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THE DETERMINANTAL IDEALS OF LINK MODULES. II

LORENZO TRALDI

Let H be the multiplicative free abelian group of rank $m \geq 1$. Suppose $0 \to B \to A \to IH \to 0$ is a short exact sequence of $\mathbf{Z}H$-modules, and the module A is finitely generated. Then B is also a finitely generated $\mathbf{Z}H$-module, and for any $k \in \mathbf{Z}$ the determinantal ideals of A and B satisfy the equality

$$E_k(A):(IH)^p = E_{k-1}(B):(IH)^q$$

for all sufficiently large values of p and q. Furthermore, if this exact sequence is the link module sequence of a tame link of m components in S^3, then

$$E_k(A) = E_{k-1}(B):(IH)^{\binom{m-1}{2}}$$

whenever $k \geq m$.

1. Introduction. Let H be the multiplicative free abelian group of rank $m \geq 1$, and $\mathbf{Z}H$ its integral group ring; if ε: $\mathbf{Z}H \to \mathbf{Z}$ is the augmentation map then its kernel is the augmentation ideal IH of $\mathbf{Z}H$. Following **[6]**, we will call a short exact sequence

$$(1) \qquad 0 \to B \xrightarrow{\phi} A \xrightarrow{\psi} IH \to 0$$

of $\mathbf{Z}H$-modules and homomorphisms an *augmentation sequence*, provided that the $\mathbf{Z}H$-module A is finitely generated. The module B is then also finitely generated, and so for any $k \in \mathbf{Z}$ there are well-defined determinantal ideals $E_k(A)$, $E_k(B) \subseteq \mathbf{Z}H$.

In **[6]** we discussed the relationship between the product ideals $E_k(A)\cdot(IH)^p$ and $E_{k-1}(B)\cdot(IH)^q$ for various values of k, p, and q. In the present paper, instead, we will consider the relationship between the various quotient ideals $E_k(A):(IH)^p$ and $E_{k-1}(B):(IH)^q$. (We recall the definition: if $U, V \subseteq \mathbf{Z}H$ are ideals then the quotient ideal $U:V$ is $\{x \in \mathbf{Z}H \mid xV \subseteq U\}$.)

At first glance, it may seem that if $U \subseteq \mathbf{Z}H$ is an ideal the quotient ideals $U:(IH)^p$ and the various product ideals $U\cdot(IH)^q$ are, in some sense, "duals" of each other, but this is not so. For the descending sequence

$$U = U \cdot (IH)^0 \supseteq U \cdot (IH)^1 \supseteq U \cdot (IH)^2 \supseteq \cdots$$

of ideals of $\mathbf{Z}H$ need not terminate, in general, while since $\mathbf{Z}H$ is noetherian the ascending sequence

$$U = U:(IH)^0 \subseteq U:(IH)^1 \subseteq U:(IH)^2 \subseteq \cdots$$

must, that is, there is a (unique least) $\rho(U)$ such that

$$U:(IH)^{\rho(U)} = U:(IH)^r \quad \forall\, r \geq \rho(U).$$

We will devote most of our attention to this terminal quotient ideal.

THEOREM (1.1). *If* (1) *is an augmentation sequence then for any* $k \in \mathbf{Z}$

$$E_k(A):(IH)^{\rho(E_k(A))} = E_{k-1}(B):(IH)^{\rho(E_{k-1}(B))}.$$

It is of interest, then, to determine the integers $\rho(E_k(A))$ and $\rho(E_{k-1}(B))$. Though this seems impracticable in general, we will prove

THEOREM (1.2). *If* (1) *is an augmentation sequence,* $n \in \mathbf{Z}$, *and* $\varepsilon E_n(A) = \mathbf{Z}$, *then* $\rho(E_k(A)) = 0$ *whenever* $k \geq n$. *Furthermore,* $\rho(E_{k-1}(B)) = 0$ *whenever* $k \geq n + \binom{m-1}{2}$, *and* $\rho(E_{k-1}(B)) \leq n + \binom{m-1}{2} - k$ *whenever* $n \leq k \leq n + \binom{m-1}{2}$. *Consequently,* $\rho(E_{k-1}(B)) \leq \binom{m-1}{2}$ *whenever* $k \geq n$.

