Section 2.3

Differentiation Rules

Although the limit definition of the derivative is quite useful, it is a bit annoying to calculate
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
each time we wish to find a derivative. Fortunately, we can generalize the process to get "shortcut" rules for differentiation. In this section we will collect many such rules.

1. Derivatives of Constant Functions: Let’s think about the derivative of the constant function \( f(x) = 4 \), graphed below:

Of course, the graph of \( f(x) = 4 \) is a horizontal line. Notice that the values for \( f(x) \) never change. Since \( f'(x) \) is a description of the rate of change of \( f(x) \), as well as of the slopes of lines tangent to \( f(x) \), it seems reasonable to guess that the derivative function for \( f(x) \) is \( f'(x) = 0 \).

Let’s check this using the limit definition:
\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - 4}{h} = \lim_{h \to 0} 0 = 0.
\]

So our guess was indeed correct. But we could make the same observations about any constant function \( f(x) = c \)—the function does not change as \( x \) varies, so its derivative should be 0. To summarize,

If \( f(x) = c \) is a constant function, then \( f'(x) = 0 \).
Example. If \( f(x) = -10 \), then
\[
f'(x) = 0.
\]

Example. If \( g(x) = \pi \), then
\[
g'(x) = 0.
\]

2. **Power rule:** Functions of the form \( f(x) = x^n \) have very nice derivatives, as indicated by the following rule:

\[
\text{If } f(x) = x^n, \text{ where } n \text{ is any real constant, then } f'(x) = nx^{n-1}.
\]

Let’s check one case, say for \( n = 3 \); then \( f(x) = x^3 \) and the rule above indicates that \( f'(x) = 3x^2 \). To check the power rule in this case, we need to evaluate

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}
= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}
= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}
= \lim_{h \to 0} \frac{3x^2 + 3xh + h^2}{h}
= 3x^2 + 3x \cdot 0 + 0^2
= 3x^2.
\]

So the theorem checks out in this case.

In the previous section, we calculated the derivative of \( f(x) = \frac{1}{x^3} \), and found that
\[
f'(x) = -\frac{3}{x^4}.
\]

Let’s compare this with the result of the power rule: in this case, rewriting \( f(x) = \frac{1}{x^3} \) as \( f(x) = x^{-3} \), we see that \( n = -3 \) and \( n - 1 = -4 \). Thus the power rule says that
\[
f'(x) = -3x^{-4} = -\frac{3}{x^4},
\]
which matches up with our earlier calculation.

Example. If \( f(x) = x^{10} \), then \( n = 10 \) and \( n - 1 = 9 \), so that
\[
f'(x) = 10x^9.
\]
Example. If \( g(x) = x \), then \( n = 1 \) and \( n - 1 = 0 \), so that
\[
g'(x) = 1 \cdot x^0 = 1.
\]

Example. If \( h(x) = \frac{1}{\sqrt{x}} \), we might wish to rewrite \( h(x) \) as \( h(x) = x^{-1/4} \). Now we see that we can use the power rule: with \( n = -1/4 \) and \( n - 1 = -5/4 \), we have
\[
h'(x) = -\frac{1}{4}x^{-5/4} = \frac{-1}{4x^{5/4}}.
\]

3. Derivatives of constant multiples of functions: If we know the derivative \( f'(x) \) of \( f(x) \), then we can also determine the derivative of any constant multiple of \( f(x) \):

If \( c \) is a constant and \( f(x) \) is differentiable, then \( \frac{d}{dx} cf(x) = cf'(x) \).

In other words, constants are really just distractions; to find the derivative of \( cf(x) \), we can calculate the derivative of \( f(x) \), then multiply the derivative by the constant.

Be careful to distinguish between constant functions and constant multiples of functions. \( f(x) = 7 \) is a constant function (since it never changes), but \( g(x) = 7x^2 \) is a constant multiple of \( x^2 \).

Example. If \( f(x) = -10x^{10} \), then
\[
f'(x) = -10 \left( \frac{d}{dx} x^{10} \right) = -10(10x^9) = -100x^9.
\]

Example. If \( g(x) = \pi x \), then
\[
g'(x) = \pi \left( \frac{d}{dx} x \right) = \pi(1) = \pi.
\]

Example. If \( h(x) = \frac{1}{\sqrt{x}} \), then
\[
h'(x) = \frac{1}{5} \left( \frac{d}{dx} \frac{1}{\sqrt{x}} \right) = \frac{1}{5} \left( \frac{-1}{4x^{5/4}} \right) = \frac{-1}{20x^{5/4}}.
\]

4. Derivatives of sums and differences: Given a pair of functions \( f(x) \) and \( g(x) \), we can create many new functions by combining these two in different ways. The functions \( f(x) + g(x) \) and \( f(x) - g(x) \) are two particular examples. If we know the derivatives of \( f(x) \) and \( g(x) \), then we hope that we can use these functions to find the derivative of \( f(x) + g(x) \) and \( f(x) - g(x) \).
This turns out to work exactly as we would expect—simply add or subtract the derivatives of \( f(x) \) and \( g(x) \) as indicated below:

\[
\text{If } f(x) \text{ and } g(x) \text{ are differentiable, then } \frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x).
\]

In other words, the derivative of a sum or difference is the sum or difference of the derivatives.

