PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 1, 1982

THE DETERMINANTAL IDEALS OF LINK MODULES. I.

LORENZO TRALDI

Let $m \geqq 1$, and let H be the multiplicative free abelian group of rank m, with integral group ring ZH and augmentation ideal IH. Suppose $0 \to B \to A \to IH \to 0$ is a short exact sequence of ZH-modules, and the module A is finitely generated. Then A and B are both finitely presented, and for any $k \in Z$ the determinantal ideals $E_k(A)$ and $E_{k-1}(B)$ satisfy the inclusions

$$E_{k-1}(B) \cdot (IH)^{m-1} \subseteq E_k(A) \text{ and } E_k(A) \cdot (IH)^{\binom{m-1}{2}} \subseteq E_{k-1}(B)\,,$$

where $\binom{m-1}{2}$ is the binomial coefficient $\left(\text{and in particular } \binom{m-1}{2} = 0 \text{ if } m \leqq 2\right)$, and $(IH)^0 = ZH$. In particular, if $m = 1$ then $E_k(A) = E_{k-1}(B)$ for any $k \in Z$. A consequence of these inclusions is the fact that the greatest common divisors $\Delta_k(A) = \Delta_{k-1}(B)$ for any $k \in Z$.

1. Introduction. Let H be the (multiplicative) free abelian group of rank $m \geqq 1$, Z the ring of integers, and ZH the integral group ring of H; let IH be the augmentation ideal of ZH, that is, if $\varepsilon: ZH \to Z$ is the homomorphism with $\varepsilon(h) = 1$ $\forall h \in H$ then $IH = \ker \varepsilon$.

A short exact sequence

$$(1) \qquad 0 \longrightarrow B \xrightarrow{\phi} A \xrightarrow{\psi} IH \longrightarrow 0$$

of ZH-modules and ZH-homomorphisms will be called an *augmentation sequence* if A is a finitely generated ZH-module. Since ZH is a noetherian ring, A is then a finitely presented ZH-module; also, the module B in an augmentation sequence is finitely presented, since it is isomorphic to a submodule of A.

For instance, if $L \subseteq S^3$ is a tame link of m components, $G = \pi_1(S^3 - L)$, and G' is the commutator subgroup of G, then $G/G' \cong H$. If $p: \tilde{X} \to X = S^3 - L$ is the universal abelian cover of X, and F is its fiber, then as discussed in [1, pp. 227–234] there is an augmentation sequence (1) in which $B \cong H_1(\tilde{X}; Z)$ is the *Alexander invariant of* L and $A \cong H_1(\tilde{X}, F; Z)$ is the *Alexander module of* L. These abelian groups are H-modules under the action given by identifying $H \cong G/G'$ with the group of covering automorphisms of $\tilde{X}$.

Given an augmentation sequence (1), determinantal ideals $E_k(A)$ and $E_k(B)$ can be defined for any $k \in Z$ (see § 2); this paper is concerned with the relationship between these ideals.

The following theorem is proven in Chapter 1 of [10].

THEOREM 1.1_1. *If $m = 1$ then for any augmentation sequence (1) and $k \in \boldsymbol{Z}$, $E_k(A) = E_{k-1}(B)$.*

THEOREM 1.1_2. *If $m = 2$ then for any augmentation sequence (1) and $k \in \boldsymbol{Z}$, $E_{k-1}(B) \cdot IH \subseteq E_k(A) \subseteq E_{k-1}(B)$. Furthermore, $E_1(A) = E_0(B) \cdot IH$.*

THEOREM 1.1_3. *If $m = 3$ then for any augmentation sequence (1) and $k \in \boldsymbol{Z}$, $E_{k-1}(B) \cdot (IH)^2 \subseteq E_k(A)$ and $E_k(A) \cdot IH \subseteq E_{k-1}(B)$. Furthermore, $E_2(A) \subseteq E_1(B)$ and $E_1(A) \subseteq E_0(B) \cdot IH$.*