(*Here* $\binom{m-1}{2}$ *is the binomial coefficient, and in particular* $\binom{0}{2} = \binom{1}{2} = 0$.)

If (1) is the module sequence of a tame link $L \subseteq S^3$ of m components (described, e.g., in **[1]**) then it is known **[5]** that $\varepsilon E_m(A) = \mathbf{Z}$. (Note: in **[5]** the notation $E_k(A) = E_k(L)$ is used in this case.) Combining this with Theorems (1.1) and (1.2), we obtain

COROLLARY (1.3). *If* (1) *is the module sequence of a tame link* $L \subseteq S^3$, *then*

$$E_k(A) = E_{k-1}(B)$$

whenever $k > \binom{m}{2}$, *and*

$$E_k(A) = E_{k-1}(B):(IH)^{m+\binom{m-1}{2}-k}$$

whenever $m \leq k \leq \binom{m}{2}$. *Consequently,*

$$E_k(A) = E_{k-1}(B):(IH)^{\binom{m-1}{2}}$$

whenever $k \geq m$.

A special case of this is particularly pleasant: if (1) is the module sequence of a tame two-component link $L \subseteq S^3$ then $E_k(A) = E_{k-1}(B)$ whenever $k \geq 2$. Since $E_1(A) = E_0(B) \cdot IH$, and $E_k(A) = E_{k-1}(B) = 0$ whenever $k \leq 0$, it follows that for any $k \in \mathbf{Z}$ $E_k(A)$ and $E_{k-1}(B)$ are equivalent as invariants of L, that is, each ideal determines the other. In this respect, the behavior of these invariants for two-component links is analogous to their behavior for knots. (Recall that if $m = 1$ and (1) is any augmentation sequence then **[6]** $E_k(A) = E_{k-1}(B)$ for every value of k.)

For links of three or more components in S^3, the relationship between the determinantal ideals of the modules A and B appearing in the link module sequence is more complex; we will discuss this further in §3.

Another result, analogous to Theorem (1.2) (though seemingly of less use in the application to the module sequences of tame links), is

THEOREM (1.4). *If* (1) *is an augmentation sequence,* $n \in \mathbf{Z}$, *and* $\varepsilon E_{n-1}(B) = \mathbf{Z}$, *then* $\rho(E_{k-1}(B)) = 0$ *whenever* $k \geq n$. *Furthermore,* $\rho(E_k(A)) = 0$ *whenever* $k \geq n + m - 1$, *and* $\rho(E_k(A)) \leq n + m - 1 - k$ *whenever* $n \leq k \leq n + m - 1$. *Consequently,* $\rho(E_k(A)) \leq m - 1$ *whenever* $k \geq n$.

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2. Proofs.

PROPOSITION (2.1). *Let* U *and* V *be ideals of* $\mathbf{Z}H$. *Then* $U : (IH)^{\rho(U)} = V : (IH)^{\rho(V)}$ *if, and only if, there are integers* $p, q \geq 0$ *such that* $U \cdot (IH)^p \subseteq V$ *and* $V \cdot (IH)^q \subseteq U$.

Proof. First, suppose that $U : (IH)^{\rho(U)} = V : (IH)^{\rho(V)}$. Then $U \cdot (IH)^{\rho(V)} \subseteq (U : (IH)^{\rho(U)}) \cdot (IH)^{\rho(V)} = (V : (IH)^{\rho(V)}) \cdot (IH)^{\rho(V)} \subseteq V$, and similarly $V \cdot (IH)^{\rho(U)} \subseteq U$.

