The Definite Integral

In the last section, we introduced the area problem: given a function $f(x) \geq 0$ on the interval $[a, b]$, can we find the area of the region below $f(x)$ above the $x$-axis?

We learned that we can think of the area of this region as a limit of sums of areas of approximating rectangles, given by the formula

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$  

This idea is extremely important; in this section, we will give the aforementioned “limit of a sum” a new name and practice calculating it.

**Definition 2.** If $f(x)$ is defined for $a \leq x \leq b$, divide the interval $[a, b]$ into $n$ equal subintervals of length $\Delta x = \frac{b-a}{n}$, and let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of the subintervals. Then the **definite integral** of $f$ from $x = a$ to $x = b$ is

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,$$

if the limit exists. We say that $f(x)$ is **integrable** if the limit exists.

The function $f(x)$ is called the **integrand**; the endpoints $a$ and $b$ of the interval are the **limits of integration**.

Since the definition of the definite integral involves the same limit of a sum that we investigated in the previous section, we should make an observation here: if $f(x) \geq 0$ on $[a, b]$ and integrable, then the area of the region under $f(x)$ from $x = a$ to $x = b$ is precisely

$$A = \int_{a}^{b} f(x) \, dx.$$  

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When attempting to evaluate a definite integral, we should be concerned about whether or not the integral actually exists, i.e. whether or not \( f(x) \) is integrable. Fortunately for us, Theorem 3 below tells us that the indicated limit does actually exist for many functions:

**Theorem 3.** If \( f(x) \) is continuous on \([a, b]\), then \( f(x) \) is integrable on \([a, b]\).

**Example.** In the last section, we investigated the area of the region under the curve \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \):

We wrote the area as a limit of sums of approximating rectangles,

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{i/n^3}.
\]

Using the definition above, we can rewrite the sum as a definite integral:

\[
\int_{0}^{1} \sqrt{x} \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{i/n^3}.
\]

**Example.** Evaluate

\[
\int_{0}^{2} 3x \, dx.
\]

We could attempt to evaluate this integral using the definition

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x,
\]

but we could also note that, since \( f(x) = 3x \) is greater than 0 from \( x = 0 \) to \( x = 3 \), we know that

\[
\int_{0}^{2} 3x \, dx
\]

is just the area of the region under \( 3x \) from \( x = 0 \) to \( x = 2 \), indicated below:
This region is just a triangle, whose area is given by $A = \frac{1}{2} \times bh$, so the area of the shaded region above is $A = \frac{1}{2} \cdot 2 \cdot 6 = 6$. Thus we conclude that

$$\int_0^2 3x \, dx = 6.$$  

Unfortunately, most definite integrals are not as easy to evaluate as the previous one; in such cases, we will have to find a way to actually evaluate the limit in

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

by hand.

To do so, it will be extremely helpful to rewrite the sum in a “closed form”; the following identities will help us do so:

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2},$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6},$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n + 1)}{2}\right)^2,$$

$$\sum_{i=1}^{n} c = nc,$$
\[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i, \]  

(2)

\[ \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i. \]

**Example.** Evaluate

\[ \int_{0}^{1} x^2 \, dx. \]

Using the formula

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

with

\[ \Delta x = \frac{1 - 0}{n} = \frac{1}{n} \]

and

\[ x_0 = 0, \; x_1 = \frac{1}{n}, \ldots, x_i = \frac{i}{n}, \]

we have

\[ \int_{0}^{1} x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1}{n} \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n} \]

\[ = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} i^2, \]

where the last step above is due to the fact that \( n \) is a constant with respect to the summation variable \( i \), thus may be factored out as in rule (2).
Next, we use rule (1):

\[
\lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}
\]

\[
= \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}
\]

\[
= \lim_{n \to \infty} \frac{(n^2 + n)(2n+1)}{6n^3}
\]

\[
= \lim_{n \to \infty} \frac{2n^3 + 2n^2 + n^2 + n}{6n^3}
\]

Notice that there is no summation left—in fact, the equation has reduced to a limit that we know how to evaluate! Let’s finish the problem:

\[
\lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3}
\]

\[
= \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6}
\]

\[
= \frac{2}{6} = \frac{1}{3}
\]

Thus we see that

\[
\int_{0}^{1} x^2 \, dx = \frac{1}{3}
\]

and since \( x^2 \geq 0 \) on \([0, 1]\), we can conclude that the area of the region below \( f(x) = x^2 \) from \( x = 0 \) to \( x = 1 \) is precisely \( 1/3 \).