Dice graphs

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Abstract

For integers $n \geq 1$, $a \leq b$ and s let $D(n,a,b,s)$ be the set of all lists of integers $(x_1,...,x_n)$ with $a \leq x_1 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. There is a graph $G(n,a,b,s)$ with vertex-set $D(n,a,b,s)$ in which two vertices are adjacent if one is stronger than the other in a natural dice game. In *The prevalence of "paradoxical" dice* [Bull. Inst. Combin. Appl. **45** (2005), 70-76] the dice which are isolated vertices of $G(n,a,b,s)$ are characterized; here we prove that all the other vertices lie in a single connected component.

1. Introduction

If n is a positive integer then an *n-sided generalized die with integer labels* is a list $X = (x_1,...,x_n)$ of integers with $x_1 \leq x_2 \leq ... \leq x_n$. For integers $a \leq b$ and s let $D(n,a,b,s)$ denote the collection of all n-sided dice with $a \leq x_1 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. Let $p = p(n,a,b,s) = \min\{x_1|(x_1,...,x_n) \in D(n,a,b,s)\}$ and $q = q(n,a,b,s) = \max\{x_n|(x_1,...,x_n) \in D(n,a,b,s)\}$; then a die $X = (x_1,...,x_n) \in D(n,a,b,s)$ has a *characteristic vector* $v = (v_p,...,v_q)$ given by $v_j = |\{i|x_i = j\}|$.

If $X, Y \in D(n,a,b,s)$ then it is natural to think of playing a game with X and Y: each die is *rolled* (i.e., one of the n labels of each is chosen at random with probability $1/n$), and the higher roll wins. X is *stronger* than Y if it is more likely that X wins than it is that Y wins; i.e., X is stronger if $|\{(i,j)|x_i > y_j\}| > |\{(i,j)|x_i < y_j\}|$. If neither of X, Y is stronger than the other then X and Y are *tied*. An element of $D(n,a,b,s)$ which is tied with all the other elements of $D(n,a,b,s)$ is *balanced*. In [4] we showed that despite the fact that every element of $D(n,a,b,s)$ has the same average roll s/n, there are surprisingly few balanced dice: if $n \geq 3$ and $q-p \geq 2$ then balanced dice must have characteristic vectors of the form $(v,w,v,w,...)$, and (consequently) few values of s permit any balanced dice at all. It is natural to wonder whether the scarcity of balanced dice reflects a general scarcity of ties among elements of $D(n,a,b,s)$ or whether it merely reflects the strictness of the requirement that balanced dice tie *all* other elements of $D(n,a,b,s)$.

Investigating this issue, we have found only one $D(n,a,b,s)$ with $n>3$ and $q-p>2$ in which most pairs of elements are tied: $D(4,1,4,8)$ has four elements, and four of the six pairs are tied. In other examples tied pairs are relatively rare, though they are certainly more common than balanced dice. For instance, in the various dice families $D(6,1,6,s)$ with $6\leq s\leq 36$ there are 5150 pairs of distinct dice, and only 949 pairs are tied. Of the 462 dice in these families only seven are balanced, and all but four of the 455 non-balanced dice have the property that most of their contests with other elements of $D(6,1,6,s)$ are not tied. These examples suggest several questions which we have not been able to answer.

Question 1. *Is $D(4,1,4,8)$ the only example of a $D(n,a,b,s)$ with $n>3$ and $q-p>2$ in which most pairs of elements are tied? If not, how many such examples are there?*

Question 2. *What is the asymptotic behavior of the relative proportion of ties among elements of $D(n,a,b,s)$ as n,s and $q-p$ grow?*

Question 3. *What is the asymptotic behavior of the relative proportion of elements of $D(n,a,b,s)$ which tie most others as n,s and $q-p$ grow?*

The theorem we present here does not answer any of these questions, but it is nevertheless a striking refutation of the "common sense" expectation that elements of $D(n,a,b,s)$ "should" usually tie. Let $G(n,a,b,s)$ be the undirected graph with vertex-set $D(n,a,b,s)$ in which two vertices are adjacent if and only if they are not tied. Balanced dice are isolated vertices in $G(n,a,b,s)$, of course, but it turns out that they are the only ones not connected to the rest: all the other vertices of $G(n,a,b,s)$ lie in a single component.

Theorem 1. *The graph $G(n,a,b,s)$ is connected, except for isolated vertices.*

There are many interesting questions to ask about the dice graphs $G(n,a,b,s)$. What is the largest possible diameter of a component of $G(n,a,b,s)$? (Our proof of Theorem 1 shows that it cannot be more than 6.) Is there a $G(n,a,b,s)$ with more than two edges whose nontrivial component fails to be 2-connected? How highly connected are these graphs in general? There are also many interesting questions about the directed dice graphs $\Delta(n,a,b,s)$, in which each edge of $G(n,a,b,s)$ is directed from the weaker to the stronger of its end-vertices. How common are sources and sinks? How common are vertices with the same in- and out-degree? It is well known [1, 2, 3] that $\Delta(n, a, b, s)$ may contain directed cycles; some directed dice graphs are covered by cycles (like $\Delta(6, 1, 6, 17)$), some are acyclic (like $\Delta(6, 1, 6, 13)$), and some are neither acyclic nor covered by cycles (like $\Delta(6, 1, 6, 18)$). Is it possible to characterize the 4-tuples (n, a, b, s) which give rise to directed graphs $\Delta(n, a, b, s)$ which are covered by cycles?

The rest of the paper is a rather long proof of Theorem 1, with many special cases. The special cases presented in Sections 2 and 5 are interesting, but those discussed in Sections 3 and 4 are verified through rather tedious detailed arguments. We hope that a clever reader will find a more concise proof, which might provide more insight into the properties of dice graphs.

2. $q - p \leq 2$

This is the most striking special case of Theorem 1. When $q - p \leq 2$, it turns out that $G(n, a, b, s)$ is either edgeless or complete – if any two distinct elements of $D(n, a, b, s)$ are tied then they all are tied.