Suppose, instead, that there are non-negative integers p and q as described. Then $(U : (IH)^{\rho(U)}) \cdot (IH)^{p+\rho(U)} \subseteq U \cdot (IH)^p \subseteq V$, and hence $U : (IH)^{\rho(U)} \subseteq V : (IH)^{p+\rho(U)} \subseteq V : (IH)^{\rho(V)}$. Similarly, $V : (IH)^{\rho(V)} \subseteq U : (IH)^{\rho(U)}$, so these two ideals coincide. □

Theorem (1.1) follows immediately from Proposition (2.1) and Theorem (1.1) of **[6]**.

LEMMA (2.2). *Let U and V be ideals of $\mathbf{Z}H$, and suppose that $\varepsilon(U) = \mathbf{Z}$. Then $U + V = U + V\cdot(IH)^k$ for any $k \geq 0$.*

Proof. Since $(IH)^0 = \mathbf{Z}H$, certainly $U + V = U + V\cdot(IH)^0$.

Since $\varepsilon(U) = \mathbf{Z}$, $U + IH = \mathbf{Z}H$, and hence $U + V = (U + V)\cdot(U + IH) \subseteq U + V\cdot IH \subseteq U + V$. Thus $U + V = U + V\cdot IH$.

Proceeding inductively, suppose $k \geq 1$ and $U + V = U + V\cdot(IH)^k$. Then $U + V = U + V\cdot(IH)^k = (U + V\cdot(IH)^k)\cdot(U + IH) \subseteq U + V\cdot(IH)^{k+1} \subseteq U + V$, and hence $U + V = U + V\cdot(IH)^{k+1}$. □

COROLLARY (2.3). *Let $U \subseteq \mathbf{Z}H$ be an ideal with $\varepsilon(U) = \mathbf{Z}$. Then $\rho(U) = 0$.*

Proof. By definition, $(U:(IH)^{\rho(U)})\cdot(IH)^{\rho(U)} \subseteq U$, and hence $U = U + (U:(IH)^{\rho(U)})\cdot(IH)^{\rho(U)}$. By the preceding lemma, then, $U = U + (U:(IH)^{\rho(U)})$, that is, $U \supseteq U:(IH)^{\rho(U)}$. Since $U \subseteq U:(IH)^{\rho(U)}$, it follows that $U = U:(IH)^{\rho(U)}$, and hence $\rho(U) = 0$. □

We may now proceed to the proof of Theorem (1.2); suppose (1) is an augmentation sequence and $\varepsilon E_n(A) = \mathbf{Z}$.

If $m = 1$, then by Theorem $(1.1)_1$ of **[6]** $E_k(A) = E_{k-1}(B)$ for any value of k. Also, if $k \geq n$ then $E_k(A) \supseteq E_n(A)$, so $\varepsilon E_k(A) = \mathbf{Z}$, so by Corollary (2.3) $\rho(E_k(A)) = 0$.

If $m = 2$, then by Theorem $(1.1)_2$ of **[6]** $E_{k-1}(B)\cdot IH \subseteq E_k(A) \subseteq E_{k-1}(B)$ for any value of $k \in \mathbf{Z}$. If $k \geq n$ then $E_k(A) \supseteq E_n(A)$, so by Corollary (2.3) $\rho(E_k(A)) = 0$. Furthermore, since $E_{k-1}(B)\cdot IH \subseteq E_k(A)$, $E_k(A) = E_k(A) + E_{k-1}(B)\cdot IH$, so by Lemma (2.2) $E_k(A) = E_k(A) + E_{k-1}(B)$, that is, $E_k(A) \supseteq E_{k-1}(B)$; since $E_k(A) \subseteq E_{k-1}(B)$, it follows that $E_k(A) = E_{k-1}(B)$.