THEOREM 1.1_m. *If $m \geqq 4$ then for any augmentation sequence (1) and $k \in \boldsymbol{Z}$, $E_{k-1}(B) \cdot (IH)^{m-1} \subseteq E_k(A)$ and $E_k(A) \cdot (IH)^{\binom{m-1}{2}} \subseteq E_{k-1}(B)$. Furthermore, if $1 \leqq k \leqq m$ then $E_k(A) \cdot (IH)^{\binom{m-1}{2}+k-m} \subseteq E_{k-1}(B)$.*

(Here if $p \geqq q \geqq 0$ $\binom{p}{q}$ denotes the binomial coefficient $\binom{p}{q} = p!/q!(p-q)!$; also, $(IH)^0 = ZH$, so that $E \cdot (IH)^0 = E$ for any ideal E of ZH.)

A full proof of Theorem 1.1 may be found in [10]. Rather than duplicate this rather lengthy proof here, we will, instead, present an argument due to J. A. Hillman [5]. This argument is sufficient to verify most of the assertions of the theorem (all but those in the final sentences of the statements of Theorem 1.1_2, 1.1_3, and 1.1_m), and has a considerable advantage over the original in brevity.

A special feature of the ring ZH is that it's a greatest common divisor (or g.c.d.) ring, that is, to any (finitely generated) ideal E of ZH there corresponds a unique minimal principal ideal $\bar{E} \supseteq E$; any generator of $\bar{E}$ is a *greatest common divisor* (or *g.c.d.*) of E, and any two g.c.d.s of E are unit multiples of each other. (Since ZH is noetherian, each of its ideals is finitely generated.) It is easily verified that if J, K are ideals of ZH with g.c.d.s j and k, respectively, then the product jk is a g.c.d. for the product ideal $J \cdot K$; also, if $m \geqq 2$ then $1 \in ZH$ is a g.c.d. for IH (see § 2). A well known result follows from this and Theorem 1.1, namely

COROLLARY 1.2. *If $m \geqq 1$ then for any augmentation sequence (1) and $k \in \boldsymbol{Z}$ the ideals $E_k(A)$ and $E_{k-1}(B)$ of ZH have the same* g.c.d.s.

Theorem 1.1 is a partial extension to the higher ideals of a result due to R. H. Crowell and D. Strauss [4], who showed that if (1) is a "link module sequence" then for $m \geqq 2$ $E_0(B)\cdot IH = E_1(A)\cdot (IH)^{\binom{m-2}{2}}$. $\left(\text{Here } \binom{0}{2} = \binom{1}{2} = 0.\right)$ As shown in [10], the definition of "link module sequence" in [4] is equivalent to the assumption that A have some presentation (as a ZH-module) with strictly fewer relations than generators. It should be emphasized that despite the use of the term "link module" in the title, we do not, in fact, assume that an augmentation sequence is a "link module sequence" in the sense of either [2] or [4].

As we have noted, Theorem 1.1 was proven in the first chapter of [10], the author's Ph.D. thesis, which was written at Yale University under the direction of W. S. Massey. We would like to express our deep gratitude to him.

2. Finitely presented modules. Let R be a commutative ring with unity, and let F and G be finitely generated free R-modules with bases $X = \{x_i\}$ and $Y = \{y_j\}$, respectively. If $f\colon F \to G$ is an R-homomorphism then $\operatorname{mat}_X^Y f$ (or simply $\operatorname{mat} f$) is the matrix $\operatorname{mat} f = (m_{ij})$ such that

$$f(x_i) = \sum_j m_{ij} y_j$$

for every basis element x_i of F.

An R-module A is *finitely presented* iff there is an exact sequence

$$F \xrightarrow{f} G \longrightarrow A \longrightarrow 0$$

of R-homomorphisms with F and G finitely generated free R-modules; this exact sequence is then a *finite presentation of* A, and the matrix of f (with respect to any choice of bases in F and G) is a *presentation matrix of* A.

