Section 4.1

The Area Problem

Our next topic of study in this class will seem to be quite a departure from our discussion of derivatives and antiderivatives: we are about to study the problem of finding areas under curves. However, over the next few sections we will see that the idea of antiderivatives is closely tied to the problem of determining the area below a curve.

As a first example, suppose that we wish to know the area under the curve \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \), i.e. the area of the shaded region in the graph below:

![Graph of function f(x) = \sqrt{x}

At present, we do not have the tools to be able to answer this particular question, as the problem is not one that we can solve using simple geometry. In fact, we must use calculus to answer the question.

To get some motivation as to how to solve this problem, let’s think back to an earlier problem; at the beginning of this class, we wanted to find the instantaneous rate of change of a function, i.e. the slope of a tangent line to the curve:

![Graph of function f(x) and tangent line

Slope of the tangent line at \( x = 1 \) is the instantaneous rate of change of \( f \) at \( x = 1 \).
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Since we could not actually calculate the slope of the tangent line above, we used secant lines to help us out. For example, we calculated the slope of the tangent line below, and used its slope as an approximation for the slope of the tangent line:

Finally, we saw that we could increase the accuracy of our approximation for the slope of the tangent line by sliding the second point of the secant closer to the point of tangency:

In particular, we used the calculus idea of limits to go from an approximation to a precise answer.

To answer the question about area under a curve, we will go through a similar process: we’ll begin by estimating the answer using what we do know, see how to make the approximations better, and then use calculus to find an exact answer.

**Example.** Estimate the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \).

We want to estimate the area of the shaded region below:
We can estimate the area under $f(x)$ using a figure whose area is easy to compute—a rectangle! Let’s use the right endpoint of the graph above, $(1, 1)$, to “anchor” a rectangle on the graph:

The rectangle above has height $h = f(1) = \sqrt{1} = 1$ and length $l = 1 - 0 = 1$, so its area is $A = l \cdot h = 1$.

Thus we estimate that the area of the shaded region is close to 1; of course, from the picture it looks as if the area under $f(x) = \sqrt{x}$ is actually a bit less than 1.

We could get a better estimate by using more rectangles. For example, we could use two rectangles instead of one, as illustrated below:
We want to estimate the area under $f(x) = \sqrt{x}$ from $x = 0$ to $x = 1$ using two rectangles, so we break the interval $[0,1]$ into two equal pieces $[0, .5]$ and $[.5, 1]$ of length

$$\frac{\text{total length of interval}}{\text{number of rectangles}} = \frac{1}{2}.$$

We then “anchor” the rectangles to the curve using the right endpoints $x = .5$ and $x = 1$ of the intervals. The height of the first box is $f(.5) = \sqrt{.5}$, and the height of the second rectangle is $f(1) = \sqrt{1}$:

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Length</th>
<th>Height</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
<td>$f(.5) = \sqrt{.5} \approx .7$</td>
<td>$.5 \times .7 = .35$</td>
</tr>
<tr>
<td>2</td>
<td>.5</td>
<td>$f(1) = \sqrt{1} = 1$</td>
<td>$.5 \times 1 = .5$</td>
</tr>
</tbody>
</table>

We add the areas of these two rectangles, $.5 + .35 = .85$, to get a better approximation for the area under $\sqrt{x}$ from $x = 0$ to $x = 1$.

Comparing the graphics from which we built the approximations, we see that the approximation using two rectangles, while still an overestimate, is much closer to the actual area under the curve than was the approximation using one rectangle:

**Example.** Estimate the area under $f(x) = \sqrt{x}$ from $x = 0$ to $x = 1$ using two rectangles, whose heights are given by the left endpoints of the intervals.
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We want to estimate the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) using two rectangles, so we again break the interval \([0, 1]\) into two equal pieces \([0, .5]\) and \([.5, 1]\) of length

\[
\frac{\text{total length of interval}}{\text{number of rectangles}} = \frac{1}{2}.
\]

We again “anchor” the rectangles to the curve using the left endpoints \( x = 0 \) and \( x = .5 \) of the intervals. The height of the first rectangle is \( f(0) = \sqrt{0} \), and the height of the second rectangle is \( f(.5) = \sqrt{.5} \):

<table>
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<th>Area</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
<td>( f(0) = 0 )</td>
<td>.5 \times 0 = 0</td>
</tr>
<tr>
<td>2</td>
<td>.5</td>
<td>( f(.5) = \sqrt{.5} \approx .7 )</td>
<td>.5 \times .7 = .35</td>
</tr>
</tbody>
</table>

We add the areas of these two rectangles, \( 0 + .35 = .35 \), to get a left endpoint approximation for the area under \( \sqrt{x} \).