If $m \geq 3$ and $k \geq n$ then $\mathbf{Z} = \varepsilon E_n(A) = \varepsilon E_k(A)$, so by Corollary (2.3) $\rho(E_k(A)) = 0$. As shown in §3 of **[6]**,

$$E_{k-1}(B) \supseteq \sum_i E_{i+m}(X)E_{k-i-1}(A),$$

where X is a $\mathbf{Z}H$-module with $E_{m-2}(X) = 0$, $E_j(X) = (IH)^{\binom{m}{2}-j}$ for $m - 1 \leq j < \binom{m}{2}$, and $E_{\binom{m}{2}}(X) = \mathbf{Z}H$.

In particular, if $k \geq n + \binom{m-1}{2}$ then $E_{k-1}(B) \supseteq E_{\binom{m}{2}}(X)E_{k-\binom{m-1}{2}}(A) = E_{k-\binom{m-1}{2}}(A) \supseteq E_n(A)$, so $\varepsilon E_{k-1}(B) = \varepsilon E_n(A) = \mathbf{Z}$, so by Corollary (2.3) $\rho(E_{k-1}(B)) = 0$.

If $n \leq k < n + \binom{m-1}{2}$, then

$$\begin{aligned} E_{k-1}(B) &\supseteq E_{k-n-1+m}(X)E_n(A) + E_{m-1}(X)E_k(A) \\ &= (IH)^{\binom{m-1}{2}+n-k} \cdot E_n(A) + (IH)^{\binom{m-1}{2}} \cdot E_k(A) \\ &= (IH)^{\binom{m-1}{2}+n-k} \cdot \left(E_n(A) + (IH)^{k-n} \cdot E_k(A)\right). \end{aligned}$$

Since $\varepsilon E_n(A) = \mathbf{Z}$, it follows from Lemma (2.2) that $E_n(A) + (IH)^{k-n} \cdot E_k(A) = E_n(A) + E_k(A)$, so since $E_n(A) \subseteq E_k(A)$ (and hence $E_n(A) = E_n(A) + E_k(A)$) we conclude that

$$E_{k-1}(B) \supseteq (IH)^{\binom{m-1}{2}+n-k} \cdot E_k(A).$$

Since $\rho(E_k(A)) = 0$ (as noted earlier), it follows from this and Theorem (1.1) that

$$E_{k-1}(B) \supseteq (IH)^{\binom{m-1}{2}+n-k} \cdot \left(E_{k-1}(B) : (IH)^{\rho(E_{k-1}(B))}\right),$$

and hence

$$E_{k-1}(B) : (IH)^{\rho(E_{k-1}(B))} \subseteq E_{k-1}(B) : (IH)^{\binom{m-1}{2}+n-k}.$$

That $\rho(E_{k-1}(B)) \leq \binom{m-1}{2} + n - k$ follows immediately.

This completes the proof of Theorem (1.2).

Turning to Theorem (1.4), suppose (1) is an augmentation sequence and $\varepsilon E_{n-1}(B) = \mathbf{Z}$.

If $m = 1$, then by Theorem $(1.1)_1$ of **[6]** $E_k(A) = E_{k-1}(B)$ for any value of k. If $k \geq n$ then $E_{k-1}(B) \supseteq E_{n-1}(B)$, and so $\varepsilon E_{k-1}(B) = \mathbf{Z}$; by Corollary (2.3), then, $\rho(E_{k-1}(B)) = 0$.

If $m \geq 2$ and $k \geq n$ then $\mathbf{Z} = \varepsilon E_{n-1}(B) = \varepsilon E_{k-1}(B)$, so by Corollary (2.3) $\rho(E_{k-1}(B)) = 0$. Also, by Lemma (2.1) of **[6]**

$$E_k(A) \supseteq \sum_i E_{k-i}(B)E_i(IH).$$

In **[2]** it is shown that $E_0(IH) = E_0(N_2(m)) = 0$, $E_j(IH) = E_j(N_2(m)) = (IH)^{m-j}$ for $1 \leq j < m$, and $E_m(IH) = E_m(N_2(m)) = \mathbf{Z}H$. ($N_2(m)$ is a presentation matrix for IH, studied in **[2]**.)

