ are identical until the removal of $e_j$. $B \cup S \in \mathfrak{F}(\mathfrak{C})$, so $\operatorname{Res}(\mathfrak{C}, S)$ does not involve the deletion of any isthmus before $e_j$ is removed. If $e_j$ is not an isthmus when it is removed then it may be deleted without resulting in the empty clutter, and this portion of $\operatorname{Res}(\mathfrak{C}, S)$ may be extended to a full resolution of $\mathfrak{C}$ which does not involve the deletion of any isthmus. Let $T$ be the set of elements contracted as non-loops in such a resolution; then $T \in \mathfrak{B}$.

As $B \cup S \in \mathfrak{F}(\mathfrak{C})$, Theorem 4.1 tells us that $\operatorname{Res}(\mathfrak{C}, B \cup S) = \operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(B \cup S))$. Consequently $\operatorname{Res}(\mathfrak{C}, S)$ and $\operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(B \cup S))$ are identical until the removal of $e_j$, and the same for $\operatorname{Res}(\mathfrak{C}, T)$ and $\operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(B \cup S))$. As $T, \beta_{\mathfrak{B}}(B \cup S) \in \mathfrak{B}$, Theorem 4.1 tells us that $T \cap \{e_{j+1}, ..., e_m\} = (\beta_{\mathfrak{B}}(B \cup S)) \cap \{e_{j+1}, ..., e_m\}$. This contradicts the definition of $\beta_{\mathfrak{B}}$, for $e_j \in \beta_{\mathfrak{B}}(S) - S$ and $e_j \notin T$; the contradiction proves that $e_j$ must be an isthmus when it is removed.

Finally, observe that $(B \cup S) \cap \{e_{j+1}, ..., e_m\} = S \cap \{e_{j+1}, ..., e_m\}$ implies $(\beta_{\mathfrak{B}}(B \cup S)) \cap \{e_{j+1}, ..., e_m\} = \beta_{\mathfrak{B}}(S) \cap \{e_{j+1}, ..., e_m\}$ and hence until $e_j$ is removed $\operatorname{Res}(\mathfrak{C}, S)$, $\operatorname{Res}(\mathfrak{C}, B \cup S)$, $\operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(B \cup S))$ and $\operatorname{Res}(\mathfrak{C}, \beta_{\mathfrak{B}}(S))$ are all the same. ■

In the situation of Proposition 4.5 either $S \cup \{e_j\} \supseteq \beta_{\mathfrak{B}}(S)$ or we may set $j = j_1$ and apply the proposition to the greatest-indexed element of

$\beta_{\mathfrak{B}}(S)-(S\cup\{e_{j_1}\})$. Repeating as many times as is necessary, we see that $\beta_{\mathfrak{B}}(S)-S=\{e_{j_1},...,e_{j_k}\}$ with $j_1>...>j_k$, where each $e_{j_i}$ is contracted as an isthmus in $\mathrm{Res}(\mathfrak{C},\beta_{\mathfrak{B}}(S))$. That is, $IO(\beta_{\mathfrak{B}}(S))\subseteq S$. Observe also that in the situation of Proposition 4.5 $\mathrm{Res}(\mathfrak{C},S)$ and $\mathrm{Res}(\mathfrak{C},\beta_{\mathfrak{B}}(S))$ coincide until $e_j$ is removed, so the elements of $S-\beta_{\mathfrak{B}}(S)$ with indices greater than $j$ are all loops in $\mathrm{Res}(\mathfrak{C},\beta_{\mathfrak{B}}(S))$. Applying this observation repeatedly, we see that $S\subseteq\beta_{\mathfrak{B}}(S)\cup EL(\beta_{\mathfrak{B}}(S))$, and hence $S\in[IO(\beta_{\mathfrak{B}}(S)),E-EO(\beta_{\mathfrak{B}}(S))]$. This shows that $\beta_{\mathfrak{B}}^{-1}(\{B\})\subseteq[IO(B),E-EO(B)]$ for every $B\in\mathfrak{B}$.