Proposition 2.1. *If $q(n, a, b, s) - p(n, a, b, s) \leq 2$ then the* stronger *relation on $D(n, a, b, s)$ is either trivial or a linear order.*

Proof. If $q - p \leq 1$ then $D(n, a, b, s)$ has only one element.

If $q - p = 2$ then for a die $X \in D(n, a, b, s)$ with characteristic vector (v_p, v_{p+1}, v_q) let the *strength* of X be

$$str(X) = \frac{v_q - v_p}{v_p + 2v_{p+1} + v_q}.$$

As noted in [5], $str(X) > str(Y)$ if and only if X is stronger than Y. The proof is not difficult: if $Y = (y_1, ..., y_n)$ has characteristic vector (w_p, w_{p+1}, w_q) then

$$\begin{aligned} & \frac{1}{2}(v_p + 2v_{p+1} + v_q)(w_p + 2w_{p+1} + w_q)(str(X) - str(Y)) \\ = \quad & \frac{1}{2}(w_p + 2w_{p+1} + w_q)(v_q - v_p) - (v_p + 2v_{p+1} + v_q)(w_q - w_p) \\ = \quad & w_p v_q + w_{p+1} v_q + v_{p+1} w_p - w_{p+1} v_p - w_q v_p - v_{p+1} w_q \\ = \quad & \left|\{(i,j) | x_i > y_j\}\right| - \left|\{(i,j) | x_i < y_j\}\right|. \end{aligned}$$

Observe that

$$str(X) = \frac{s - n(p+1)}{n + v_{p+1}}.$$

If $s = n(p+1)$ it follows that $str(X) = 0\ \forall X \in D(n,a,b,s)$, and hence all the elements of $D(n,a,b,s)$ are tied. If $s \neq n(p+1)$ then $str(X) = str(Y)$ if and only if $v_{p+1} = w_{p+1}$; as $v_p + v_{p+1} + v_q = n = w_p + w_{p+1} + w_q$ and $pv_p + (p+1)v_{p+1} + qv_q = s = pw_p + (p+1)w_{p+1} + qw_q$, $v_{p+1} = w_{p+1}$ if and only if $X = Y$. That is, if $s \neq n(p+1)$ then $str : D(n,a,b,s) \to \mathbb{Q}$ is injective, and consequently *stronger* is a linear order on $D(n,a,b,s)$. ■

3. $n \leq 4$

Proposition 3.1. *If $n \leq 2$ then every element of $D(n,a,b,s)$ is balanced.*

Proof. If $n = 1$ then $D(n,a,b,s)$ has only one element. If $n = 2$ then two distinct elements of $D(n,a,b,s)$ are (x_1,x_2) and (y_1,y_2) with $x_1 < y_1 \leq y_2 < x_2$; clearly they are tied. ■

Proposition 3.2. *If $n = 3$ then $G(n,a,b,s)$ is connected, except for isolated vertices. Moreover, every component has diameter ≤ 4.*

Proof. Suppose that $s = 3k+1$. Certainly $3p \leq s \leq 3q$ and hence $p \leq k$ and $k+1 \leq q$; consequently $(k,k,k+1) \in D(n,a,b,s)$. It turns out that $(k,k,k+1)$ is not tied with any other element of $D(n,a,b,s)$. To see why, suppose $X = (x_1,x_2,x_3) \in D(n,a,b,s)$. If $x_1 \geq k$ then $X = (k,k,k+1)$. If $x_1 < k$ and $x_2 \leq k$ then X is weaker than $(k,k,k+1)$, because X loses at least four rolls (all three rolls of x_1 and also a roll of x_2 against $k+1$) and X wins no more than three rolls (the three rolls of x_3). If $x_1 < k < x_2$ then X is stronger than $(k,k,k+1)$.

Similarly, if $s = 3k+2$ then $(k,k+1,k+1)$ does not tie any other element of $D(n,a,b,s)$, and consequently $G(n,a,b,s)$ has diameter ≤ 2.

Suppose $s = 3k$. If $p \geq k-1$ and $q \leq k+1$ then $D(n,a,b,s) \subseteq \{(k,k,k),(k-1,k,k+1)\}$ and every element is balanced.

If $p \leq k-2$ or $q \geq k+2$ then $D(n,a,b,s)$ contains (k,k,k) and at least one $X \in \{(k-1,k-1,k+2)\,,\ (k-2,k+1,k+1)\}$; X is not tied with (k,k,k), so they are adjacent in $G(n,a,b,s)$. If Y is a vertex of $G(n,a,b,s)$ which is not adjacent to (k,k,k) then the die Y is tied with (k,k,k) and hence $Y = (y_1,k,2k-y_1)$. If $y_1 < k-2$ or $y_1 = k-1$ then Y is weaker than $(k-2,k+1,k+1)$ and stronger than $(k-1,k-1,k+2)$. Consequently every vertex $Y \neq (k-2,k,k+2)$ of $G(n,a,b,s)$ is adjacent to an element of

$\{(k,k,k),X\}$. The die $(k-2,k,k+2)$ is balanced if $p \geq k-3$ and $q \leq k+3$; otherwise at least one of $(k-4,k+2,k+2),(k-2,k-2,k+4)$ is a vertex of $G(n,a,b,s)$ adjacent to both $(k-2,k,k+2)$ and (k,k,k). As the distance between (k,k,k) and an arbitrary non-isolated vertex of $G(n,a,b,s)$ cannot be more than 2, the diameter of a component of $G(n,a,b,s)$ cannot be more than 4. ■

Proposition 3.3. *If $n = 4$ then $G(n,a,b,s)$ is connected, except for isolated vertices. Moreover, every component has diameter ≤ 4.*

Proof. Suppose first that $s = 4k+1$; then $4p \leq s \leq 4q$ and hence necessarily $p \leq k < q$. We claim that the distance between $(k,k,k,k+1)$ and an arbitrary vertex of $G(n,a,b,s)$ cannot be more than 2. If $p \geq k-2$ and $q \leq k+3$ then the claim is verified exhaustively. First, note that $(k,k,k,k+1)$ is adjacent to $(k-1,k,k+1,k+1)$, $(k-1,k,k,k+2)$, $(k-1,k-1,k+1,k+2)$, $(k-2,k+1,k+1,k+1)$, $(k-2,k,k+1,k+2)$, and $(k-1,k-1,k,k+3)$. There are only four other possible elements of $D(n,a,b,s)$. One of them, $(k-2,k-1,k+2,k+2)$, is weaker than $(k-1,k,k,k+2)$. The other three, $(k-2,k,k,k+3)$, $(k-2,k-1,k+1,k+3)$ and $(k-2,k-2,k+2,k+3)$, are all neighbors of $(k-1,k-1,k,k+3)$.

