Derivatives of Trigonometric Functions

In this section, we will learn the derivatives of the six trigonometric functions. To do so, we must recall two limits that we analyzed in our first Mathematica lab:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \text{ and } \lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Let's find the derivative of $f(x) = \sin x$. So far, we don't have any information about this function that can immediately lead to a "shortcut" rule, so we must revert to the limit definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

In our case, this means that

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}.$$

The limit above looks rather intractable–in particular, keep in mind that

$$\sin(x+h) \neq \sin x + \sin h,$$

so there does not appear to be a good way to rewrite the function.

However, an identity that you learned in trigonometry *can* help us out:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Let's use this identity to rewrite $\sin(x+h)$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h - \sin x + \cos x\sin h}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h} \\
&= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right) \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}.
\end{aligned}$$

Let's tackle the two limits separately. Starting with

$$\lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h},$$

notice that $\sin x$ does not change when h changes–in other words, it is a *constant* with respect to h. Since constants can be "pulled out" of limits, we may rewrite the limit as

$$\begin{aligned} \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} \\ &= \sin x \cdot 0 \\ &= 0, \end{aligned}$$

since we have already seen that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Now let's look at the second limit,

$$\lim_{h\to 0} \frac{\cos x \sin h}{h}.$$

Again, $\cos x$ is a constant with respect to the limiting variable h, which means that we can rewrite the limit as

$$\begin{aligned} \lim_{h\to 0} \frac{\cos x \sin h}{h} &= \cos x \lim_{h\to 0} \frac{\sin h}{h} \\ &= \cos x \cdot 1 \\ &= \cos x, \end{aligned}$$

since we know that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Let's put all of this information together: we have just seen that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\ &= 0 + \cos x \\ &= \cos x. \end{aligned}$$

This solves the problem: we have just seen that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x.$$

Using similar reasoning, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x.$$

To get the derivatives of the remaining four trig functions, it is helpful to recall that each of them can be built from $\sin x$ and $\cos x$:

$$\tan x = \tfrac{\sin x}{\cos x} \qquad \sec x = \tfrac{1}{\cos x}$$

$$\cot x = \tfrac{\cos x}{\sin x} \qquad \csc x = \tfrac{1}{\sin x}$$

In other words, we can find the derivatives of each of the functions above by using the derivatives of $\sin x$ and $\cos x$, along with the quotient rule. Let's start by finding

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x}.$$

Recall that the quotient rule says that

$$\frac{\mathrm{d}}{\mathrm{dx}}\frac{f(x)}{g(x)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.$$

Thinking of $f(x) = \sin x$ and $g(x) = \cos x$, let's write out the pieces that we'll need:

$$\begin{array}{lll} f(x) = \sin x & f'(x) = \cos x & \\ g(x) = \cos x & g'(x) = -\sin x & (g(x))^2 = \cos^2 x \end{array}$$

Using the quotient rule, we see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x. \end{aligned}$$

Thus our rule for the derivative of $\tan x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x.$$

Let's find

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos x}.$$

Using the quotient rule

$$\frac{\mathrm{d}}{\mathrm{dx}}\frac{f(x)}{g(x)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

with $f(x) = 1$ and $g(x) = \cos x$, we have:

$$\begin{array}{lll} f(x) = 1 & f'(x) = 0 & \\ g(x) = \cos x & g'(x) = -\sin x & (g(x))^2 = \cos^2 x \end{array}$$

So the derivative is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos x} \\ &= \frac{0\cdot\cos x - 1(-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \\ &= \sec x\tan x. \end{aligned}$$

Thus our rule for the derivative of $\sec x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x.$$

Using similar reasoning, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x.$$

To summarize, we have seen that:

$$\begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x & \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \\ \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x \\ \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x & \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x \end{array}$$

Example. Differentiate each of the functions below.

1. $f(x) = \cos x \tan x$
2. $g(x) = x^2 \csc x$
3. $h(x) = \sin^2 x$
4. $j(x) = \frac{x \sec x}{\sqrt{x}+1}$
5. $k(x) = x^2 - \pi \cot x + \sqrt{2}$
6. $m(x) = \frac{\cos x}{x} + \frac{x}{\cos x}$

1. Since $f(x)$ is a product, we will use the product rule to differentiate it. The derivative of $\cos x$ is $-\sin x$ and the derivative of $\tan x$ is $\sec^2 x$, so

$$\begin{aligned} f'(x) &= -\sin x \tan x + \cos x \sec^2 x \\ &= -\sin x \cdot \frac{\sin x}{\cos x} + \cos x \cdot \frac{1}{\cos^2 x} \\ &= \frac{-\sin^2 x}{\cos x} + \frac{1}{\cos x} \\ &= \frac{1 - \sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x. \end{aligned}$$

2. The function $g(x)$ is also a product, so we will again differentiate using the product rule. The derivative of x^2 is $2x$ and the derivative of $\csc x$ is $-\csc x \cot x$, so

$$\begin{aligned} g'(x) &= 2x \csc x + x^2(-\csc x \cot x) \\ &= 2x \csc x - x^2 \csc x \cot x \\ &= x \csc x(2 - x \cot x). \end{aligned}$$

3. Since $h(x) = \sin^2 x$ can be rewritten as $h(x) = \sin x \cdot \sin x$, it is clear that we can use the product rule here, too. The derivative of $\sin x$ is $\cos x$, so the product rule says that

$$\begin{aligned} h'(x) &= \cos x \sin x + \sin x \cos x \\ &= 2 \sin x \cos x \\ &= \sin(2x). \end{aligned}$$

4. We will need the quotient rule here. However, when we differentiate the numerator $x \sec x$, we will also need to use the product rule! Let's start by finding the derivative of $x \sec x$, then use this derivative in our chart for the quotient rule. Since the derivative of x is 1 and the derivative of $\sec x$ is $\sec x \tan x$, the product rule says that

$$\frac{\mathrm{d}}{\mathrm{d}x} x \sec x = \sec x + x \sec x \tan x = \sec x(1 + x \tan x).$$

Now we need to go back and use the quotient rule to differentiate the original function,

$$j(x) = \frac{x \sec x}{\sqrt{x} + 1}.$$

Let's start with a chart to help us out:

$$f(x) = x \sec x \qquad f'(x) = \sec x(1 + x \tan x)$$

$$g(x) = \sqrt{x} + 1 = x^{1/2} + 1 \qquad g'(x) = \tfrac{1}{2} x^{-1/2} = \tfrac{1}{2\sqrt{x}} \qquad (g(x))^2 = (\sqrt{x} + 1)^2,$$

so that

$$j'(x) = \frac{\sec x(1 + x \tan x)(\sqrt{x} + 1) - \frac{x \sec x}{2\sqrt{x}}}{(\sqrt{x} + 1)^2}.$$

5. We can differentiate each term separately. Note that π and $\sqrt{2}$ are both constants. However, the derivative of $\sqrt{2}$ is 0, while the derivative of $\pi \cot x$ is π *multiplied by the derivative* of $\cot x$. We have

$$k'(x) = 2x - \pi(-\csc^2 x) + 0 = 2x + \pi \csc^2 x.$$

6. Again, we may differentiate each term separately. Each one will require the quotient rule; since the derivative of $\cos x$ is $-\sin x$ and the derivative of x is 1, the derivative of $m(x)$ is

$$m'(x) = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}.$$