In particular, if $k \geq n + m - 1$ then $E_k(A) \supseteq E_{k-m}(B)E_m(IH) = E_{k-m}(B) \supseteq E_{n-1}(B)$, so $\varepsilon E_k(A) = \mathbf{Z}$, and hence by Corollary (2.3) $\rho(E_k(A)) = 0$.

If $n \leq k < n+m-1$, then

$$\begin{aligned} E_k(A) &\supseteq E_{n-1}(B)E_{k-n+1}(IH) + E_{k-1}(B)E_1(IH) \\ &= (IH)^{m-k+n-1}\cdot E_{n-1}(B) + (IH)^{m-1}\cdot E_{k-1}(B) \\ &= (IH)^{m-k+n-1}\cdot \left(E_{n-1}(B) + (IH)^{k-n}\cdot E_{k-1}(B)\right). \end{aligned}$$

Since $\varepsilon E_{n-1}(B) = \mathbf{Z}$, it follows from Lemma (2.2) that

$$E_{n-1}(B) + (IH)^{k-n}\cdot E_{k-1}(B) = E_{n-1}(B) + E_{k-1}(B) = E_{k-1}(B);$$

hence

$$E_k(A) \supseteq (IH)^{n+m-1-k}\cdot E_{k-1}(B).$$

Since $\rho(E_{k-1}(B)) = 0$, it follows from this and Theorem (1.1) that

$$E_k(A) \supseteq (IH)^{n+m-1-k}\cdot \left(E_k(A) : (IH)^{\rho(E_k(A))}\right).$$

We may conclude from this that $\rho(E_k(A)) \leq n+m-1-k$.

This completes the proof of Theorem (1.4).

We may note here, without going into detail, that Theorems (1.1), (1.2), and (1.4) hold in a broader context, with $\mathbf{Z}H$ replaced by an arbitrary noetherian commutative ring with unity R, and IH replaced by the ideal of R generated by the elements of some R-sequence $\{r_1, \ldots, r_m\}$. (The hypotheses $\varepsilon E_n(A) = \mathbf{Z}$ and $\varepsilon E_{n-1}(B) = \mathbf{Z}$ of Theorems (1.2) and (1.4) should be replaced by the equivalent hypotheses $\mathbf{Z}H = E_n(A) + IH$ and $\mathbf{Z}H = E_{n-1}(B) + IH$, respectively, prior to any such generalization.) An analogous generalization is discussed, in greater depth, in §5 of [**6**].

3. Links of three or more components. A simple consequence of Corollary (1.3) is: if (1) is the module sequence of a tame link of m components in S^3, then for $k \geq m$ the ideal $E_k(A)$ is determined by $E_{k-1}(B)$. A natural question to ask, especially in view of the cases $m = 1$ and $m = 2$ (discussed in §1) is: does $E_k(A)$, in turn, determine $E_{k-1}(B)$, for $k \geq m$? That the answer to this question is "no" may be seen by considering the three-component links 6_2^3 and 8_5^3 (as they are named in Appendix C of [**4**]). As W. S. Massey has shown, if (1) is the link module sequence of the former then $E_3(A) = \mathbf{Z}H$ and $E_2(B) = IH$ [**3**, Example 1], while if (1) is the link module sequence of the latter then $E_3(A) = \mathbf{Z}H = E_2(B)$ [**3**, Example 2].

Another natural question is: can the result of Theorem (1.1) be made more definitive for $1 < k < m$, as it can for $k \geq m$ (Corollary (1.3)) and

$k = 1$ [**2**]? Though we shall not answer this question, we will consider an example of a three-component link for which the relationship between $E_2(A)$ and $E_1(B)$ is particularly complex.