Let’s compare the graphics from the right endpoint and left endpoint approximations:

Our right endpoint approximation for the area was .85, and the left endpoint approximation for the area is .35. It is clear from the graphics that the first approximation is an overestimate of the actual area, while the second approximation is an underestimate.

If we wish to make our approximation for the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) even better, we should try using more rectangles. For example, compare the area approximation using two rectangles to an approximation using four rectangles:
Clearly the approximation with four rectangles is much closer to the actual area; it includes less “excess” than does the two rectangle approximation.

Example. Estimate the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) using:

1. 4 rectangles, with heights determined by the right endpoints of the intervals
2. 4 rectangles, with heights determined by the left endpoints of the intervals

1. To make our estimate, we will determine the area of each rectangle below, and then add the areas together:

Since we wish to estimate using four rectangles with right endpoints, we need to break the interval \([0, 1]\) into four equal pieces; the length of each piece is

\[
\frac{\text{total length of interval}}{\text{number of rectangles}} = \frac{1}{4} = .25.
\]

Our four intervals are thus

\([0, .25], [.25, .5], [.5, .75], \text{ and } [.75, 1]\.\]
We then “anchor” the rectangles to the curve using the right endpoints \( x = .25, \ x = .5, \ x = .75, \) and \( x = 1 \) of the intervals. We can determine the heights of each rectangle by evaluating \( f(x) \) at each of the endpoints. The chart below records the area of each rectangle:

<table>
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<th>Height</th>
<th>Area</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>.25</td>
<td>( f(.25) = \sqrt{.25} = .5 )</td>
<td>(.25 \times .5 = .125)</td>
</tr>
<tr>
<td>2</td>
<td>.25</td>
<td>( f(.5) = \sqrt{.5} \approx .7 )</td>
<td>(.25 \times .7 = .175)</td>
</tr>
<tr>
<td>3</td>
<td>.25</td>
<td>( f(.75) = \sqrt{.75} \approx .87 )</td>
<td>(.25 \times .87 = .2175)</td>
</tr>
<tr>
<td>4</td>
<td>.25</td>
<td>( f(1) = \sqrt{1} = 1 )</td>
<td>(.25 \times 1 = .25)</td>
</tr>
</tbody>
</table>

We add the areas of these four rectangles to get an approximation for the area:

\[
A \approx .125 + .175 + .2175 + .25 = .7675.
\]

2. To make our left endpoint estimate, we will determine the area of each rectangle below, and then add the areas together:

Since we wish to estimate using four rectangles with left endpoints, we need to break the interval \([0, 1]\) into four equal pieces; the length of each piece is

\[
\frac{\text{total length of interval}}{\text{number of rectangles}} = \frac{1}{4} = .25.
\]

Our four intervals are thus

\([0, .25], \ [.25, .5], \ [.5, .75], \) and \([.75, 1]\).

We then “anchor” the rectangles to the curve using the left endpoints \( x = 0, \ x = .25, \ x = .5, \) and \( x = .75 \) of the intervals. We can determine the heights of each rectangle by evaluating \( f(x) \) at each of the endpoints. The chart below records the area of each rectangle:

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<tr>
<td>1</td>
<td>.25</td>
<td>( f(0) = \sqrt{0} = 0 )</td>
<td>(.25 \times 0 = 0)</td>
</tr>
<tr>
<td>2</td>
<td>.25</td>
<td>( f(.25) = \sqrt{.25} = .5 )</td>
<td>(.25 \times .5 = .125)</td>
</tr>
<tr>
<td>3</td>
<td>.25</td>
<td>( f(.5) = \sqrt{.5} \approx .7 )</td>
<td>(.25 \times .7 = .175)</td>
</tr>
<tr>
<td>4</td>
<td>.25</td>
<td>( f(.75) = \sqrt{.75} \approx .87 )</td>
<td>(.25 \times .87 = .2175)</td>
</tr>
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</table>
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We add the areas of these four rectangles to get an approximation for the area:

\[ A \approx 0 + .125 + .175 + .2175 = .5175. \]