Suppose now that $p < k-2$ or $q > k+3$, and let $X = (x_1,x_2,x_3,x_4) \in D(n,a,b,s)$. If $x_1 > p$ or $x_4 < q$ then we may assume inductively that the distance between X and $(k,k,k,k+1)$ is not more than 2. If $x_3 \leq k$ and $p \leq k-2$ then X is weaker than $(k-2,k+1,k+1,k+1)$. If $x_3 \leq k$ and $p \geq k-1$ then the assumption that $x_4 = q > k+3$ implies that $X = (k-1,k-1,k-1,k+4)$. In any case, if $x_3 \leq k$ then the claim is satisfied.

If $x_3 > k$ and $x_2 \geq k$ then X is stronger than $(k,k,k,k+1)$. If $x_3 > k$ and $x_2 = k-1$ then either $x_3 > k+1$, in which case $p = x_1 \leq k-2$ and X is stronger than $(k-2,k+1,k+1,k+1)$, or $x_3 = k+1$, in which case X is weaker than $(k,k,k,k+1)$.

Suppose $x_3 > k$ and $x_2 < k-1$. If $k - x_2 > x_3 - k$ then it must be that $x_4 > x_3$; it follows that $Y = (x_1+1, x_2+x_3-k, k-1, x_4)$ is weaker than $(k,k,k,k+1)$ and stronger than X. If $k - x_2 \leq x_3 - k$ then consider $Y = (x_1, k, x_3 - k + x_2 + 1, x_4 - 1)$. Y is stronger than $(k,k,k,k+1)$, because it cannot lose more than 5 rolls (the four rolls of x_1 and a roll of k against $k+1$) and it must win at least 6 rolls (those of $x_3 - k + x_2 + 1$ or $x_4 - 1$ against k). If $x_1 < x_2$ then X is stronger than Y, because X wins at least 8 of the 16 rolls (all four rolls of x_4, at least three rolls of x_3, and the roll of x_2 against x_1) and at least one roll is tied. If $x_1 = x_2$ then $k - x_1 = k - x_2 \leq x_3 - k \leq x_4 - k$ and $s = 4k+1$ imply that

$x_3 - k = k - x_2 \geq 2$ and $x_4 = x_3 + 1$, so $Y = (x_1, k, k+1, x_3)$. Once again X is stronger than Y, because X wins at least 7 rolls (four of x_4 and three of x_3) and at least three rolls are tied (two of $x_1 = x_2$ and one of x_3).

This completes the proof when $s = 4k + 1$. If $s = 4k + 3$ then note that $(x_1, x_2, x_3, x_4) \mapsto (-x_4, -x_3, -x_2, -x_1)$ defines a *stronger*-reversing bijection between $D(n, a, b, s)$ and $D(n, -b, -a, -s)$, which gives an isomorphism between $G(n, a, b, s)$ and $G(n, -b, -a, -s)$; as $-s = 4(-k - 1) + 1$, the argument given above applies to $G(n, -b, -a, -s)$.

Suppose now that $s = 4k + 2$. We claim that the distance between $(k, k, k+1, k+1)$ and a vertex of $G(n, a, b, s)$ representing a non-balanced die cannot be more than 2. If $p \geq k - 2$ and $q \leq k + 2$ then the claim is verified by first noting that $(k, k, k+1, k+1)$ is adjacent to $(k, k, k, k+2)$, $(k-1, k+1, k+1, k+1)$, $(k-2, k+1, k+1, k+2)$ and $(k-2, k, k+2, k+2)$. The only other dice which might appear are $(k-1, k-1, k+2, k+2)$, which is adjacent to $(k, k, k, k+2)$, and $(k-1, k, k+1, k+2)$, which is balanced unless $p = k - 2$, in which case it is adjacent to $(k-2, k, k+2, k+2)$.

Suppose now that $p < k - 2$ or $q > k + 2$, and let $X = (x_1, x_2, x_3, x_4) \in D(n, a, b, s)$. If $x_2 > k$ then X is stronger than $(k, k, k, k+2)$, and if $x_3 \leq k$ then X is weaker than $(k-1, k+1, k+1, k+1)$.

Suppose $x_2 \leq k < x_3$. If $x_2 = k$ and $x_3 > k+1$ then X is stronger than $(k, k, k+1, k+1)$. If $x_2 = k$, $x_3 = k+1$ and $x_1 \leq k - 2$ then X is weaker than $(k-1, k+1, k+1, k+1)$. If $x_2 = k$, $x_3 = k+1$ and $x_1 \geq k-1$ then X is either $(k, k, k+1, k+1)$ or $(k-1, k, k+1, k+2)$. As $p < k-2$ or $q > k+2$, $G(n, a, b, s)$ has at least one vertex adjacent to both $(k-1, k, k+1, k+2)$ and $(k, k, k+1, k+1)$, namely $(k-3, k+1, k+2, k+2)$ or $(k-1, k, k, k+3)$.

Suppose $x_2 < k < x_3$. If $x_3 \leq k+2$ then $(k, k, k, k+2)$ is stronger than X. If $x_3 \geq k+3$ and $x_2 \geq k-2$ then X is stronger than $(k-2, k, k+2, k+2)$.