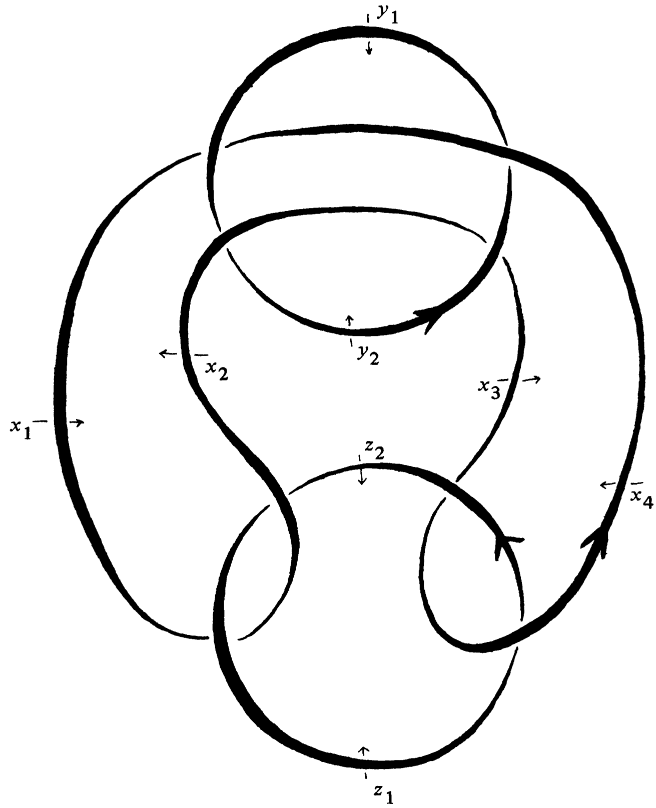

FIGURE 1

Pictured in Figure 1 is the link 8^3_{10} [**4**, Appendix C]. The Wirtinger presentation [**4**, p. 56] of the fundamental group G of the complement of this link in S^3 is

$$\langle x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2;\ x_1z_1 = z_1x_2,\ y_2x_2 = x_3y_2,\\ x_3z_2 = z_2x_4,\ y_1x_4 = x_1y_1,\ x_2y_1 = y_2x_2,\\ x_4y_2 = y_1x_4,\ z_1x_4 = x_4z_2,\ z_2x_2 = x_2z_1\rangle.$$

Since any one of the relations in this presentation is redundant, we may simply delete the seventh. Also, we may remove the fourth relation and the generator x_1, replacing any occurrence of x_1 in another relation by an occurrence of $y_1x_4y_1^{-1}$; similarly, we may remove the third relation and the generator x_3, replacing x_3 by $z_2x_4z_2^{-1}$ in the remaining relations. What results, after some simple rewriting of relations, is the presentation

$$\langle x_2, x_4, y_1, y_2, z_1, z_2;\ x_4 = y_1^{-1}z_1x_2z_1^{-1}y_1,\ y_1 = x_2^{-1}y_2x_2,$$
$$x_2^{-1}y_2^{-1}z_2x_4z_2^{-1}y_2 = 1,\ x_4y_2x_4^{-1}y_1^{-1} = 1,\ z_1 = x_2^{-1}z_2x_2\rangle.$$

After deleting the first relation and the generator x_4, and replacing x_4 by $y_1^{-1}z_1x_2z_1^{-1}y_1$ in the remaining relations, we may delete the second and fifth relations and the generators y_1 and z_1, substituting $x_2^{-1}y_2x_2$ for y_1 and $x_2^{-1}z_2x_2$ for z_1, and obtain the presentation

$$\langle x_2, y_2, z_2;\ x_2^{-1}y_2^{-1}z_2x_2^{-1}y_2^{-1}z_2x_2z_2^{-1}y_2x_2z_2^{-1}y_2 = 1,$$
$$y_2^{-1}z_2x_2z_2^{-1}y_2x_2y_2x_2^{-1}y_2^{-1}z_2x_2^{-1}z_2^{-1} = 1\rangle.$$