EXAMPLE. It is shown in [6, p. 189] that to any elements $t_1, \cdots, t_m \in H$ which freely generate H (as an abelian group) there corresponds an exact sequence

$$(2) \qquad Y_3 \xrightarrow{\alpha_3} Y_2 \xrightarrow{\alpha_2} Y_1 \xrightarrow{\alpha_1} IH \longrightarrow 0$$

of ZH-homomorphisms such that Y_1, Y_2, and Y_3 are free ZH-modules with bases $\{e_i \mid 1 \leqq i \leqq m\}$, $\{e_{ij} \mid 1 \leqq i < j \leqq m\}$, and $\{e_{ijk} \mid 1 \leqq i < j < k \leqq m\}$, respectively, $\alpha_1(e_i) = t_i - 1$ whenever $1 \leqq i \leqq m$, $\alpha_2(e_{ij}) = (t_i - 1)e_j - (t_j - 1)e_i$ whenever $1 \leqq i < j \leqq m$, and $\alpha_3(e_{ijk}) = (t_i - 1)e_{jk} - (t_j - 1)e_{ik} + (t_k - 1)e_{ij}$ whenever $1 \leqq i < j < k \leqq m$. (If $m = 1$ then $Y_3 = Y_2 = 0$, and if $m = 2$ then $Y_3 = 0$.) We follow [4] and let

$N_2(m)$, $N_3(m)$ denote the matrices of α_2 and α_3, respectively, with respect to the indicated bases.

It follows from the exactness of this sequence that $\{t_1 - 1, \cdots, t_m - 1\}$ is an algebraically independent subset of ZH (considered as an algebra over Z), and also that $t_i - 1$ and $t_j - 1$ are relatively prime whenever $i \neq j$; hence if $m \geqq 2$ then 1 is a g.c.d. for IH.

DEFINITION. If $p, q \geqq 1$ and M is a $p \times q$ matrix with entries in R then determinantal (or "elementary") ideals $E_k(M)$, $k \in Z$, are defined by:

if $k \geqq q$ then $E_k(M) = R$,

if $k < q - p$ or $k < 0$ then $E_k(M) = 0$, and

if $k \geqq 0$ and $q - p \leqq k < q$ then $E_k(M) \subseteq R$ is the ideal generated by the determinants of the $(q - k) \times (q - k)$ submatrices of M.

In [4] it is shown that if $m \geqq 2$ then $E_0(N_2(m)) = 0$, $E_k(N_2(m)) = (IH)^{m-k}$ for $1 \leqq k < m$, and $E_m(N_2(m)) = ZH$. Also, if $m \geqq 3$ then $E_{m-2}(N_3(m)) = 0$, $E_k(N_3(m)) = (IH)^{\binom{m}{2}-k}$ for $m - 1 \leqq k < \binom{m}{2}$, and $E_{\binom{m}{2}}(N_3(m)) = ZH$.

If A is a finitely presented R-module, and M is any presentation matrix of A, then the determinantal ideals[1] $E_k(A)$ are defined to be $E_k(A) = E_k(M)$ $\forall k \in Z$. It can be shown [7, p. 58] that these ideals are independent of the choice of M.

LEMMA 2.1. *Some properties of the determinantal ideals are*:

(a) $E_k(A) \subseteq E_{k+1}(A)$ $\forall\, k \in Z$ *and* $E_k(A) = 0$ $\forall\, k < 0$;

(b) *if A can be generated by p of its elements (as an R-module) then* $E_p(A) = R$;

(c) *if $p \geqq 1$ and F is a free R-module of rank p then* $E_k(F) = 0$ $\forall\, k < p$; *and*

(d) *if* $0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0$ *is a short exact sequence of R-modules then*

$$E_k(A) \supseteq \sum_i E_i(A'')E_{k-i}(A')$$

for any $k \in Z$, and equality holds if the sequence splits.

Proof. Proofs are given in [7, pp. 58, 59, 90, and 91].