Suppose $x_3 \geq k + 3$ and $x_2 \leq k - 3$. If $x_3 - k > k - x_2$ then let $Y = (x_1, k, x_3 + x_2 - k + 1, x_4 - 1)$; Y is stronger than $(k, k, k+1, k+1)$. If $x_4 - 1 \neq x_3$ then X and Y share exactly one label, so one must be stronger than the other (they cannot share 15 decisive rolls evenly). If $x_4 - 1 = x_3$ then X is stronger than Y, as Y cannot win more than 6 rolls against X (x_1 cannot win at all, and no label of Y is greater than x_3 or x_4) and X wins at least 7 rolls against Y (x_4 wins four rolls and x_3 wins three). If $x_3 - k \leq k - x_2$ then let $Z = (x_1 + 1, x_2 + x_3 - k - 1, k, x_4)$; Z is weaker than $(k, k, k+1, k+1)$. If $x_1 + 1 \neq x_2$ then X and Z share exactly one label and hence cannot tie. If $x_1 + 1 = x_2$ then Z loses 6 rolls against X

(three against x_4 and three against x_3) and wins at least 7 (four against x_1, three against x_2 and perhaps x_4 against x_3), so Z is stronger than X.

Finally, suppose $s = 4k$. We claim that the distance from (k,k,k,k) to a vertex of $G(n,a,b,s)$ representing a non-balanced die cannot be more than 2. If $p \geq k-1$ and $q \leq k+1$ the claim is vacuous – all the elements of $G(n,a,b,s)$ are balanced. If $p \geq k-2$ and $q \leq k+2$ the claim is verified by observing that $(k-1,k-1,k+1,k+1)$ is balanced, that (k,k,k,k) is adjacent to $(k-1,k-1,k,k+2)$ and $(k-2,k,k+1,k+1)$, and that both of the latter are adjacent to $(k-1,k,k,k+1)$, $(k-2,k,k,k+2)$, $(k-2,k-1,k+1,k+2)$ and $(k-2,k-2,k+2,k+2)$.

Suppose $p < k-2$ or $q > k+2$, and let $X = (x_1,x_2,x_3,x_4) \in D(n,a,b,s)$. X is adjacent to (k,k,k,k) if $x_2 > k$, or $x_3 < k$, or precisely one of x_2, x_3 is equal to k. If $x_2 = k = x_3$ and $X \neq (k,k,k,k)$ then X is stronger than $(k-1,k-1,k,k+2)$ and weaker than $(k-2,k,k+1,k+1)$. It remains to consider X with $x_2 < k < x_3$.

Suppose $x_2 = k-1$ and $k < x_3$. If $p \geq k-2$ then $q > k+2$ implies that $Y = (k-1,k-1,k-1,k+3) \in D(n,a,b,s)$; Y is weaker than (k,k,k,k) and either weaker than X (if $x_1 = k-1$) or stronger than X (if $x_1 = k-2$). Suppose $p < k-2$. Then $Z = (k-3,k+1,k+1,k+1) \in D(n,a,b,s)$; Z is stronger than (k,k,k,k). If $x_3 > k+1$ then X is stronger than Z. If $x_3 = k+1$ and $x_4 > k+3$ then X is stronger than Y; if $x_3 = k+1 = x_4$ then X is weaker than Z; if $x_3 = k+1$ and $x_4 = k+2$ then X is stronger than Z. If $x_3 = k+1$ and $x_4 = k+3$ then $X = (k-3,k-1,k+1,k+3)$ is balanced for $p \geq k-4$ and $q \leq k+4$, stronger than $(k-2,k-2,k-1,k+5)$ for $q \geq k+5$, and weaker than $(k-5,k+1,k+2,k+2)$ for $p \leq k-5$.

If $x_2 < k$ and $x_3 = k+1$ then very similar arguments show that if X is not balanced then the distance between X and (k,k,k,k) in $G(n,a,b,s)$ is no more than 2.

It remains to consider X with $x_2 < k-1$ and $x_3 > k+1$. If $k - x_2 \leq x_3 - k$ then let $Y = (x_1, k, x_3 - k + x_2 + 1, x_4 - 1)$; Y is stronger than (k,k,k,k). If $x_1 < x_2$ then Y is weaker than X because it loses at least 8 rolls (four against x_4, at least three against x_3, and x_1 against x_2) and cannot win more than 7. If $x_1 = x_2$ then it must be that $x_3 = x_4$; it follows again that X is stronger than Y. Similarly, if $k - x_2 > x_3 - k$ then X is adjacent to $(x_1 + 1, x_2 - k + x_3 - 1, k, x_4)$ and hence the distance between X and (k,k,k,k) in $G(n,a,b,s)$ is no more than 2. ■

4. $s \leq (n-3)p+3q$

Proposition 4.1. *If $n \geq 5$, $q-p \geq 3$ and $s \leq (n-2)p+2q$ then $G(n,a,b,s)$ is connected, except for isolated vertices. Moreover, every component has diameter ≤ 4.*

Proof. The definitions of p and q imply that $s \geq (n-1)p+q$. If $s=(n-1)p+q$ then $(p,p,...,p,q)$ is weaker than every other element of $D(n,a,b,s)$.

Suppose $(n-1)p+q < s \leq (n-2)p+2q$ and $n \geq 7$. If $X=(x_1,...,x_n)$ and $Y=(y_1,...,y_n)$ have $x_{n-2}=p$ and $y_{n-2}>p$ then X loses at least $3(n-2)$ rolls against Y, and wins no more than $2n$; $3(n-2)>2n$, so X is weaker than Y. Every element of $D(n,a,b,s)$ is either such an X or such a Y, and there are dice of both types; hence $G(n,a,b,s)$ is connected and its diameter is no more than 2.

Suppose $(n-1)p+q < s \leq (n-2)p+2q$ and $n \in \{5,6\}$. If $X=(x_1,...,x_n)$ and $Y=(y_1,...,y_n)$ have $x_{n-2}=p$ and $y_{n-3}>p$ then X loses at least $4(n-2)$ rolls against Y, and wins no more than $2n$; $4(n-2)>2n$, so X is weaker than Y. Such Y exist because $s \geq (n-1)p+q+1 \geq np+4$, and such X exist because $s \leq (n-2)p+2q$. To complete the proof it suffices to show that an arbitrary non-balanced $Z=(z_1,...,z_n) \in D(n,a,b,s)$ with $z_{n-3}=p<z_{n-2}$ is adjacent to at least one such Y or X.

