Section 3.7

**Optimization**

Our ability to use calculus to find absolute extremes of a function can be extremely useful in practical situations in which we need to optimize a quantity. For example, we may wish to minimize a cost or maximize the volume of an object. In this section we will investigate the ways in which calculus can be be used to solve such real-world problems.

The following is one possible strategy for solving optimization problems:

1. Draw a picture.
2. Name all variables.
3. Write an equation for the given constraints that relates the variables.
4. Identify the quantity that needs to be optimized and write an equation for it; the equation will generally need to take the constraints into account.
5. If necessary, rewrite the equation in terms of one variable using the constraints.
6. Determine the endpoints of the interval of interest and find the absolute max/min of the equation over this interval using the techniques from 3.1. Remembering to check the endpoints of the interval if necessary. Throw out any ”solutions” that don’t make sense.

A farmer wants to devote 150 $m^2$ of land to a crop of corn. He is planning to fence off the resulting rectangle, and add in one more fence parallel to one side of the rectangle in order to distinguish between two different fertilization systems. What dimensions of the plot of land will minimize the amount of fencing needed?

Let’s start with a graph of some possible choices:

The farmer could choose to fence off a long skinny rectangle, a square, etc.; but regardless of how he chooses the lengths of the rectangle, its area should be $A = 150$.

It is clear that we can vary the side lengths $x$ and $y$, so these are the variables we will work with.
Since the area of a rectangle is $A = xy$, it is clear that the equation

$$xy = 150$$

provides a constraint.

We wish to minimize the length of fencing needed; let’s write an equation for this function:

$$P = 2x + 3y.$$ 

We wish to find the absolute extremes of $P$, but this will be difficult to do since $P$ is currently a function of two variables $x$ and $y$. Fortunately, the constraint $xy = 150$ gives us a relationship between $x$ and $y$: we can rewrite $y$ as

$$y = \frac{150}{x},$$

so that the formula for $P$ can be rewritten as

$$P = 2x + \frac{450}{x}.$$ 

In this case, there is no natural “smallest” $x$ (since $x = 0$ is not in the domain of $P$) or “largest” $x$ (since we could potentially make $x$ as large as we want). So we do not have to check endpoints when finding the extremes of $P$.

To find the extremes, we’ll need to start by locating the critical numbers of $P$. The derivative of $P$ is

$$P' = 2 - \frac{450}{x^2}.$$ 

To find the type 1 critical numbers, we’ll set $P' = 0$ and solve

$$2 - \frac{450}{x^2} = 0.$$ 

This is equivalent to

$$\frac{450}{x^2} = 2,$$

so that

$$x^2 = 225.$$
Thus the type 1 critical numbers are \( x = -15 \) and \( x = 15 \); since we can’t use a negative amount of fencing, we’ll throw away the first critical number.

The only type 2 critical number is \( x = 0 \), which is not in the domain of \( P \), so we may ignore it as well.

Since the only critical number of our function is \( x = 15 \), it must correspond to a local extreme of \( P \). In fact, since \( P' < 0 \) on \((0, 15)\) and \( P' > 0 \) on \((15, \infty)\), we can be certain that \( x = 15 \) does correspond to the desired minimum value for the perimeter \( P \). So choosing \( x = 15 \) and \( y = 150/15 = 10 \) yields the desired minimum amount of fencing.

A 400 meter racetrack is shaped like a rectangle with a semicircle at each end. Find the dimensions that will maximize the area enclosed by the track.

The perimeter \( P \) of the track is our constraint; we are told that \( P = 400 \). Since the perimeter of a circle is \( 2\pi r \), the formula for \( P \) is

\[
P = 2x + 2\pi r.
\]

Thus we have

\[
2x + 2\pi r = 400,
\]

or \( x = 200 - \pi r \).

We wish to maximize the area \( A \) of the track. The area of a circle is \( \pi r^2 \); since the rectangular part of the track has sides of length \( x \) and \( 2r \), its area is \( 2rx \). Thus the formula for \( A \) is

\[
A = \pi r^2 + 2rx,
\]

which we rewrite (using the constraint \( x = 200 - \pi r \)) as

\[
A = 400r - \pi r^2.
\]

The smallest possible value for \( r \) is 0, and the largest occurs when \( x = 0 \), that is when \( r = \frac{200}{\pi} \). So we wish to maximize area over \([0, \frac{200}{\pi}]\).
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Since $A' = 400 - 2\pi r$ is never undefined, there are no type 2 critical numbers; we just need to find the type 1 critical numbers, i.e. determine when $A' = 0$. This occurs if

$$2\pi r = 400,$$

so that $r = \frac{200}{\pi}$.

The two possible $r$ values at which $A$ could have an absolute extreme are at the endpoints $r = 0$ and $r = \frac{200}{\pi}$. We need to check each possibility in $A$ to determine the maximum area; so we need to calculate

1. $A(0) = 0$
2. $A\left(\frac{200}{\pi}\right) = \frac{40000}{\pi}$.

Notice that the maximum area occurred at the endpoint of the interval, when $r = \frac{200}{\pi}$ and $x = 0$; so the track enclosing the maximum area is actually a circle.

If 1200 cm$^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

The surface area $S$ of the box is the constraint here; $S = 1200$. The box has a square base with area $x^2$ and four sides, each of area $xh$; so a formula for the surface area is $S = x^2 + 4xh$. Because of the constraint, we know that $x^2 + 4xh = 1200$.

We wish to optimize the volume $V = x^2h$. Since $h$ can be rewritten as

$$h = \frac{1200 - x^2}{4x},$$

we can in turn rewrite our volume formula as

$$V = \frac{1200x - x^3}{4}.$$

The smallest possible value for $x$ is 0, and the largest value would occur if $h = 0$, that is if $x^2 = 1200$; so the largest possible $x$ is $x = \sqrt{1200}$. We wish to find the absolute maximum value of

$$V = \frac{1200x - x^3}{4}$$

on $[0, \sqrt{1200}]$. 

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Since
\[ V' = \frac{1200 - 3x^2}{4} \]
is always defined, there are no type 2 critical numbers; so we’ll focus on type 1 critical numbers, when \( V' = 0 \).

If \( V' = 0 \), we must have \( 1200 - 3x^2 = 0 \), that is
\[ x^2 = \frac{1200}{3} = 400 \]
or \( x = \pm 20 \). Obviously \(-20\) can not be a the length of a side, so we throw out that option. Thus the only critical number is \( x = 20 \).

The three possible \( x \) values at which \( V \) could have an absolute extreme are at the endpoints \( x = 0 \) and \( x = \sqrt{1200} \), as well as at the critical number \( x = 20 \). We need to check each possibility in \( V \) to determine the maximum volume:

1. \( V(0) = 0 \)
2. \( V(20) = 4000 \)
3. \( A(\sqrt{1200}) = 0 \).

So \( x = 20 \) will maximize the area of the box; with this choice of \( x \), the height \( h \) must be \( h = 10 \).