Theorem 1.1_1 follows immediately from Lemma 2.1. If $m = 1$ and (1) is an augmentation sequence then IH is a principal ideal of ZH, so since ZH is an integral domain (that is, a commutative ring without zero divisors), IH is a free ZH-module of rank one, and hence projective. Then the sequence (1) must split, so by

[1] Also known as elementary ideals, or Fitting invariants.

part (d) of Lemma 2.1

$$E_k(A) = \sum_i E_i(B)E_{k-i}(IH)$$

for any $k \in Z$. By part (c) of Lemma 2.1, $E_i(B)E_{k-i}(IH) = 0$ $\forall\, i \geqq k$, and by part (b), $E_i(B)E_{k-i}(IH) = E_i(B)$ $\forall\, i < k$; since $E_i(B) \subseteqq E_{i+1}(B)$ $\forall\, i \in Z$ by part (a), it follows that for any $k \in Z$

$$E_k(A) = \sum_i E_i(B)E_{k-i}(IH) = E_{k-1}(B) .$$

3. Proof of Theorem 1.1$_m$, $m \geqq 2$. In this section we will assume that $m \geqq 2$.

If (1) is an augmentation sequence then since $E_1(IH) = E_1(N_2(m)) = (IH)^{m-1}$, $E_k(A) \supseteqq E_{k-1}(B) \cdot (IH)^{m-1}$ $\forall\, k \in Z$ by part (d) of Lemma 2.1.

Recall the exact sequence (2) of § 2. Let $X = \operatorname{coker} \alpha_3$; then there is a short exact sequence

$$0 \longrightarrow X \xrightarrow{\bar{\alpha}_2} Y_1 \xrightarrow{\alpha_1} IH \longrightarrow 0$$

of ZH-modules. (Note that if $m = 2$ then $X = Y_2 \cong ZH$.)

If (1) is an augmentation sequence, then since Y_1 is free there is a ZH-homomorphism $\bar{\delta}\colon Y_1 \to A$ with $\psi\bar{\delta} = \alpha_1$. In particular, $\psi\overline{\delta\alpha_2} = \alpha_1\bar{\alpha}_2 = 0$, so $\overline{\delta\alpha_2}(X) \subseteqq \ker \psi = \phi(B)$. Since ϕ is injective, there is then a ZH-homomorphism $\bar{\gamma}\colon X \to B$ such that $\phi\bar{\gamma} = \overline{\delta\alpha_2}$. A simple diagram-chase suffices to show that if $\delta\colon B \oplus Y_1 \to A$ is the map $\delta(b, y) = \bar{\delta}(y) + \phi(b)$, and $\gamma\colon X \to B \oplus Y_1$ is the map $\gamma(x) = (-\bar{\gamma}(x), \bar{\alpha}_2(x))$, then

$$(3) \qquad 0 \longrightarrow X \xrightarrow{\gamma} B \oplus Y_1 \xrightarrow{\delta} A \longrightarrow 0$$

is a short exact sequence of ZH-modules.

Applying Lemma 2.1 to the sequence (3), we conclude that

$$E_{k+m}(B \oplus Y_1) \supseteqq \sum_i E_{i+m}(X)E_{k-i}(A) \quad \forall\, k \in Z .$$

Since Y_1 is a free ZH-module of rank m, $E_{k+m}(B \oplus Y_1) = E_k(B)$ $\forall k \in Z$. Hence in particular $E_k(B) \supseteqq E_{m-1}(X)E_{k+1}(A)$ $\forall\, k \in Z$.

If $m \geqq 3$, then $E_{m-1}(X) = E_{m-1}(N_3(m)) = (IH)^{\binom{m}{2}-(m-1)} = (IH)^{\binom{m-1}{2}}$, and if $m = 2$ then $E_{m-1}(X) = E_1(X) = E_1(ZH) = ZH$. □

4. An application. If $L \subseteqq S^3$ is a tame link of m components, then its elementary ideals $E_k(L) \subseteqq ZH$ are defined to coincide with those of its Alexander module. In this section we show that for any $m, n \geqq 1$ there is a tame link $L \subseteqq S^3$ of m components with the property that whenever $1 \leqq k \leqq n$ $E_k(L)$ is a *proper* ideal of

ZH (i.e., $0 \neq E_k(L) \neq ZH$).

