The Mean Value Theorem

In this section, we will look at two more theorems that tell us about the way that derivatives affect the shapes of graphs: Rolle’s Theorem and the Mean Value Theorem.

Rolle’s Theorem

Rolle’s Theorem gives us information about $f'(x)$ on the interval $[a, b]$ if $f(a) = f(b)$. Consider the three examples below:

- The function graphed below is $f(x) = \cos x$, $\pi/4 \leq x \leq 9\pi/4$:

![Graph of cos(x) from pi/4 to 9pi/4](image)

- The function graphed below is $g(x) = x^2 + 3$, $-1 \leq x \leq 1$:

![Graph of x^2 + 3 from -1 to 1](image)

- Next, we have $h(x) = 4$, $-2 \leq x \leq 2$:
Even though the three graphs are significantly different, they have a commonality: each starts and ends at the same height. They have another common feature, which we can see by comparing the three following graphs:
Each of the three functions has at least one point at which there is a horizontal tangent line, i.e. the derivative is 0. In fact, it turns out that it is impossible to graph a differentiable function $f(x)$ whose starting and ending points have the same height without forcing $f'(x) = 0$ at some point.

This phenomenon is described by the following theorem:

**Rolle's Theorem.** Suppose that $f(x)$ is a function satisfying the three conditions below:

- $f(x)$ is continuous on $[a, b]$,
- $f(x)$ is differentiable on $(a, b)$,
- $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ so that $f'(c) = 0$.

Notice that Rolle’s Theorem doesn’t work if the two endpoints of the interval have different $y$ coordinates; for example, on the interval $[1, 2]$, the function $f(x) = x$ has $f(1) = 1 \neq 2 = f(2)$; clearly $f(x)$ has no points with $f'(x) = 0$: 

Mean Value Theorem

Recall that the **average rate of change** of a function \( f(x) \) over an interval \([a, b] \) is the same as the slope of a secant line joining \((a, f(a))\) to \((b, f(b))\), and is calculated using the formula

\[
\frac{f(b) - f(a)}{b - a}.
\]

On the other hand, **instantaneous rate of change** of \( f(x) \) at the number \( x = c \) is the same as the slope of a tangent line to \( f(x) \) at the point \((c, f(c))\), and is calculated using the formula

\[
\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = f'(c).
\]

**Example.** As a motivating example for the ideas behind the Mean Value Theorem, suppose you manage to drive your car a total of 35 miles in 1/2 hr. Can we draw any conclusions about your instantaneous velocity from this information?

We can easily calculate your **average velocity** over the 30 minute time period; since you drove 35 miles in 1/2 hr, your average velocity was

\[
\frac{35 - 0}{\frac{1}{2} - 0} = 70 \text{ mi/hr}.
\]

Now we’ve only found your **average velocity** over the time period; can we draw any conclusions about your **instantaneous velocity** from this information? It seems reasonable to guess that you were traveling 70 mi/hr at some point in time. This is precisely what is predicted by the Mean Value Theorem below.

The Mean Value Theorem below tells us that we can make predictions about a function’s instantaneous rate of change on an interval if we know the function’s average rate of change. In fact, Rolle’s Theorem is actually just a special case of this theorem:

**Mean Value Theorem.** If \( f(x) \) is continuous on \([a, b] \) and differentiable on \((a, b)\), then there is a number \( c \) in the interval \((a, b)\) so that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

The Mean Value Theorem simply says that a function must attain its average rate of change over an interval as the instantaneous rate of change at some point in the interval. As an example, consider the function \( f(x) = -x^3 + 6x - 2 \), graphed below on the interval \([-3, 3]\):
The average rate of change of $f(x)$ on the interval $[-3, 3]$ is precisely the slope of the secant line joining $(-3, f(-3))$ to $(3, f(3))$:

The Mean Value Theorem says that there is at least one number $c$ between $-3$ and $3$ so that the slope of the tangent line to $f(x)$ at $c$ (i.e. $f'(c)$) matches up with the slope of this secant. In this example, there are actually two such points, marked below:
Notice that the tangent lines at these points appear to be parallel to the secant line, meaning that their slopes are the same:

**Example.** Given the function $f(x) = -x^3 + 6x - 2$ on the interval $[-3, 3]$, find the point(s) $c$ guaranteed by the Mean Value Theorem so that

$$f'(c) = \frac{f(3) - f(-3)}{3 - (-3)}.$$
Let’s start by computing the average rate of change of $f(x)$ on the interval $[-3, 3]$:

$$
\frac{f(3) - f(-3)}{3 - (-3)} = \frac{-11 - 7}{6} = \frac{-18}{6} = -3.
$$

The Mean Value Theorem says that there is a number $c$ between $-3$ and $3$ so that $f'(c) = -3$. To find $c$, we’ll need to know $f'(x)$. Since

$$
f'(x) = -3x^2 + 6,
$$
we want to find $c$ so that

$$
-3c^2 + 6 = -3;
$$
solving, we see that

$$
-3c^2 + 6 = -3
$$

$$
-3c^2 = -9
$$

$$
c^2 = 3
$$

$$
c = \pm\sqrt{3}.
$$

Thus we know that $x = -3$ and $x = 3$ are numbers so that

$$
f'(-3) = \frac{f(3) - f(-3)}{3 - (-3)} \quad \text{and} \quad f'(3) = \frac{f(3) - f(-3)}{3 - (-3)};
$$
equivalently, the slopes of tangent lines to $f(x)$ at $x = -3$ and $x = 3$ match up with the slope of the secant line joining the endpoints of $f(x)$ on $[-3, 3]$:
Example. Sheriff Smith clocks a car going 50 mi/h in a 55 mi/h zone on Highway 22; 30 minutes later, Deputy Jones, who is 35 miles down Highway 22, clocks the same car going 50 mi/hr. After conferring with the Sheriff, Deputy Jones decides to give the driver a ticket. Why?

We can easily calculate the car’s average rate of change over the 30 minute time period; since the car drove 35 miles in 1/2 hr, its average rate of change was

\[
\frac{35 - 0}{\frac{1}{2} - 0} = 70 \text{ mi/hr.}
\]

The Mean Value Theorem says that, at some point between time \( t = 0 \) and time \( t = .5 \), the car’s instantaneous velocity was the same as its average velocity over the 1/2 hour time period. Since the average velocity was 70, there was some point in time where the car was actually driving 70 mi/hr.