Exponential Functions and their Derivatives

Exponential functions are functions of the form

\[ f(x) = a^x, \]

where \( a \) is a positive constant referred to as the base. The functions

\[ f(x) = 2^x, \quad g(x) = e^x, \quad \text{and} \quad h(x) = \left( \frac{1}{2} \right)^x \]

are all exponential functions.

Of course, if we choose \( x \) to be a nonnegative integer, then \( a^x \) has a special interpretation. Consider \( h(x) \) above; if we evaluate \( h \) at \( x = 5 \), we have

\[ h(5) = \left( \frac{1}{2} \right)^5 = \frac{1}{32} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}. \]

In a sense, \( x \) just counts how many copies of the \( 1/2 \) appear.

It is extremely important to distinguish between exponential functions and polynomial functions. For example, let’s compare the exponential function \( f(x) = 2^x \) to the polynomial function \( p(x) = x^2 \). These two functions are completely different:

- \( f(x) \) raises the constant 2 to the variable power \( x \),

while

- \( p(x) \) raises the variable \( x \) to the constant power 2.

In other words, exponential functions have constant bases but variable powers, while polynomial functions have variable bases but constant powers.

It will be extremely helpful to understand the general shape of the graph of an exponential function. The following graph illustrates several exponential functions with bases \( a > 1 \):
The graph below illustrates some exponential functions with bases $0 < a < 1$:

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**Properties of Exponential Functions**

Exponential functions have many of the same properties that we are used to seeing when working with polynomials. The following list details the domain, range, and rules for combining exponential functions.

Let $a > 0$, $b > 0$, $a \neq 1$.

1. $a^x$ is continuous
2. $a^x$ has domain $(-\infty, \infty)$
3. $a^x$ has range $(0, \infty)$
4. If $a > 1$, then $a^x$ is an increasing function.
5. If $0 < a < 1$, then $a^x$ is an decreasing function.
6. $a^{x+y} = a^x a^y$
7. $a^{x-y} = \frac{a^x}{a^y}$
8. $(a^x)^y = a^{xy}$
9. $(ab)^x = a^x b^x$
Calculus Properties of Exponential Functions

Of course, we would like to know what calculus has to say about exponential functions. In particular, we would like to understand:

1. limits,
2. derivatives, and
3. integrals of exponential functions.

Limits of Exponential Functions

Throughout the rest of the section, assume that \( a > 0 \).

Since exponential functions are continuous, finite limits agree with function values:

\[
\lim_{x \to c} a^x = a^c, \text{ for any real number } c.
\]

We would like to understand limits at infinity as well. You may have already guessed the following fact based on the graphs above:

1. If \( a > 1 \), then
   (a) \( \lim_{x \to \infty} a^x = \infty \)
   (b) \( \lim_{x \to -\infty} a^x = 0 \)
2. If \( 0 < a < 1 \), then
   (a) \( \lim_{x \to \infty} a^x = 0 \)
   (b) \( \lim_{x \to -\infty} a^x = \infty \)

Example. Evaluate

\[
\lim_{x \to \infty} 3 - \left(\frac{1}{2}\right)^x.
\]

Since \( 1/2 < 1 \), we know that

\[
\lim_{x \to \infty} \left(\frac{1}{2}\right)^x = 0;
\]

thus

\[
\lim_{x \to \infty} 3 - \left(\frac{1}{2}\right)^x = \lim_{x \to \infty} 3 - \lim_{x \to \infty} \left(\frac{1}{2}\right)^x = (\lim_{x \to \infty} 3) - 0 = 3,
\]

since the limit of a constant is the constant. Thus the line \( y = 3 \) is a horizontal asymptote of \( 3 - (1/2)^x \).
Derivative of the Natural Exponential Function

We will actually put off learning about the derivatives of most exponential functions until section 3.4; however, in this section, we will learn the derivative of one special exponential function, $e^x$, which we call the natural exponential function.

**Derivative of $e^x$.** The number $e$ is the number so that

$$\frac{d}{dx}e^x = e^x.$$  

In other words, $e^x$ is a function that describes its own rate of change.

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**Example.** Evaluate $\frac{d}{dx}e^{x/2}$.

Since the form of our function is not the same as the form of the rule we’ve just learned, we must think of it as a product, quotient, or composition. Clearly $e^{x/2}$ is a composition function; so we’ll differentiate using the chain rule.