As in [8, p. 179], if K' and K'' are tame knots (i.e., one-component links) in S^3, with Alexander invariants B' and B'', respectively, then the Alexander invariant of $K = K' \# K''$ is $B = B' \oplus B''$. Hence by Theorem 1.1_1 and Lemma 2.1

$$E_k(K) = E_{k-1}(B) = \sum_i E_i(B')E_{k-1-i}(B'') = \sum_i E_{i+1}(K')E_{k-i}(K'') \ \forall\, k \in Z .$$

In [3, pp. 124 – 131] examples are given of knots K such that $E_1(K)$ is a proper ideal of ZH; any knot with nontrivial Alexander polynomial (e.g., any of the 250 knots listed in Appendix C of [8]) also has this property. If K_0 is any such knot and we let $K_p = K_{p-1} \# K_0$ for $p \geqq 1$, then it follows that $E_k(K_p)$ is a proper ideal of ZH whenever $p \geqq 1$ and $1 \leqq k \leqq 2p$. Thus for any $n \geqq 1$ there is a tame one-component link $K \subseteq S^3$ with $E_k(K)$ a proper ideal of ZH for $1 \leqq k \leqq n$.

Proceeding inductively, suppose $m \geqq 2$ and $n \geqq 1$. By hypothesis, there is a tame $(m-1)$-component link $L' \subseteq S^3$ such that $E_k(L')$ is a proper ideal of ZH' whenever $1 \leqq k \leqq n$ (H' is the free abelian group with basis $t_1, \cdots, t_{m-1}$).

Let K be any embedded circle in S^3 which has nonzero linking number with some component of L', and let $L = L' \cup K$. If $\phi: ZH \to ZH'$ is the (unique) homomorphism with $\phi(t_i) = t_i$ whenever $i < m$, and $\phi(t_m) = 1$, then it is a consequence of [11] or [10, Chapter 2] that $\phi E_1(L) \neq 0$ and for any $k \geqq 1$ $E_{k-1}(L') \subseteq \phi E_k(L) \subseteq E_k(L')$. Hence whenever $1 \leqq k \leqq n$ $\phi E_k(L)$ is a proper ideal of ZH' (and so certainly $E_k(L)$ is a proper ideal of ZH).

5. A Generalization. Let R be a commutative ring with unity, $m \geqq 1$, and let $\{r_1, \cdots, r_m\}$ be a sequence of elements of R. Then [7, p. 131] $\{r_1, \cdots, r_m\}$ is an *R-sequence* iff $r_1 \neq 0$ is not a divisor of zero in R, and for $2 \leqq i \leqq m$ the map

$$\text{multiplication by } r_i\colon R/(r_1, \cdots, r_{i-1}) \longrightarrow R/(r_1, \cdots, r_{i-1})$$

is injective.

PROPOSITION 5.1. *Let R be a commutative ring with unity, $m \geqq 1$, and let $\{r_1, \cdots, r_m\}$ be an R-sequence. Let $I = (r_1, \cdots, r_m)$ be the ideal of R generated by the elements of this R-sequence. Then there is an exact sequence*

$$Y_3 \xrightarrow{\alpha_3} Y_2 \xrightarrow{\alpha_2} Y_1 \longrightarrow I \longrightarrow 0$$

of R-homomorphisms, in which Y_1, Y_2, and Y_3 are, respectively, free

R-modules of ranks m, $\binom{m}{2}$, *and* $\binom{m}{3}$ (*if* $m = 1$ *then* $Y_3 = Y_2 = 0$, *and if* $m = 2$ *then* $Y_3 = 0$). *Furthermore, bases of* Y_3, Y_2, *and* Y_1 *can be chosen, with respect to which the matrix of* α_3 *is* $N_3(r_1, \cdots, r_m)$ (i.e., $N_3(m)$ *with each entry* $\pm(t_i - 1)$ *replaced by* $\pm r_i$) *and the matrix of* α_2 *is* $N_2(r_1, \cdots, r_m)$ (i.e., $N_2(m)$ *with* $\pm(t_i - 1)$ *replaced by* $\pm r_i$ *for each i*).

