Section 2.2

The Derivative as a Function

In the last section, we calculated the derivative \( f'(1) \) of the function \( f(x) = x^{-3} \) at \( x = -1 \). We found that \( f'(-1) \) is

(a) The instantaneous rate of change of \( f(x) = x^{-3} \) at \( x = -1 \), and

(b) The slope of the tangent line to \( f(x) = x^{-3} \) at \( x = -1 \). The tangent line is graphed below.

\[
\text{Slope of tangent line is } f'(-1).
\]

If we had wanted to know the slope of the line tangent to \( f(x) \) at \( x = 2 \), or \( x = 10 \), or \( x = -3 \), we would have needed to calculate

\[
f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h},
\]
\[
f'(10) = \lim_{h \to 0} \frac{f(10 + h) - f(10)}{h}, \quad \text{or}
\]
\[
f'(-3) = \lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h},
\]

separately.

In fact, at any \( x \) point other than \( x = 0 \), we could have tried to calculate the slope of the tangent line \( f'(x) \) (\( f(x) = x^{-3} \) is undefined when \( x = 0 \), so \( f'(0) \) doesn’t make sense). In other words, we can think about the derivative as being not just a number, but a function in its own right:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]
Definition. The derivative \( f'(x) \) of the function \( f(x) \) is the function defined by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},
\]

whenever this limit exists. The derivative \( f'(x) \) is the function that describes

(a) The instantaneous rate of change of \( f(x) \) at any point \( x \) where the limit exists, and

(b) The slope of the tangent line to \( f(x) \) at any point \( x \) where the limit exists.

Example. Find the derivative \( f'(x) \) of \( f(x) = x^{-3} \), and use it to write the equation of the line tangent to \( f(x) \) at \( x = 1 \).
To find the derivative, we need to evaluate the limit below:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}
\]

\[
= \lim_{h \to 0} \frac{\frac{1}{(x+h)^3} \cdot x^3 - \frac{1}{x^3} \cdot (x+h)^3}{h}
\]

\[
= \lim_{h \to 0} \frac{x^3 - (x+h)^3}{h \cdot x^3 - (x+h)^3}
\]

\[
= \lim_{h \to 0} \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{hx^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{hx^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{hx^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{hx^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{hx^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{x^3(x + h)^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2}{x^3 \cdot x^3}
\]

\[
= \lim_{h \to 0} \frac{-3x^2}{x^6}
\]

\[
= \frac{-3}{x^4}
\]

So the derivative of \( f(x) = \frac{1}{x^2} \) is the function \( f'(x) = \frac{-3}{x^4} \).

We would like to write the equation of the line tangent to \( f(x) \) at \( x = 1 \). Recall that, in order to write the equation of a line, we must know

- A point through which the line passes
- The slope of the line.

We know that our tangent line should touch the curve when \( x = 1 \), i.e. at the point \((1, f(1)) = (1, 1)\), so we will use the point \((1, 1)\) to help us write the desired equation.
In order to determine the slope of the line tangent to \( f(x) \), we need simply use the derivative! We know that \( f'(1) \) is precisely the slope of the line tangent to \( f(x) \) at \( x = 1 \). Since \( f'(x) = \frac{-3}{x^4} \), we know that the slope of the desired tangent line is

\[
f'(1) = \frac{-3}{1} = -3.
\]

So the equation of the tangent line is given by

\[
y - 1 = -3(x - 1) \text{ or } y = 4 - 3x.
\]

**Example:** Find the function that is the derivative of \( f(x) = \frac{x}{x+4} \).

We need to calculate

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h+4} - \frac{x}{x+4}}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+h)(x+4) - x(x+h+4)}{h(x+h+4)(x+4)}
\]

\[
= \lim_{h \to 0} \frac{x^2 + xh + 4x + 4h - x^2 - xh - 4x}{h(x+h+4)(x+4)}
\]

\[
= \lim_{h \to 0} \frac{4h}{h(x+h+4)(x+4)}
\]

\[
= \frac{4}{(x + 4)(x + 4)}
\]

\[
= \frac{4}{(x + 4)^2}.
\]

So the derivative of \( f(x) = \frac{x}{x+4} \) is the function \( f'(x) = \frac{4}{(x+4)^2} \).

**Example.** Use the graph of \( f(x) \) below to sketch a rough graph of the function \( f'(x) \).
In order to draw our rough graph of $f'(x)$, we should interpret the values for $f'(x)$ as the slopes of tangent lines to $f(x)$.

The first thing to note here is that, at around $x = -2.25$, $x = 0$, and $x = 2.25$, the function appears to have horizontal tangent lines, as indicated below:

In other words, we know that

$$f'(-2.25) = 0, \quad f'(0) = 0, \quad \text{and} \quad f'(2.25) = 0.$$ 

Let’s go ahead and plot these points on our graph:
Next, note that slopes of tangent lines are *negative* when \( x < -2.25 \); similarly, slopes are positive when \(-2.25 < x < 0\), negative when \(0 < x < 2.25\), and positive again when \(2.25 < x\):

In other words, we know that \( f'(x) < 0 \) when \( x < -2.25 \); \( f'(x) > 0 \) when \(-2.25 < x < 0\), \( f'(x) < 0 \) when \(0 < x < 2.25\); and \( f'(x) > 0 \) when \(2.25 < x\). Using this information, we get the following rough sketch of the graph of \( f'(x)\):
Alternate Notation for the Derivative