To complete the proof of Theorem 2.5 it suffices to show that $[IO(B),E-EO(B)]\subseteq\beta_{\mathfrak{B}}^{-1}(\{B\})$ for every $B\in\mathfrak{B}=\mathfrak{B}(\mathfrak{C})$. Suppose instead that $B\in\mathfrak{B}$, $S\in[IO(B),E-EO(B)]$ and $\beta_{\mathfrak{B}}(S)\neq B$. Let $j$ be the largest index with $e_j\in B\Delta\beta_{\mathfrak{B}}(S)$. Then $\mathrm{Res}(\mathfrak{C},B)$ and $\mathrm{Res}(\mathfrak{C},\beta_{\mathfrak{B}}(S))$ are the same until $e_j$ is removed, and they differ in their treatment of $e_j$. As $B$ and $\beta_{\mathfrak{B}}(S)$ are both in $\mathfrak{B}$, $e_j$ cannot be deleted as an isthmus or contracted as a loop in either resolution; hence it cannot be removed as either an isthmus or a loop in either resolution, because it is contracted in one resolution and deleted in the other. If $e_j\in S$ then $e_j\in\beta_{\mathfrak{B}}(S)$ by definition, and hence $e_j\notin B$; but $S\in[IO(B),E-EO(B)]$ and hence $S-B\subseteq E-EO(B)-IO(B)$, implying that $e_j$ is deleted as a loop in $\mathrm{Res}(\mathfrak{C},B)$, a contradiction. If $e_j\notin S$ then $e_j\notin\beta_{\mathfrak{B}}(S)$ by definition, so $e_j\in B-S$; but $S\in[IO(B),E-EO(B)]$ implies that every element of $B-S$ is contracted as an isthmus in $\mathrm{Res}(\mathfrak{C},B)$, another contradiction. This completes the proof of Theorem 2.5. ■

Theorem 2.5 has an interesting consequence regarding our definition of clutter resolutions. The recursive calculation of $\mathrm{rel}(\mathfrak{C})$ mentioned in the introduction might seem to motivate a different definition of *resolution*, in which isthmuses are never deleted and loops never contracted. The latter change would not affect anything, by Lemma 3.2, but the former would dramatically alter the resolutions corresponding to sets $S\notin\mathfrak{F}(\mathfrak{C})$; no resolution of $\mathfrak{C}\neq\emptyset$ would ever terminate in $\emptyset$ if deletions of isthmuses were disallowed. However Theorem 2.5 tells us that this different kind of resolution would give rise to the same partition of $2^E$ as our definition, namely the partition determined by $\beta_{\mathfrak{B}(\mathfrak{C})}$.

## 5. Clutters without fake minpaths

To prove Theorem 2.6 we must show that if $\mathfrak{C}$ is a nonempty clutter on $E$, and no order of $E$ produces a fake minpath for $\mathfrak{C}$, then $\mathfrak{C}$ is a matroid basis clutter. This is easily proven using a characterization of matroids given by Dawson [9]: $\mathfrak{C}$ is a matroid basis clutter if and only if whenever $S,T\in\mathfrak{C}$ and $S'\subseteq S$, there is a $U\in\mathfrak{C}$ with $S'\subseteq U\subseteq S'\cup T$.