Suppose $n=6$; then $Z=(p,p,p,z_4,z_5,z_6)$ with $z_4>p$. If $z_6 \geq p+3$ then Z is weaker than $Y=(p+1,p+1,p+1,z_4,z_5,z_6-3)$; the labels of Y should be reordered as $(p+1,p+1,p+1,z_4,z_6-3,z_5)$, $(p+1,p+1,p+1,z_6-3,z_4,z_5)$ or $(z_6-3,p+1,p+1,p+1,z_4,z_5)$ if necessary. It's impossible for z_6 to be less than $p+2$, because $s>np+3$; the only remaining possibility is that $z_6=p+2$. If $z_5=p+1$ and $z_4=p+1$ then $X=(p,p,p,p,p+2,p+2)$ is weaker than Z, and if $z_5=p+2$ and $z_4=p+1$ then $X=(p,p,p,p,p+2,p+3)$ is weaker than Z. Finally, if $z_5=p+2$ and $z_4=p+2$ then $Z=(p,p,p,p+2,p+2,p+2)$ is balanced for $q-p=3$, and stronger than $X=(p,p,p,p,p+2,p+4)$ for $q-p \geq 4$.

Suppose now that $n=5$; then $Z=(p,p,z_3,z_4,z_5)$ with $z_3>p$. If $z_5-1>z_4$ then Z is weaker than $Y=(p,p+1,z_3,z_4,z_5-1)$. If $z_5=z_4>z_3+1$ then $Y=(p+1,p+1,z_3,z_4-2,z_5)$ is stronger than Z. If $z_5=z_4 \leq z_3+1$ then $Y=(p+1,p+1,z_3,z_4-1,z_5-1)$ is weaker than Z unless $z_5=z_4=z_3+1=p+2$, in which case $Z=(p,p,p+1,p+2,p+2)$ is stronger than $X=(p,p,p,p+2,p+3)$. (The labels of Y should be reordered if necessary.) If $z_5-1=z_4=z_3$ then $Y=(p,p+1,z_3,z_4,z_4)$ is adjacent to Z. Suppose $z_5-1=z_4>z_3$. If $z_3>p+2$ then $Y=(p+1,p+1,z_3-2,z_4,z_5)$

is stronger than Z; if $z_3 = p+2$ then $Y = (p+1, p+1, p+1, z_4, z_4)$ is stronger than Z; and if $z_3 = p+1$ then $X = (p, p, p, z_5, z_5)$ is weaker than Z. ■

If $X = (x_1, ..., x_n) \in D(n, a, b, s)$ and $p \leq k \leq q$ then let $f_X(k)$ give the win-loss difference of a roll of k against X:

$$f_X(k) = |\{i|x_i < k\}| - |\{i|x_i > k\}| .$$

For $Y = (y_1, ..., y_n) \in D(n, a, b, s)$ let $f_X(Y) = \sum_{i=1}^{n} f_X(y_i)$. Then $f_X(Y)$ is positive, negative or 0 according to whether X is weaker than Y, stronger than Y, or tied with Y.

Lemma 4.2. *Suppose $n \geq 5$, $q - p \geq 3$ and $(n-2)p + 2q < s \leq (n-3)p + 3q$. Suppose $Z \in D(n, a, b, s)$ ties all the non-balanced dice $X = (x_1, ..., x_n) \in D(n, a, b, s)$ such that $x_{n-3} = p$ and $x_n \geq q - 1$, and Z also ties all the non-balanced dice $Y = (y_1, ..., y_{n-1}, q) \in D(n, a, b, s)$ such that $y_{n-4} = p < y_{n-3}$ and $y_{n-1} \geq q - 1$. Then Z is balanced.*

Proof. Let $\sigma = s - 2q - (n-3)p$; then $p < \sigma \leq q$. We claim that if Z ties all the dice X described in the statement of the lemma then $f_Z(\tau+1) - f_Z(\tau) = f_Z(q) - f_Z(q-1)$ for all τ with $\sigma \leq \tau < q$; if $\sigma \geq q-1$ then there is nothing to prove. If $\sigma < q - 1$ then for $\sigma \leq \tau \leq q-1$ let $X_\tau^0 = (p, ..., p, \tau, q + \sigma - \tau, q)$ and $X_\tau^1 = (p, ..., p, \tau + 1, q + \sigma - \tau, q - 1) \in D(n, a, b, s)$, with labels reordered if necessary. As Z ties both X_τ^0 and X_τ^1, $f_Z(X_\tau^0) = 0 = f_Z(X_\tau^1)$ and hence $f_Z(\tau) + f_Z(q) = f_Z(\tau+1) + f_Z(q-1)$.

We claim that similarly, if Z ties X_σ^0 and all the dice Y described in the statement of the lemma then $f_Z(\tau+1) - f_Z(\tau) = f_Z(q) - f_Z(q-1)$ for all τ with $p \leq \tau < \sigma$. If $p \leq \tau < \sigma$ then let $Y_\tau^0 = (p, ..., p, \tau, p + \sigma - \tau, q, q)$ and $Y_\tau^1 = (p, ..., p, \tau+1, p + \sigma - \tau, q-1, q) \in D(n, a, b, s)$, with labels reordered if necessary. Note that $Y_p^0 = (p, ..., p, \sigma, q, q) = X_\sigma^0$ is not one of the Y described in the statement, but every other Y_τ^i is. As Z ties both Y_τ^0 and Y_τ^1, $f_Z(Y_\tau^0) = 0 = f_Z(Y_\tau^1)$ and hence $f_Z(\tau) + f_Z(q) = f_Z(\tau+1) + f_Z(q-1)$.