Recall that we must find the “inside” function $g(x)$ and the outside function $f(x)$; then the chain rule says that

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)).$$

Let’s set up the chart:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$e^x$</th>
<th>$f'(x) = e^x$</th>
<th>$f'(g(x)) = e^{x/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>$x/2$</td>
<td>$g'(x) = 1/2$</td>
<td></td>
</tr>
</tbody>
</table>

So

$$\frac{d}{dx}e^{x/2} = \frac{1}{2}e^{x/2}.$$  

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**Example.** Evaluate $\frac{d}{dx}e^{x \sin x}$.

Again, we realize that this function is a composition, so that we must use the chain rule to find its derivative. Thinking of $g(x)$ as the inside function and $f(x)$ as the outside function, we have

<table>
<thead>
<tr>
<th>$f(x)$</th>
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<th>$f'(x) = e^x$</th>
<th>$f'(g(x)) = e^{x \sin x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>$x \sin x$</td>
<td>$g'(x) = ?$</td>
<td></td>
</tr>
</tbody>
</table>

Of course, to find the derivative of $x \sin x$, we’ll need to use the product rule:

$$\frac{d}{dx}x \sin x = \sin x + x \cos x.$$  

Finishing off the chart, we have

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$e^x$</th>
<th>$f'(x) = e^x$</th>
<th>$f'(g(x)) = e^{x \sin x}$</th>
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<tbody>
<tr>
<td>$g(x)$</td>
<td>$x \sin x$</td>
<td>$g'(x) = \sin x + x \cos x$</td>
<td></td>
</tr>
</tbody>
</table>
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Using the chain rule
\[ \frac{d}{dx} f(g(x)) = g'(x)f'(g(x)), \]
we see that
\[ \frac{d}{dx} e^{x \sin x} = (\sin x + x \cos x)e^{x \sin x}. \]

**Example.** Evaluate \( \frac{d}{dx} e^{e^x} \).

We have yet another composition function to differentiate. Let’s set up the chart for the chain rule:

\[
\begin{array}{c|c|c|c}
 f(x) &= e^x & f'(x) &= e^x & f'(g(x)) &= e^{e^x} \\
g(x) &= e^{e^x} & g'(x) &= ? \\
\end{array}
\]

Unfortunately, we don’t know how to differentiate \( g(x) \) immediately; we’ll have to apply the chain rule again:

\[
\begin{array}{c|c|c|c|c}
 out(x) &= e^x & out'(x) &= e^x & out'(in(x)) &= e^x \\
in(x) &= e^x & in'(x) &= e^x \\
\end{array}
\]

Thus
\[ g'(x) = e^x e^{e^x}, \]

returning to the original chart, we have

\[
\begin{array}{c|c|c|c|c}
 f(x) &= e^x & f'(x) &= e^x & f'(g(x)) &= e^{e^x} \\
g(x) &= e^{e^x} & g'(x) &= e^x e^{e^x}, \\
\end{array}
\]

so that
\[ \frac{d}{dx} e^{e^x} = e^x e^x e^{e^x}. \]

**Integral of the Natural Exponential Function**

The rule for integrating \( e^x \) should be clear from the the rule for differentiating it:
\[ \int e^x \, dx = e^x + C. \]

**Example.** Evaluate
\[ \int \frac{e^{\cot x}}{\sin^2 x} \, dx. \]

Clearly, we will need to use u-substitution to evaluate the integral. The most natural substitution to make seems to be
\[ u = \cot x \text{ so that } du = -\csc^2 x \, dx. \]
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Considering the original integral, however, it appears that we have an issue with this substitution:

\[ \int \frac{e^{\cot x}}{\sin^2 x} \, dx. \]

How do we replace the factor of \(1/\sin^2 x\)? Perhaps rewriting the original function will help. We know that

\[ \frac{1}{\sin x} = \csc x, \]

so

\[ \int \frac{e^{\cot x}}{\sin^2 x} \, dx = \int \csc^2 x e^{\cot x} \, dx. \]

Now we can replace all of the terms containing \(x\)s with terms containing \(u\)s!

We have

\[
\begin{align*}
\int \frac{e^{\cot x}}{\sin^2 x} \, dx &= \int \csc^2 x e^{\cot x} \, dx \\
&= - \int e^u \, du \\
&= -e^u + C \\
&= -e^{\cot x} + C.
\end{align*}
\]

Thus

\[ \int \frac{e^{\cot x}}{\sin^2 x} \, dx = -e^{\cot x} + C. \]