Dawson's characterization is trivially satisfied if $S' = S$ or $S' \subseteq T$. Otherwise, consider an order of $E$ such that $S' = \{e_{|E|+1-|S'|}, ..., e_{|E|}\}$. No element of $S'$ is contracted as a loop in $\mathrm{Res}(\mathfrak{C}, S' \cup T)$, because $S' \subseteq S \in \mathfrak{C}$. It follows that $S' \subseteq (S' \cup T) - IL(S' \cup T)$; as $\mathfrak{C}$ has no fake minpath, it must be that $(S' \cup T) - IL(S' \cup T) \in \mathfrak{C}$. ■

The reader familiar with [10] will observe that in the terminology used there, Theorem 2.6 states: no order of $E$ produces a fake minpath for $\mathfrak{C}$ if, and only if, every order of $E$ produces an increasing $\alpha$ for $\mathfrak{C}$. There is also a striking similarity between the proof of Theorem 2.6 we have just given and the proof of Theorem 4.3 in [10]. Moreover, our Theorem 4.1 and Theorem 4.2 of [10] together tell us that if $\mathfrak{C}$ has an increasing $\alpha$ then it has no fake minpath. Given all this, it is natural to wonder whether what we mean by *having no fake minpath* is simply equivalent to what Dawson means by *having an increasing* $\alpha$. Consider the clutter $\mathfrak{C} = \{\{1,2,3\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}$; we will show below that $\mathfrak{C}$ has no fake minpath with respect to the natural order on $E = \{1,2,3,4,5\}$. However, $\beta_{\mathfrak{C}}(\{1,4\}) = \{2,3,4\}$ does not contain $\{1,4\}$, so Theorem 3.2 of [10] implies that $\mathfrak{C}$ does not have an increasing $\alpha$ with respect to this order. Hence *having no fake minpath* is genuinely a weaker hypothesis than *having an increasing* $\alpha$, and consequently our Theorem 2.6 is somewhat stronger than Theorem 4.3 of [10]. Similarly, our Corollary 2.7 (which follows immediately from Theorem 4.1 and Theorem 2.6) is a result about Dawson's $\beta_{\mathfrak{C}}$ which does not seem to follow from [8, 10, 11].

To prove the first assertion of Theorem 2.8 it suffices to prove that if $\mathfrak{C}$ is a clutter on $E = \{e_1, ..., e_m\}$ which has no fake minpath with respect to this ordering of $E$ then for every $i \in \{1, ..., m\}$, $\mathfrak{C} - e_i$ is a clutter on $E - \{e_i\} = \{e_1, ..., e_{i-1}, e_{i+1}, ..., e_m\}$ which has no fake minpath with respect to this ordering of $E - \{e_i\}$. The theorem is obviously valid if $\mathfrak{C} - e_i = \emptyset$, so we may as well assume that $\mathfrak{C} - e_i \neq \emptyset$. Suppose that $e_i \notin S \subseteq E$, and let $\mathrm{Res}(\mathfrak{C}, S)$ be $\mathfrak{C} = \mathfrak{C}_m, \mathfrak{C}_{m-1}, ..., \mathfrak{C}_0$. $\mathrm{Res}(\mathfrak{C} - e_i, S)$ is $\mathfrak{C}_m - e_i, ..., \mathfrak{C}_{i+1} - e_i, \mathfrak{C}_{i-1}, ..., \mathfrak{C}_0$. If $S \in \mathfrak{F}(C - e_i)$ and $S \notin \mathfrak{C} - e_i$ then $S \in \mathfrak{F}(C)$ and $S \notin \mathfrak{C}$. $S$ cannot be a fake minpath of $\mathfrak{C}$, so there must be some $j$ such that $e_j \in S$ is a loop of $\mathfrak{C}_j$; $e_i \notin S$, so $j \neq i$. If $j < i$ then $\mathfrak{C}_j$ appears in $\mathrm{Res}(\mathfrak{C} - e_i, S)$, and $e_j$ is a loop of $\mathfrak{C}_j$. If $j > i$ then $\mathfrak{C}_j - e_i$ appears in $\mathrm{Res}(\mathfrak{C} - e_i, S)$ and $e_j$ is a loop of $\mathfrak{C}_j - e_i$, as it is a loop of $\mathfrak{C}_j$. Either way, $S$ cannot be a fake minpath of $\mathfrak{C} - e_i$.