Combining the two claims, we see that $f_Z(\tau+1) - f_Z(\tau) = f_Z(q) - f_Z(q-1)$ whenever $p \leq \tau \leq q-1$. Theorem 2 of [4] implies that Z is balanced. ■

Proposition 4.3. *If $n \geq 5$, $q - p \geq 3$ and $(n-2)p + 2q < s \leq (n-3)p + 3q$ then $G(n, a, b, s)$ is connected, except for isolated vertices. Moreover, every component has diameter ≤ 6.*

Proof. Every non-balanced die is adjacent to at least one of the dice X, Y described in Lemma 4.2, so to prove this proposition it suffices to show that there are $W, W' \in D(n,a,b,s)$ such that every die X or Y of Lemma 4.2 is adjacent to at least one of W, W' and some die is adjacent to both W and W'. In most cases we show that there is a single die W adjacent to all the X and Y of Lemma 4.2.

We begin the proof with an initial observation. Consider any die $W = (w_1, ..., w_n) \in D(n,a,b,s)$ with $w_{n-6} > p$; as $s \geq np+2(q-p)+1 \geq np+7$, such dice W exist for $n \geq 7$. Against a die $X = (x_1, ..., x_n)$ with $x_{n-3} = p$, such a W wins at least $7(n-3)$ rolls and loses no more than $3n$; if $n \geq 7$ then $7(n-3) > 3n$ so W is stronger than X. Against a die $Y = (y_1, ..., y_n)$ with $y_{n-4} = p$, such a W wins at least $7(n-4)$ rolls and loses no more than $4n$; if $n \geq 10$ then $7(n-4) > 4n$ so W is stronger than Y. If $n \geq 10$ this observation completes the proof.

Suppose $n = 9$. If $s - np > 7$ then there is a die $W = (w_1, ..., w_9)$ with $w_2 > p$. Against a die $Y = (y_1, ..., y_9)$ with $y_5 = p$, W wins at least 40 of the 81 possible rolls; moreover Y cannot win any of the rolls of $y_1 = y_2 = y_3 = y_4 = y_5 = p$ against w_1, so Y cannot win more than 36 rolls; hence W is stronger than Y. If $s - np = 7$ then consider $W = (p, p, p+1, ..., p+1)$; against a die $Y = (y_1, ..., y_9)$ with $y_5 = p$, W wins precisely 35 of the 81 possible rolls, so if W and Y are tied then Y also wins 35 rolls. Consequently, if W and Y are tied then there are precisely 11 tied rolls. There cannot be an odd number of tied rolls involving the label p, as W has precisely two labels equal to p, so if W and Y are tied then Y must have at least one label equal to $p+1$. But there are 8 tied rolls involving the label p, which occurs twice on W and four times on Y, and there cannot be fewer than 7 tied rolls involving the label $p+1$, which appears 7 times on W. Hence there cannot be precisely 11 tied rolls, so W is adjacent to Y.

Suppose $n = 8$ and $s - np \geq 8$. Then there is a die $W = (w_1, ..., w_8)$ with $w_1 > p$; as $s \leq (n-3)p + 3q$ we may choose such a W so that $w_1 = p+1$. Against a die $Y = (y_1, ..., y_7, q)$ with $y_4 = p$, W wins every roll involving a label p of Y, so Y can only tie W if $w_8 < y_5$. Consequently if there is any Y as described in Lemma 4.2 which ties W then $w_8 < q$, and hence $W' = (p, w_2, ..., w_7, w_8+1) \in D(n,a,b,s)$. If Y ties both W and W' then $0 = 0 - 0 = f_Y(W) - f_Y(W') = f_Y(p+1) + f_Y(w_8) - f_Y(p) - f_Y(w_8+1) = f_Y(p+1) - f_Y(p) + f_Y(w_8) - f_Y(w_8+1)$. As $y_4 = p$, $f_Y(p+1) - f_Y(p) \geq 4$; hence $f_Y(w_8+1) - f_Y(w_8) \geq 4$ too. Y has no label equal to w_8, so $f_Y(w_8+1) - f_Y(w_8) \geq 4$ implies that Y must have at least 4 labels equal to $w_8 + 1$. Y has four labels equal to p and at least one label equal to q; $w_8 + 1 > p$ so Y must have four labels equal to q, an impossibility as $s < 4q + (n-4)p$. Consequently every Y described in Lemma 4.2 is adjacent to at least one of W, W'. As noted in the initial observation of the proof, W and W' are both stronger than $X^0_\sigma = (p, p, p, p, p, \sigma, q, q)$.

Suppose instead that $n = 8$ and $s - np = 7$; as $q - p \geq 3$ and $s - np > 2(q - p)$ it must be that $q - p = 3$. A die $Y = (y_1, ..., y_7, q)$ as described in Lemma 4.2 has $y_7 \geq q - 1 = p + 2$, and hence can only be $(p, p, p, p, p + 1, p + 1, p + 2, p + 3)$. This die is weaker than $W = (p, p + 1, ..., p + 1)$.

Suppose $n = 7$, and let $W' = X^0_\sigma = (p, p, p, p, \sigma, q, q)$. A die $Y = (p, p, p, y_4, y_5, q, q)$ as described in Lemma 4.2 is stronger than W'. Consider a die $Y = (p, p, p, y_4, y_5, q-1, q)$ as described in Lemma 4.2. If $\sigma < q-1$ then W' is weaker than Y, and if $\sigma = q$ then W' is stronger than Y. Suppose $\sigma = q - 1$. If $q - p = 3$ then $Y = (p, p, p, p + 1, p + 2, p + 2, p + 3)$ is weaker than $W = (p + 1, p + 1, p + 1, p + 1, p + 1, p + 1, p + 2)$. If $q - p = 4$ then $Y \in \{(p, p, p, p+1, p+3, p+3, p+4), (p, p, p, p+2, p+2, p+3, p+4)\}$; Y is weaker than $W = (p + 1, p + 1, p + 1, p + 1, p + 1, p + 2, p + 4)$. If $q - p \geq 5$ then $W = (p + 1, p + 1, p + 1, p + 1, q - 4, q - 1, q)$ is stronger than Y. As W is stronger than W' in each case, this suffices.

If $n \leq 6$ then we do not rely on the initial observation given in the second paragraph of the proof. We prove instead that the distance between X^0_σ and any other die X or Y of Lemma 4.2 is no more than 2.

