Derivatives and Integrals of Logarithmic and Exponential Functions

In this section, we record and work with the derivatives and integrals of general logarithmic and exponential functions.

1. Derivative Rules:
   (a) \( \frac{d}{dx} \log_a x = \frac{1}{x \ln a} \), \( a > 0 \), \( a \neq 1 \)
   (b) \( \frac{d}{dx} \ln x = \frac{1}{x} \)
   (c) \( \frac{d}{dx} a^x = a^x \ln a \)

2. Integration Rules:
   (a) \( \int \frac{1}{x} \, dx = \ln |x| + C \)
   (b) \( \int a^x \, dx = \frac{a^x}{\ln a} + C \), \( a > 0 \), \( a \neq 1 \)

Example. Find \( \frac{d}{dx} 4^x^4 \).

We will need to use the chain rule here:

- \( f(x) = 4^x \)
- \( f'(x) = 4^x \ln 4 \)
- \( f'(g(x)) = 4^{x^4} \ln 4 \)

So \( \frac{d}{dx} 4^{x^4} = f'(g(x))g'(x) = 4x^34^{x^4} \ln 4 \).

Example. Evaluate \( \int \cot x \, dx \).

Since we don’t know the integral of \( \cot x \) immediately, we should try to rewrite the function:

\( \cot x = \frac{\cos x}{\sin x} \).

It seems like a u-substitution might be an appropriate choice here: setting \( u = \sin x \) so that \( du = \cos x \, dx \),

we have

\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx \\
= \int \frac{1}{u} \, du \\
= \ln |u| + C \\
= \ln |\sin x| + C.
\]
So
\[ \int \cot x \, dx = \ln |\sin x| + C. \]

**Example.** Evaluate
\[ \int \frac{1}{x \ln x} \, dx. \]

It is clear that we will need a substitution here: setting
\[ u = \ln x \] so that \( du = \frac{1}{x} \, dx, \)
we have
\[ \int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\ln x| + C. \]

**Example.** Evaluate
\[ \int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}} \, dx. \]

Again, we will need to use \( u \)-substitution here; unfortunately, there are several different possibilities for the choice of \( u \). However, notice that if we set
\[ u = \ln(\sec x + \tan x) \] so that \( du = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx, \)
we have
\[ du = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx = \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} \, dx = \sec x \, dx. \]
Thus the integral becomes
\[ \int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}} \, dx = \int \frac{1}{\sqrt{u}} \, du = \int u^{-1/2} \, du = 2u^{1/2} + C = 2(\ln(\sec x + \tan x))^{1/2} + C. \]
Thus

\[
\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}} \, dx = 2(\ln(\sec x + \tan x))^{1/2} + C.
\]