To verify the second assertion of Theorem 2.8 we give an example of a clutter $\mathfrak{C}$ with no fake minpath such that some minor $\mathfrak{C}/e_i$ has a fake minpath with respect to every order of $E - \{e_i\}$. Consider the clutter $\mathfrak{C} = \{\{1,2,3\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}$ on $E =$

$\{1,2,3,4,5\}$; $E$ is ordered in the natural way. If $S$ is a true or fake minpath of $\mathfrak{C}$ which contains both 4 and 5 then $(\mathfrak{C}/5)/4 = \{\{1\},\{2\},\{3\}\}$ appears in the resolution of $\mathfrak{C}$ corresponding to $S$, and clearly then $S \in \{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}$. If $S$ is a true or fake minpath of $\mathfrak{C}$ which contains 5 but not 4 then $(\mathfrak{C}/5)-4 = \{\{2,3\}\}$ appears in the resolution of $\mathfrak{C}$ corresponding to $S$, and $S$ must be $\{2,3,5\}$. Finally, if $S$ is a true or fake minpath of $\mathfrak{C}$ which does not contain 5 then $\mathfrak{C}-5 = \{\{\{1,2,3\},\{2,3,4\}\}$ appears in the resolution of $\mathfrak{C}$ corresponding to $S$, and then either $4 \in S$ (in which case $S = \{2,3,4\}$) or else $4 \notin S$ (in which case $S = \{1,2,3\}$). This verifies the claim that $\mathfrak{C}$ has no fake minpath. In contrast, $\mathfrak{C}/1 = \{\{2,3\},\{4,5\}\}$ has a fake minpath with respect to every order on $\{2,3,4,5\}$. ■

## 6. Closing comments

It is natural to seek interval partitions of $\mathfrak{F}(\mathfrak{C})$ which involve as few intervals as possible. A fake minpath is the union of a true minpath with terminal segments of other true minpaths, so it seems reasonable to guess that if we are given a clutter $\mathfrak{C}$ on a set $E$ and we wish to order $E$ so that relatively few fake minpaths are created, then we should concentrate the true minpaths at the upper end of the order. Examples 6.1 and 6.2 illustrate this guess.

**Example 6.1**. Let $E = \{e_1, ..., e_m\}$ and $\mathfrak{C} = \{C_1, C_2\}$, where $C_1 \cup C_2 = E$, $C_1 \cap C_2 = \emptyset$, $e_1 \in C_1$, and $s$ is the least index with $e_s \in C_2$. A fake minpath of $\mathfrak{C}$ must properly contain either $C_1$ or $C_2$ and must have no element which is a loop in the corresponding resolution, so it is either the union of $C_1$ and a proper terminal segment of $C_2$ or the union of $C_2$ and a proper terminal segment of $C_1$ which does not include $e_{s-1}$. It follows that there are $|C_2| - 1$ fake minpaths which contain $C_1$ and $|C_1 \cap \{e_s, ..., e_m\}| = |C_1| - (s-1)$ which contain $C_2$. Altogether there are $m - s$ fake minpaths, a number between $\min\{|C_1|, |C_2|\} - 1$ and $m - 2$. ■

**Example 6.2**. Let $E = \{e_1, ..., e_{2m}\}$ and suppose $\mathfrak{C}$ consists of $m$ pairwise disjoint 2-element subsets of $E$. We call $e_i \in E$ a *bottom* or *top element* according to whether $\{e_i, e_j\} \in \mathfrak{C}$ has $i < j$ or $i > j$; e.g., $e_1$ is a bottom element and $e_{2m}$ is a top element. A fake minpath of $\mathfrak{C}$ is the union of a true minpath and a nonempty set of additional top elements which have greater indices than the bottom element of the true minpath. Hence each bottom element $e_i$ appears in $2^{p-1} - 1$ fake minpaths, where $p$ is the number of top elements $e_j$ with $j > i$, so the number of fake minpaths is between $\sum_{k=1}^{m} 2^{k-1} - m = 2^m - 1 - m$ and $m2^{m-1} - m$. The smallest number occurs when $\mathfrak{C} = \{\{e_1, e_2\}, \{e_3, e_4\}, ..., \{e_{2m-1}, e_{2m}\}\}$, and the largest number occurs when $\mathfrak{C} = \{\{e_1, e_{m+1}\}, \{e_2, e_{m+2}\}, ..., \{e_m, e_{2m}\}\}$. ■