Suppose $n = 6$; then $X^0_\sigma = (p, p, p, \sigma, q, q)$. If $X = (p, p, p, x_4, x_5, x_6)$ then either $x_5 = q$, in which case $X = X^0_\sigma$, or $x_5 < q$, in which case X is weaker than X^0_σ. Suppose $Y = (p, p, y_3, y_4, q, q)$ has $y_3 > p$; then Y is stronger than X^0_σ unless $\sigma = q$. If $\sigma = q$ then $(p+1, p+1, p+1, q-2, q-1, q)$ is stronger than both X^0_σ and Y unless $y_3 = y_4 = p + 2$, in which case Y is balanced. Suppose now that $Y = (p, p, y_3, y_4, q - 1, q)$ has $y_3 > p$. If $\sigma = q$ then Y is weaker than X^0_σ. If $\sigma = q - 1 > y_4$ then Y is weaker than X^0_σ; if $\sigma = q - 1 \leq y_4$ then $Y = (p, p, p + 1, q - 1, q - 1, q)$ and X^0_σ are both weaker than $(p + 1, p + 1, p + 1, q - 2, q - 1, q - 1)$ unless $q - p = 3$, in which case Y is balanced. If $\sigma < q - 1$ and $\sigma \leq y_4$ then Y is stronger than X^0_σ, and if $y_4 < \sigma < q - 1$ then $(p + 1, p + 1, y_3, y_4, q - 3, q)$ is stronger than both X^0_σ and Y.

Finally, suppose $n = 5$; then $X^0_\sigma = (p, p, \sigma, q, q)$. If $X = (p, p, x_3, x_4, x_5)$ then either $x_4 = q$, in which case $X = X^0_\sigma$, or $x_4 < q$, in which case X is weaker than X^0_σ. If $Y = (p, y_2, y_3, q, q)$ has $y_2 > p$ then Y is weaker than X^0_σ if $\sigma = q$ and stronger than X^0_σ if $\sigma < q$. Suppose $Y = (p, y_2, y_3, q - 1, q)$ has $y_2 > p$. Y is weaker than X^0_σ if $\sigma \geq q - 1$, and if $\sigma < q - 2$ then both Y and X^0_σ are stronger than $(p, p, \sigma + 1, q - 1, q)$. If $\sigma = q - 2 \neq y_3$ then X^0_σ is stronger than Y or weaker than Y according to whether $\sigma > y_3$ or $\sigma < y_3$. If $\sigma = q - 2 = y_3 = y_2$ then Y is stronger than X^0_σ. Suppose

$\sigma = q-2 = y_3 > y_2$. Then $y_2 = p+1$, so $Y = (p, p+1, q-2, q-1, q)$; if $q-2 = p+2$ then Y is balanced. If $q-2 > p+2$ then $(p+2, p+2, q-2, q-2, q-2)$ is weaker than both X^0_σ and Y, and if $q-2 = p+1$ then $X^0_\sigma = (p, p, p+1, p+3, p+3)$ and $Y = (p, p+1, p+1, p+2, p+3)$ are both weaker than $(p+1, p+1, p+1, p+2, p+2)$. ■

5. The general case

So far we have verified Theorem 1 for $n \leq 4$, for $q - p \leq 2$, and for greater values of n and $q-p$ when $s \leq (n-3)p + 3q$. Observe that if $s > \frac{n}{2}(p+q)$ then the map $(x_1, ..., x_n) \mapsto (p+q-x_n, ..., p+q-x_1)$ defines a *stronger*-reversing bijection between $D(n, a, b, s) = D(n, p, q, s)$ and $D(n, p, q, n(p+q) - s)$. This bijection gives an isomorphism between $G(n, a, b, s)$ and $G(n, p, q, n(p+q) - s)$, so it suffices to prove Theorem 1 under the hypothesis that $s \leq \frac{n}{2}(p+q)$. In particular, having verified Theorem 1 for $s \leq (n-3)p + 3q$ we need not concern ourselves any longer with $n \in \{5, 6\}$.

It remains to prove Theorem 1 when $n \geq 7$, $q - p \geq 3$ and $(n-3)p + 3q < s \leq \frac{n}{2}(p+q)$. The proof consists of three lemmas, two of which are valid in greater generality.

Lemma 5.1. *Suppose that $X \in D(n, a, b, s)$ has characteristic vector $(v_p, ..., v_q)$, and suppose there is a $j \in \{p+1, ..., q-1\}$ such that $v_{j-1} \neq v_{j+1}$ and $v_j \geq 2$. Then X has a neighbor X' in $G(n, a, b, s)$ such that every $Y \in D(n, a, b, s)$ with different numbers of labels equal to $j-1$ and $j+1$ is adjacent to X or X'.*

Proof. Let X' be the die obtained from X by replacing two labels equal to v_j with a pair of labels, one equal to $v_j - 1$ and the other equal to $v_j + 1$. The characteristic vector of X' is then $(v'_p, ..., v'_q) = (v_1, ..., v_{j-2}, v_{j-1} + 1, v_j - 2, v_{j+1} + 1, v_{j+2}, ..., v_q)$.

Suppose $Y \in D(n, a, b, s)$ has characteristic vector $(w_1, ..., w_q)$, and recall that $f_Y(k)$ gives the win-loss difference of a roll of k against Y:

$$\begin{aligned} f_Y(k) &= |\{i | y_i < k\}| - |\{i | y_i > k\}| \\ &= \sum_{i=p}^{k-1} w_i - \sum_{i=k+1}^{q} w_i. \end{aligned}$$

Then

$$\begin{aligned} f_Y(X) - f_Y(X') &= \sum_{k=p}^{q} v_k f_Y(k) - \sum_{k=p}^{q} v'_k f_Y(k) \\ &= 2f_Y(j) - f_Y(j-1) - f_Y(j+1) \\ &= w_{j-1} - w_{j+1}. \end{aligned}$$

Consequently if $w_{j-1} \neq w_{j+1}$ then $f_Y(X) \neq f_Y(X')$ and hence at least one of $f_Y(X), f_Y(X')$ is nonzero, so Y cannot tie both X and X'. In particular, if we take $Y = X'$ and note that (like every die) X' ties itself, we conclude that X and X' are not tied.