This sequence is the tail end of the Koszul complex associated to the R-sequence $\{r_1, \cdots, r_m\}$. Its exactness is verified in Proposition IV. 2 of [9].

Call an exact sequence

$$0 \longrightarrow B \xrightarrow{\phi} A \xrightarrow{\psi} I \longrightarrow 0$$

of R-homomorphisms an *I-sequence* iff A is finitely generated as an R-module. *Then Theorem* 1.1_m *holds, with I replacing IH throughout.*

The arguments needed to verify this statement are not precisely analogous to those of §§ 2 and 3. We must use a definition of the determinantal ideals of a module which applies to finitely generated modules, rather than only finitely presented ones. Such a definition may be found in [7]. Lemma 2.1 still holds with this more general definition.

Also, we must replace the calculation in [4] of the elementary ideals of $N_2(m)$ and $N_3(m)$. Their argument cannot be used directly, since it only applies when the ambient ring is an integral domain, and we have made no such assumption about R. However, this problem can be circumvented with a trick. If $\boldsymbol{Z}[x_1, \cdots, x_m]$ is the ring of integral polynomials in m commuting variables, then $\{x_1, \cdots, x_m\}$ is a $\boldsymbol{Z}[x_1, \cdots, x_m]$-sequence, and arguments paralleling those of [4] can be used to calculate the elementary ideals of $N_2(x_1, \cdots, x_m)$ and $N_3(x_1, \cdots, x_m)$. Then if $f\colon \boldsymbol{Z}[x_1, \cdots, x_m] \to R$ is a ring homomorphism with $f(x_i) = r_i$ for each $i \in \{1, \cdots, m\}$, $N_2(r_1, \cdots, r_m)$ and $N_3(r_1, \cdots, r_m)$ are the entry-by-entry images under f of $N_2(x_1, \cdots, x_m)$ and $N_3(x_1, \cdots, x_m)$, respectively, and so their determinantal ideals are generated by the images under f of generators of the determinantal ideals of $N_2(x_1, \cdots, x_m)$ and $N_3(x_1, \cdots, x_m)$.

REFERENCES

1. R. H. Crowell, *The derived module of a homomorphism*, Advances in Math., **6** (1971), 210-238.

2. ———, *Torsion in link modules*, J. Math. Mech., **14** (1965), 289-298.

3. R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Springer-Verlag, New York, 1977.

4. R. H. Crowell and D. Strauss, *On the elementary ideals of link modules*, Trans.

Amer. Math. Soc., **142** (1969), 93-109.
5. J. A. Hillman, letter to the author, dated August 12, 1980.
6. S. MacLane, *Homology*, Springer-Verlag, New York, 1967.
7. D. G. Northcott, *Finite Free Resolutions*, Cambridge University Press, Cambridge, 1976.
8. D. Rolfsen, *Knots and Links*, Publish or Perish, Inc., Berkeley, California, 1976.
9. J.-P. Serre, *Algèbre Locale-Multiplicités*, Springer-Verlag, New York, 1965.
10. L. Traldi, *On the Determinantal Ideals of Link Modules and A Generalization of Torres' Second Relation*, Ph. D. Dissertation, Yale University, 1980.
11. ———, *A generalization of Torres' second relation*, Trans. Amer. Math. Soc., **269** (1982), 593–610.

Received April 24, 1981.

LAFAYETTE COLLEGE
EASTON, PA 18042