The Alexander matrix M of this presentation [1, §3] is the transpose of the matrix

$$\begin{pmatrix} (1 + t_1^{-1}t_2^{-1}t_3)(t_1^{-1}t_2^{-1}t_3 - t_1^{-1}) & (1 - t_2)(t_1 + t_2^{-1}t_3) \\ (1 - t_1^{-1})(t_2^{-1} + t_1^{-1}t_2^{-2}t_3) & (t_1 - 1)(t_1 + t_2^{-1}) \\ (t_1^{-1} - 1)(t_2^{-1} + t_1^{-1}t_2^{-2}t_3) & (t_1 - 1)(1 - t_2^{-1}) \end{pmatrix}.$$

(Here t_1, t_2, and t_3 are the elements of $G/G' = H$ determined by the elements of G represented by x_2, y_2, and z_2, respectively.) If (1) is the module sequence of the link 8_{10}^3, then M is a presentation matrix for the $\mathbf{Z}H$-module A [1, §3], and hence, in particular, the ideal of $\mathbf{Z}H$ generated by the entries of M is

$$E_2(A) = (1 + t_1^{-1}t_2^{-1}t_3)\cdot IH + (t_1 + 1, t_2 - 1)\cdot (t_1 - 1).$$

The matrix M can be factored as a product $M = M' \cdot N_2(3)$, where

$$N_2(3) = \begin{pmatrix} 1 - t_2 & t_1 - 1 & 0 \\ 1 - t_3 & 0 & t_1 - 1 \\ 0 & 1 - t_3 & t_2 - 1 \end{pmatrix}$$

and

$$M' = \begin{pmatrix} t_1^{-1}t_2^{-1}(1 + t_1^{-1}t_2^{-1}t_3) & -t_1^{-1}t_2^{-1}(1 + t_1^{-1}t_2^{-1}t_3) & 0 \\ t_1 + t_2^{-1}t_3 & 0 & t_2^{-1}(t_1 - 1) \end{pmatrix}.$$

(Here $N_2(3)$ is a matrix discussed by Crowell and Strauss [**2**], with columns corresponding to the integers 1, 2, and 3 (in order), and rows corresponding to the pairs 12, 13, and 23 (in order).) It follows [**2**, p. 106] that the module B of the link module sequence of 8^3_{10} has the presentation matrix

$$P = \begin{pmatrix} M' \\ N_3(3) \end{pmatrix}$$

$$= \begin{pmatrix} t_1^{-1}t_2^{-1}(1 + t_1^{-1}t_2^{-1}t_3) & -t_1^{-1}t_2^{-1}(1 + t_1^{-1}t_2^{-1}t_3) & 0 \\ t_1 + t_2^{-1}t_3 & 0 & t_2^{-1}(t_1 - 1) \\ t_3 - 1 & 1 - t_2 & t_1 - 1 \end{pmatrix}.$$

($N_3(3)$ is another matrix discussed in [**2**]; its columns correspond to the pairs 12, 13, and 23 (in order).) Thus the ideal of $\mathbf{Z}H$ generated by the determinants of the two-by-two submatrices of P is

$$E_1(B) = E_2(A) + \left(1 + t_1^{-1}t_2^{-1}t_3\right)^2.$$

In particular, the $\mathbf{Z}H$-modules A and B of the link module sequence of 8^3_{10} have the property that

$$\begin{aligned}(E_2(A) : IH)\cdot IH &= (E_1(B) : IH)\cdot IH \\ &\subset E_2(A) \subset E_1(B) \subset E_2(A) : IH = E_1(B) : IH,\end{aligned}$$

in which all three indicated inclusions are strict. The relationship between $E_2(A)$ and $E_1(B)$ does not, then, seem to fall into the pattern of the simple relationships between $E_k(A)$ and $E_{k-1}(B)$ for $k \neq 2$ (namely, $E_k(A) = E_{k-1}(B) \cdot IH$ for $k < 2$, and $E_k(A) = E_{k-1}(B) : IH$ for $k > 2$).

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LAFAYETTE COLLEGE
EASTON, PA 18042