By the way, Ball and Nemhauser [1] have shown that in Example 6.2 no interval partition of $\mathfrak{F}(\mathfrak{C})$ involves fewer than $2^m - 1$ intervals. We think of this exponential lower bound as a signal of the intractability of reliability calculations for planar networks [13].

We observe that even though it's traditional to follow a linear order of $E$ when using activities to analyze reliability or the Tutte polynomial, it's not actually necessary. Instead one may use a rooted binary tree to determine which element of $E$ is to be eliminated after each possible sequence of deletions and contractions. Each resolution which follows that binary tree corresponds to a subset of $E$ in the usual way: the elements of the subset are the elements which are contracted in the resolution. Activities are defined just as in Sections 2 and 3 above. There are clutters for which appropriately chosen binary trees produce partitions with fewer intervals than any linear orders, but we do not trouble to present such an example in detail. Similarly, Dawson's $\beta$ may be defined with respect to a rooted binary tree rather than a linear order. All of our results generalize immediately to these "unordered" constructions, with essentially the same proofs.

In our last two examples, the optimal interval partitions of $\mathfrak{F}(\mathfrak{C})$ are not $\mathcal{S}$-partitions and hence cannot be found using activities, whether ordered or unordered. See [19] for a more complete discussion.

**Example 6.3.** The clutter $\mathfrak{C} = \{\{e_1, e_2, e_3\}, \{e_1, e_4, e_5\}, \{e_2, e_4, e_5\}, \{e_3, e_4, e_5\}\}$ on $E = \{e_1, e_2, e_3, e_4, e_5\}$ has a fake minpath $\{e_1, e_2, e_3, e_5\}$. It is not difficult to verify that $\mathfrak{C}$ has a similar four-element fake minpath (a true minpath with the greatest element of $E$ adjoined) with respect to any other order of $E$, and hence no activities partition of $\mathfrak{F}(\mathfrak{C})$ involves only four intervals. However $\mathfrak{F}(\mathfrak{C})$ can be partitioned into four intervals: $[\{e_1, e_2, e_3\}, \{e_1, e_2, e_3, e_4, e_5\}]$, $[\{e_1, e_4, e_5\}, \{e_1, e_2, e_4, e_5\}]$, $[\{e_2, e_4, e_5\}, \{e_2, e_3, e_4, e_5\}]$, and $[\{e_3, e_4, e_5\}, \{e_1, e_3, e_4, e_5\}]$. This is essentially Example 4.1 of Chari [6]; as we work with filters rather than complexes, our version involves the complements of the elements of Chari's. ■

**Example 6.4.** Let $E = \{e_1, ..., e_{3m}\}$ and suppose $\mathfrak{C}$ has three pairwise disjoint $m$-element minpaths $A = \{a_1, ..., a_m\}$, $B = \{b_1, ..., b_m\}$ and $C = \{c_1, ..., c_m\}$. The minpath containing $e_1$ will give rise to $m^2 - 1$ fake minpaths, unions of this minpath with terminal segments of the other two. (Neither of the terminal segments may include the entire other minpath, and at least one of the terminal segments must be nonempty.) However $\mathfrak{F}(\mathfrak{C})$ may be partitioned into $3m+1$ intervals: $[A, E-\{b_1\}]$, $[A\cup\{b_1\}, E-\{b_2\}]$, $[A \cup \{b_1, b_2\}, E - \{b_3\}]$, ..., $[A \cup \{b_1, ..., b_{m-1}\}, E - \{b_m\}]$, $[B, E - \{c_1\}]$, $[B\cup\{c_1\}, E-\{c_2\}]$, ..., $[B\cup\{c_1, ..., c_{m-1}\}, E-\{c_m\}]$, $[C, E-\{a_1\}]$, $[C\cup\{a_1\}, E - \{a_2\}]$, ..., $[C \cup \{a_1, ..., a_{m-1}\}, E - \{a_m\}]$, and $\{E\}$. ■

We would like to thank G. Gordon, J. S. Provan, A. Sokal and the referee, who have discussed these ideas with us.