In the examples above, we approximated the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) using one rectangle; then we made our approximation better by using two rectangles:

Then we noted that we could make our approximation even better by using more rectangles; above, we approximated using four rectangles:

The approximation with four rectangles is much closer to the actual area, but we could do even better by using, say, 10 rectangles, as indicated below:
As you may have already guessed, we can go from an approximation for the area under the curve to the exact value by increasing the number of rectangles that we use in the approximation—more precisely, by *taking a limit as the number of rectangles goes to infinity*.

In order to be able to do this, we must develop some helpful notation. Consider the function $f(x)$ graphed below on the interval $[a, b]$:

We break the interval $[a, b]$ into $n$ equal pieces of length $\Delta x = \frac{b-a}{n}$, and use these pieces as the bases of $n$ rectangles over the graph of $f(x)$; here we use right endpoints of the intervals to determine the heights of the rectangles:
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We call the first rectangle $R_1$, the second $R_2$, etc. It is clear that the area of the $i$th rectangle is

$$A_i = f(x_i)\Delta x.$$  

Our approximation for the area under $f(x)$ from $x = a$ to $x = b$ is the sum of the areas of these rectangles, given by

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$  

Unfortunately, this notation is quite clumsy, so we use sigma notation to streamline our work: the notation

$$\sum_{i=1}^{n} a_i$$

simply means to add up the terms $a_1$, $a_2$, etc., through $a_n$:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$  

Thus we can write

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

instead of

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

to indicate our approximation for the area under $f(x)$ from $x = a$ to $x = b$:

As indicated earlier, we can find the exact value for the area under the curve by using limits—send the number of rectangles to infinity!

**Definition.** If $f(x)$ is continuous on $(a, b)$ and $f(c) \geq 0$ for every $c$ in $[a, b]$, then the area $A$ of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is the limit of a sum of areas of approximating rectangles, given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$
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It turns out that the points \( x_i \) chosen above can actually be any point in the \( i \)th interval—they do not have to be the right endpoints; we'll still get the correct area.

**Example.** Express the area under \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) as a limit of a sum of areas of rectangles.

We wish to use the formula

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,
\]

so we will need to calculate \( f(x_i) \) and \( \Delta x \). As usual, we will break the interval \([a, b] = [0, 1]\) into \( n \) pieces, as indicated below:

The length of each piece is

\[
\Delta x = \frac{b - a}{n} = \frac{1 - 0}{n} = \frac{1}{n}.
\]

So the intervals indicated above are \([0, 1/n], [1/n, 2/n], [2/n, 3/n], \) etc.:
We are going to use the right endpoints of the intervals to calculate the heights of the rectangles, which we can do easily using the height of the function $f(x) = \sqrt{x}$:

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Right Endpoint</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$\frac{1}{n}$</td>
<td>$f(\frac{1}{n}) = \sqrt{\frac{1}{n}}$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$\frac{2}{n}$</td>
<td>$f(\frac{2}{n}) = \sqrt{\frac{2}{n}}$</td>
</tr>
<tr>
<td>$R_3$</td>
<td>$\frac{3}{n}$</td>
<td>$f(\frac{3}{n}) = \sqrt{\frac{3}{n}}$</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_i$</td>
<td>$\frac{i}{n}$</td>
<td>$f(\frac{i}{n}) = \sqrt{\frac{i}{n}}$</td>
</tr>
</tbody>
</table>

Thus the height of a typical rectangle is

$$f(x_i) = \sqrt{\frac{i}{n}}.$$  

Finally, the area of the region under $f(x) = \sqrt{x}$ from $x = 0$ to $x = 1$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left( \sqrt{\frac{i}{n}} \right) \frac{1}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{i}{n^3}}$$

This limit is difficult to evaluate, so we will put off actually finding the area until the next section.