In sum, X and X' are adjacent vertices in $G(n, a, b, s)$, and every $Y \in D(n, a, b, s)$ with different numbers of labels equal to $j-1$ and $j+1$ is adjacent to at least one of them. ■

Lemma 5.2. *Suppose $n \geq 7$, $q-p \geq 3$ and $(n-3)p+3q < s \leq \frac{n}{2}(p+q)$. Then for every $j \in \{p+1, ..., q-1\}$ there is an $X_j \in D(n, a, b, s)$ whose characteristic vector $(v_p, ..., v_q)$ has $v_{j-1} \neq v_{j+1}$ and $v_j \geq 2$.*

Proof. Suppose $p < j < q$. Then $(n-3)p + 3j < (n-3)p + 3q < s \leq \frac{n}{2}(p+q) < \frac{n}{2}(j+q) < (n-3)q + 3j$, so there is a unique $k \in \{p, ..., q-1\}$ with $(n-3)k + 3j < s \leq (n-3)(k+1) + 3j$. Then $(n-3)k + 3j + 1 \leq s \leq (n-4)(k+1) + k + 3j + 1$, so there is a die $X \in D(n, a, b, s)$ with two labels equal to j, one label equal to $j+1$, and every other label equal to either k or $k+1$. If X has distinct numbers of labels equal to $j-1$ and $j+1$ then it satisfies the lemma.

If not, it must be that $j - 1 \in \{k, k+1\}$ and X has precisely one label equal to $j-1$. If $k+1 = j-1$ then X has $n-4$ labels equal to $j-2$, one equal to $j-1$, two equal to j, and one equal to $j+1$; the lemma is satisfied by the $X_j \in D(n, a, b, s)$ which has $n-5$ labels equal to $j-2$, two equal to $j-1$, and three equal to j. If $k = j-1$ then X has one label equal to $j-1$, $n-2$ equal to j, and one equal to $j+1$. If $p < j-1$ then the lemma is satisfied by the $X_j \in D(n, a, b, s)$ which has one label equal to $j-2$, $n-3$ equal to j, and two equal to $j+1$. If $p = j-1$ then $q \geq p+3 = j+2$ and the lemma is satisfied by the $X_j \in D(n, a, b, s)$ which has two labels equal to $j-1$, $n-3$ equal to j, and one equal to $j+2$. ■

Lemma 5.3. *Suppose $n \geq 5, q-p \geq 3$ and $(n-2)p + 2q \leq s \leq (n-2)q + 2p$. Then for every choice of $j \neq j' \in \{p+1, ..., q-1\}$ there is an*

$X_{j,j'} \in D(n,a,b,s)$ whose characteristic vector $(v_p,...,v_q)$ has $v_{j-1} \neq v_{j+1}$ and $v_{j'-1} \neq v_{j'+1}$.

Proof. Suppose $p < j < j' < q$. Then $(n-2)p+j+j' < (n-2)p+2q \leq s \leq (n-2)q+2p < (n-2)q+j+j'$, so there is a unique $k \in \{p,...,q-1\}$ with $(n-2)k+j+j' < s \leq (n-2)(k+1)+j+j'$. Consequently there is a die $X \in D(n,a,b,s)$ with one label equal to $j-1$, one label equal to $j'+1$, and every other label equal to either k or $k+1$; X satisfies the lemma unless the labels equal to k or $k+1$ include precisely one equal to $j+1$ or precisely one equal to $j'-1$.

If the labels equal to k or $k+1$ include precisely one equal to $j+1$ then either $s = j-1+j'+1+j+1+(n-3)j$ or else $s = j-1+j'+1+j+1+(n-3)(j+2)$. In the former case $s = j'+1+(n-1)j$ and the lemma is satisfied by $X_{j,j'}$ with $n-3$ labels equal to j, two labels equal to $j+1$ and one label equal to $j'-1$. In the latter case $s = 2j+2+j'-1+(n-3)(j+2)$ and the lemma is satisfied by $X_{j,j'}$ with two labels equal to $j+1$, $n-3$ labels equal to $j+2$ and one label equal to $j'-1$.

If the labels equal to k or $k+1$ include precisely one equal to $j'-1$ then either $s = j-1+j'+1+j'-1+(n-3)j'$ or else $s = j-1+j'+1+j'-1+(n-3)(j'-2)$. In the former case $s = 2j'-2+j+1+(n-3)j'$ and the lemma is satisfied by $X_{j,j'}$ with $n-3$ labels equal to j', two labels equal to $j'-1$ and one label equal to $j+1$. In the latter case $s = j+3+(n-1)(j'-2)$ and the lemma is satisfied by $X_{j,j'}$ with one label equal to $j+1$, $n-3$ labels equal to $j'-2$ and two labels equal to $j'-1$. ■

Proposition 5.4. *If $n \geq 7$, $q-p \geq 3$ and $(n-3)p+3q < s \leq \frac{n}{2}(p+q)$ then the distance between two arbitrary non-isolated vertices of $G(n,a,b,s)$ is not more than 6.*

Proof. Suppose X and X' are non-isolated vertices of $G(n,a,b,s)$, i.e., non-balanced dice in $D(n,a,b,s)$. Let their characteristic vectors be $(v_p,...,v_q)$ and $(v'_p,...,v'_q)$ respectively. As they are not balanced, Theorem 2 of [4] implies that there are $j,j' \in \{p+1,...,q-1\}$ with $v_{j-1} \neq v_{j+1}$ and $v'_{j'-1} \neq v'_{j'+1}$.

Suppose $X_j, X_{j'}$ are dice described in Lemma 5.2, and let $X'_j, X'_{j'}$ be their neighbors discussed in Lemma 5.1. Then X is adjacent to at least one of X_j, X'_j and X' is adjacent to at least one of $X_{j'}, X'_{j'}$. If $j = j'$ then the distance between X and X' cannot be more than 3. If $j \neq j'$ then the die $X_{j,j'}$ of Lemma 5.3 is adjacent to at least one of X_j, X'_j and also adjacent to at least one of $X_{j'}, X'_{j'}$, so the distance between X and X' is no more than 6. ■

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