## References

[1] M. O. Ball and G. L. Nemhauser, Matroids and a reliability analysis problem, *Math. Oper. Res.* **4** (1979), 132-143.

[2] M. O. Ball and J. S. Provan, Disjoint products and efficient computation of reliability, *Oper. Res.* **36** (1988), 703-715.

[3] R. A. Bari, Chromatic polynomials and the internal and external activities of Tutte, *in* "Graph Theory and Related Topics" (J. A. Bondy and U. S. R. Murty, eds.), Academic Press, New York, 1979, pp. 41-52.

[4] A. Björner and M. L. Wachs, Nonpure shellable complexes and posets I, *Trans. Amer. Math. Soc.* **348** (1996), 1299-1327.

[5] A. Björner and M. L. Wachs, Nonpure shellable complexes and posets II, *Trans. Amer. Math. Soc.* **349** (1997), 3945-3975.

[6] M. K. Chari, Steiner complexes, matroid ports, and shellability, *J. Combin. Thry.* (Ser. B) **59** (1993), 41-68.

[7] H. H. Crapo, The Tutte polynomial, *Aequationes Math.* **3** (1969), 211-229.

[8] J. E. Dawson, A construction for a family of sets and its application to matroids, *in* "Combinatorial Mathematics VIII. Proceedings 1980" (K. L. McAveney, ed.), Lect. Notes in Math. **884**, Springer-Verlag, New York and Berlin, 1981, pp.136-147.

[9] J. E. Dawson, A simple approach to some basic results in matroid theory, *J. Math. Anal. Appl.* **84** (1981), 555-559.

[10] J. E. Dawson, Matroid bases, opposite families and some related algorithms, *in* "Combinatorial Mathematics IX. Proceeedings 1981" (E. Billington, S. Oates-Williams and A. P. Street, eds.), Lect. Notes in Math. **952**, Springer-Verlag, New York and Berlin, 1982, pp.225-238.

[11] J. E. Dawson, A collection of sets related to the Tutte polynomial of a matroid *in* "Graph Theory, Singapore 1983" (K. M. Koh, H. P. Yap, eds.), Lect. Notes in Math. **1073**, Springer-Verlag, New York and Berlin, 1984, pp.193-204.

[12] G. Gordon and L. Traldi, Generalized activities and the Tutte polynomial, *Discrete Math.* **85** (1990), 167-176.

[13] J. S. Provan, The complexity of reliability computations in planar and acyclic graphs, *SIAM J. Comput.* **15** (1986), 694-702.

[14] J. S. Provan and L. J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra, *Math. Oper. Res.* **5** (1980), 579-594.

[15] L. Traldi, Generalized activities and $K$-terminal reliability, *Discrete Math.* **96** (1991), 131-149.

[16] L. Traldi, Generalized activities and $K$-terminal reliability. II, *Discrete Math.* **135** (1994), 381-385.

[17] L. Traldi, Two universal reliability problems, *Congr. Numer.* **132** (1998), 199-204.

[18] L. Traldi, Visualizing clutters, *Congr. Numer.* **154** (2002), 13-20.

[19] L. Traldi, Non-minimal sums of disjoint products, preprint, Lafayette College, 2003.

[20] W. T. Tutte, "Graph Theory," Cambridge University Press, Cambridge, 1984.

[21] N. White, ed., "Matroid Applications," Cambridge Univ. Press, Cambridge, 1992.