**Example.** If \( f(x) = \pi x - 10x^{10} \), then

\[
f'(x) = \pi - 100x^9.
\]

**Example.** If \( g(x) = \frac{1}{\sqrt{x}} - x^{10} \), then

\[
g'(x) = -\frac{1}{20x^{5/4}} - 10x^9.
\]

**Example.** If \( h(x) = \pi + 3x \), then

\[
h'(x) = 0 + 3 = 3.
\]

5. **Product rule:** Given the last rule, we would hope that the derivative of the product \( f(x) \cdot g(x) \) would simply be the product of the derivatives; unfortunately this is not true. The derivative is a bit more complicated:

\[
\text{If } f(x) \text{ and } g(x) \text{ are differentiable, then } \\
\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x).
\]

**Example.** Use the product rule to differentiate \( h(x) = x^3x^{10} \).

Since we are going to use the product rule, we need to pick out the "pieces" \( f(x) \) and \( g(x) \) of \( h(x) \), then find the derivatives of each of the pieces:

\[
\begin{align*}
  f(x) &= x^3 \\
  g(x) &= x^{10} \\
  f'(x) &= 3x^2 \\
  g'(x) &= 10x^9
\end{align*}
\]

Since

\[
\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x),
\]

\[
\frac{d}{dx}(x^3 \cdot x^{10}) = 3x^2 \cdot x^{10} + x^3 \cdot 10x^9
\]

\[
= 3x^{12} + 10x^{12} = 13x^{12}
\]
then

\[ h'(x) = 3x^2x^{10} + x^3(10x^9) \]
\[ = 3x^{12} + 10x^{12} \]
\[ = 13x^{12}. \]

Notice that if we had rewritten \( h(x) \) as \( h(x) = x^3x^{10} = x^{13} \), then we could have used the power rule immediately to find \( h'(x) = 13x^{12} \).

**Example.** Use the product rule to differentiate \( h(x) = \frac{3x^2+1}{x} \).

If we rewrite

\[ h(x) = (3x^2 + 1)x^{-1}, \]

we see that \( h(x) \) is a *product* of \( f(x) = 3x^2 + 1 \) and \( g(x) = x^{-1} \), so we may use the product rule to differentiate:

\[
\begin{align*}
f(x) &= 3x^2 + 1 \\
g(x) &= x^{-1} \\
f'(x) &= 6x \\
g'(x) &= -x^{-2}
\end{align*}
\]

Since

\[
\frac{d}{dx} (f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x),
\]

then

\[
\begin{align*}
h'(x) &= 6x \cdot x^{-1} + (3x^2 + 1)(-x^{-2}) \\
&= \frac{6x}{x} - \frac{3x^2 + 1}{x^2} \\
&= \frac{6x}{x} \cdot \frac{x}{x} - \frac{3x^2 + 1}{x^2} \\
&= \frac{6x^2}{x^2} - \frac{3x^2 + 1}{x^2} \\
&= \frac{6x^2 - 3x^2 - 1}{x^2} \\
&= \frac{3x^2 - 1}{x^2}.
\end{align*}
\]

6. **Quotient rule:** Since derivatives of products do not work out particularly nicely, we would not expect derivatives of quotients to be simple either. In fact,
If \( f(x) \) and \( g(x) \) are differentiable, and if \( g(x) \neq 0 \), then
\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}.
\]

**Example:** Use the quotient rule to differentiate \( h(x) = \frac{3x^2+1}{x} \).
We’ll think of \( f(x) = 3x^2 + 1 \) and \( g(x) = x \). Writing out all of the "pieces" of \( h(x) \) gives us:
\[
\begin{align*}
    f(x) &= 3x^2 + 1, & f'(x) &= 6x, & (g(x))^2 &= x^2 \\
    g(x) &= x, & g'(x) &= 1.
\end{align*}
\]
Since
\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2},
\]
we have
\[
h'(x) = \frac{6x \cdot x - (3x^2 + 1)(1)}{x^2} = \frac{6x^2 - 3x^2 - 1}{x^2} = \frac{3x^2 - 1}{x^2},
\]
which is exactly what we calculated when we used the product rule.

**Example.** Use the quotient rule to differentiate \( h(x) = \frac{1}{x+\frac{1}{x}} \).

Let’s start with a chart. With \( f(x) = 1 \) and \( g(x) = x + \frac{1}{x} \), we have
\[
\begin{align*}
    f(x) &= 1, & f'(x) &= 0, & (g(x))^2 &= (x + \frac{1}{x})^2 \\
    g(x) &= x + \frac{1}{x} = x + x^{-1}, & g'(x) &= 1 - x^{-2} = 1 - \frac{1}{x^2}.
\end{align*}
\]
So according to the chain rule,
\[
h'(x) = \frac{0 \cdot (x + \frac{1}{x}) - 1 \cdot (1 - \frac{1}{x^2})}{(x + \frac{1}{x})^2}
= \frac{-1 + \frac{1}{x^2}}{(x + \frac{1}{x})^2}
= \frac{-x^2 + 1}{x^2} \cdot \frac{1}{(x + \frac{1}{x})^2}
= \frac{1 - x^2}{x^2 (x + \frac{1}{x})^2}.
\]
Note that when we use the product rule, the order of $f$ and $g$ does not matter since

$$f'(x)g(x) + f(x)g'(x) = f(x)g'(x) + f'(x)g(x);$$

but when we need the quotient rule we must be extremely careful with the order of $f$ and $g$ since

$$f'(x)g(x) - f(x)g'(x) \neq f(x)g'(x) - f'(x)g(x).$$

Below we summarize the differentiation rules that we have learned in this section. Recall that $c$ and $n$ are constants.

\[
\begin{align*}
\frac{d}{dx} c &= 0 \\
\frac{d}{dx} cx^n &= nx^{n-1} \\
\frac{d}{dx} c f(x) &= cf'(x) \\
\frac{d}{dx} (f(x) \pm g(x)) &= f'(x) \pm g'(x) \\
\frac{d}{dx} (f(x) \cdot g(x)) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\
\frac{d}{dx} \frac{f(x)}{g(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{align*}
\]