We will occasionally use alternate notation to indicate the derivative \( f'(x) \) of the function \( y = f(x) \). All of the following indicate that we are referring to the derivative of \( f(x) \):

\[
\frac{df}{dx} = \frac{dy}{dx} = y' = \frac{df}{dx}.
\]

Differentiability

Earlier, we noted that the function \( f(x) = x^{-3} \) does not have a derivative at \( x = 0 \), since \( f(x) \) is not defined at \( x = 0 \). However, \( f(x) \) does have a derivative at all other \( x \) values. The following definition allows us to distinguish between these two possibilities:

**Definition 3.** The function \( f(x) \) is differentiable at \( x = a \) if \( f'(a) \) exists.

Using the definition, we see that \( f(x) = x^{-3} \) is differentiable for all \( x \) except for \( x = 0 \).

**Example.** Determine if \( f(x) = |x| \) is differentiable at \( x = 0 \).

To answer the question, we must determine whether or not \( f'(0) \) exists, which amounts to checking the limit

\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}.
\]

To do so, we need to rewrite \( f(x) = |x| \) as a piecewise defined function:

\[
f(x) = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0.
\end{cases}
\]
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Since the rule for \( f(x) \) changes at \( x = 0 \), we must break up the general limit into left-hand and right-hand limits.

To check the right-hand limit, we must start by noting that as \( h \to 0 \) from the right, \( 0 + h > 0 \). So we need to use the rule \( f(x) = x \) for the limit from the right:

\[
\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.
\]

On the other hand, as \( h \to 0 \) from the left, \( 0 + h < 0 \). So we need to use the rule \( f(x) = -x \) for the limit from the left:

\[
\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.
\]

Since

\[
\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} \neq \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h},
\]

the general limit

\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \text{ DNE},
\]

which means that \( f(x) = |x| \) is not differentiable at \( x = 0 \).

This example is interesting because, even though \( f(x) = |x| \) is defined at \( x = 0 \), it is not differentiable at \( x = 0 \). Below, we discuss the various ways in which a function can fail to be differentiable at a point.

The following list enumerates the ways in which a function \( f(x) \) can fail to be differentiable at a point \( x = a \):

1. If \( f(x) \) is not defined at \( x = a \), then \( f(x) \) is not differentiable at \( x = a \).
2. If \( f(x) \) has a corner at \( x = a \), then \( f(x) \) is not differentiable at \( x = a \).
3. If \( f(x) \) has a vertical tangent line at \( x = a \), then \( f(x) \) is not differentiable at \( x = a \).
4. If \( f(x) \) is discontinuous at \( x = a \), then \( f(x) \) is not differentiable at \( x = a \).

Let’s look at examples of each type of discontinuity.

1. The function \( f(x) = x^{-3} \) is undefined at \( x = 0 \), so \( f(x) \) is not differentiable at \( x = 0 \). This makes sense since there is no way to draw a line tangent to \( f(x) \) at \( x = 0 \):
2. Earlier, we saw that the function $f(x) = |x|$ is not differentiable at $x = 0$. Inspecting the graph of $f(x)$, we see that this makes sense:

At $x = 0$, the function has a corner. There is no hope of being able to draw a line tangent to $f(x)$ at $x = 0$.

3. The function $f(x) = \sqrt[3]{x}$ has a vertical tangent line at $x = 0$:
Since the slope of a vertical line is infinite, \( f(x) \) is not differentiable at \( x = 0 \).

4. Let \( f(x) \) be the piecewise defined function given by

\[
f(x) = \begin{cases} 
2 & \text{if } x < -1 \\
1 & \text{if } x \geq -1.
\end{cases}
\]

It is impossible to draw a tangent line to \( f(x) \) at \( x = -1 \) since \( f \) is not continuous at \( x = -1 \), so \( f \) is not differentiable at \( x = -1 \).

Notice that, in examples 2 and 3, \( f(x) \) was continuous at \( x = 0 \), but not differentiable at \( x = 0 \). Thus it is entirely possible for a function to be continuous at a point without being differentiable at the point.

On the other hand, the following theorem tells us that differentiability is actually a guarantee of continuity.
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**Theorem 4.** If \( f(x) \) is differentiable at \( x = a \), then \( f \) is continuous at \( x = a \).

In a sense, the theorem just says that, if we can draw a tangent line to \( f(x) \) at \( x = a \) (differentiability), then \( f(x) \) must be fairly smooth near \( x = a \) (continuity).

**Higher-order derivatives:**

If \( f(x) \) is a function, then \( f'(x) \) is also a function in its own right. As we saw earlier, we can try graphing \( f'(x) \), and we might want to know information about slopes of tangent lines to \( f'(x) \). Thus it makes sense to think about the derivative of \( f'(x) \). If \( f'(x) \) is differentiable, then its derivative is denoted by \( f''(x) \). We can continue the process (as long as we keep getting differentiable functions) of taking derivatives:

\[
\begin{array}{c|c}
   f(x) & \text{first derivative of } f(x) \\
   f'(x) & \\
   f''(x) & \text{second derivative of } f(x), \text{first derivative of } f'(x) \\
   f'''(x) & \text{third derivative of } f(x), \text{first derivative of } f''(x) \\
   f^{(4)}(x) & \text{fourth derivative of } f(x) \\
   f^{(n)}(x) & \text{nth derivative of } f(x) \\
\end{array}
\]